Lecture 1: Introduction

Overview

Vector bundles arise in many parts of geometry, topology, and physics. The tangent bundle $TM \to M$ of a smooth manifold M is the first example one usually encounters. The tangent space T_pM is the linearization of the nonlinear space M at the point $p \in M$. Similarly, a nonlinear map between smooth manifolds has a linearization which is a map of their tangent bundles. Vector bundles needn't be tied to the intrinsic geometry of their base space. In quantum field theory, for example, extrinsic vector bundles are used to model subatomic particles. Sections of vector bundles are generalized vector-valued functions. For example, sections of the tangent bundle $TM \to M$ are vector fields on the manifold M.

The set $\mathrm{Vect}(X)$ of isomorphism classes of complex vector bundles on a topological space X is a *homotopy invariant* of X. It is a commutative monoid: a set with a commutative associative composition law with unit, represented by the zero vector bundle. The universal abelian group formed out of this monoid is $K(X)$, the *K-theory group* of X. It is a homotopy invariant of X. Topological K-theory was introduced in the late 1950s by Atiyah-Hirzebruch [AH], following Grothendieck's ideas [BS] in the sheaf theory context related to the Riemann-Roch problem (which in turn was solved in 1954 by Hirzebruch [Hi]). The abelian group $K(X) = K^0(X)$ is part of a generalized cohomology theory, and standard computational techniques in algebraic topology can be brought to bear. K-theory, which is constructed directly from linear algebra, is in many ways more natural than ordinary cohomology and turns out to be more powerful in many situations. Also, because of its direct connection to linear algebra it appears often in geometry and physics. One of the first notable sightings is in the 1963 Atiyah-Singer index theorem for elliptic operators [AS1].

Over the past few decades there has been renewed interest in K-theory, in large part due to its appearance in quantum field theory and string theory. For example, K-theory is the generalized cohomology theory which quantizes D-brane charges in superstring theory [MM], [W1]. (A truncation of real K-theory plays the same role for the B-field [DFM1], [DFM2].) The Atiyah-Singer index theorem, refined to include differential-geometric data, expresses the *anomaly* in the partition function of fermionic fields [AS2]. These two occurrences of K-theory combine [F1] to refine the original Green-Schwarz anomaly cancellation [GS], which catalyzed the first superstring revolution. In a different direction, K-theory enters into "geometric quantization in codimension two". These ideas are surveyed in [F2]. Modern applications often involve *twistings* of complex K-theory,[1] and fleshing out the theory of twisted K-theory has been one focus of recent mathematical activity. Twistings date from the 1960s in work of Donovan-Karoubi [DK], and from an operator point of view somewhat later in work of Rosenberg [Ro]. There are many modern treatments which develop a wide variety of models, ranging from the operator-theoretic to the geometric to the abstract homotopy-theoretic. In geometry twisted K-theory appears in the representation theory of loop groups [FHT1, FHT2, FHT3]: the fusion ring of positive energy representations of the loop group $LG = \mathrm{Map}(S^1, G)$ of a compact Lie group G at a fixed level is a twisted version of $K_G(G)$,

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[1] Real K-theory can be viewed as a particular twisting of complex K-theory, for example.

the equivariant K-theory of G acting on itself by conjugation. Recently K-theory has appeared in condensed matter physics as part of the classification of phases of matter [Ho, Ki, FM].

The course may cover a bit of this recent activity, but we will begin for awhile with basics and some classical results, especially Bott periodicity.

Vector spaces and linear representations

(1.1) *The "trivial" vector space* $\mathbb{C}^n$*.* Let n be a nonnegative integer. As a set $\mathbb{C}^n = \{(\xi^1, \dots, \xi^n) : \xi^i \in \mathbb{C}\}$. The vector space $\mathbb{C}^n$ has a canonical basis $(1,0,0,\dots,0)$, $(0,1,0,\dots,0)$, ... If we view the basis as part of the structure, then $\mathbb{C}^n$ is *rigid*: the only linear symmetry of $\mathbb{C}^n$ which fixes the canonical basis is the identity map $\mathrm{id}_{\mathbb{C}^n}$.

(1.2) *Abstract finite dimensional vector spaces.* In general vector spaces are not equipped with canonical bases, and so have nontrivial linear symmetries. Let $\mathbb{E}$ be a complex vector space, assumed finite dimensional. Its group of linear symmetries is denoted $\mathrm{Aut}(\mathbb{E}) = GL(\mathbb{E})$. The automorphism group $\mathrm{Aut}(\mathbb{C}^n)$ of $\mathbb{C}^n$ (without necessarily fixing the canonical basis) is identified with the group of invertible $n \times n$ matrices. A basis of $\mathbb{E}$ is an isomorphism $b\colon \mathbb{C}^n \to \mathbb{E}$, where $n = \dim \mathbb{E}$. The set $\mathcal{B}(\mathbb{E})$ of bases of $\mathbb{E}$ is a *right torsor*[2] for $\mathrm{Aut}(\mathbb{C}^n)$. There is a unique Hausdorff topology on $\mathbb{E}$ for which vector addition $+\colon \mathbb{E} \times \mathbb{E} \to \mathbb{E}$ is continuous. A basis determines a homeomorphism to the standard topology on $\mathbb{C}^n$. The vector space $\mathrm{End}(\mathbb{E})$ of all linear operators on $\mathbb{E}$ is finite dimensional, so also has a unique vector topology. The subset $\mathrm{Aut}(\mathbb{E})$ of invertible endomorphisms is open and is a topological group in the induced topology. In a natural way $\mathbb{E}$ can be given the structure of a smooth manifold and $\mathrm{Aut}(\mathbb{E})$ the structure of a Lie group.

A geometric structure on $\mathbb{E}$ cuts down the group $\mathrm{Aut}(\mathbb{E})$ of symmetries to a subgroup G. The Kleinian[3] point of view is that the subgroup G defines the geometric structure. We take a slightly more general point of view and take a geometric structure on $\mathbb{E}$ to be a homomorphism $G \to \mathrm{Aut}(\mathbb{E})$ from a Lie group G, which then acts linearly on $\mathbb{E}$.

Example 1.4. The most familiar examples occur for *real* vector spaces $\mathbb{E}$. For example, an inner product $\langle -,- \rangle$ on $\mathbb{E}$ has the orthogonal group $O(\mathbb{E}) \subset \mathrm{Aut}(\mathbb{E})$ as its group of symmetries. A symplectic form on $\mathbb{E}$ determines the symplectic subgroup of symmetries. On the other hand, a spin structure on $\mathbb{E}$ has a group of symmetries which does not act effectively on $\mathbb{E}$: the map $\mathrm{Spin}(\mathbb{E}) \to \mathrm{Aut}(\mathbb{E})$ is a 2:1 covering of a subgroup of $\mathrm{Aut}(\mathbb{E})$.

Remark 1.5 (Infinite dimensional vector spaces)*.* There is not a unique vector topology in infinite dimensions, but rather very different sorts of topological vector spaces: Hilbert spaces, Banach spaces, Fréchet spaces, ... In this course we mostly deal with Hilbert spaces. Recall that a Hilbert space $\mathcal{H}$ is a complex vector spaces equipped with a *complete* Hermitian inner product; it can be finite or infinite dimensional. There are many possible topologies on its group $\mathrm{Aut}(\mathcal{H})$ of

[2] The map

$$\begin{aligned} \mathcal{B}(\mathbb{E}) \times \mathrm{Aut}(\mathbb{C}^n) &\longrightarrow \mathcal{B} \times \mathcal{B} \\ (b, g) &\longmapsto (b, b \circ g) \end{aligned} \tag{1.3}$$

is an isomorphism, which is to say that $\mathrm{Aut}(\mathbb{C}^n)$ acts simply transitively on $\mathcal{B}(\mathbb{E})$.

[3] Felix, that is: *Erlanger Programm*.

automorphisms (and also on the subgroup $U(\mathcal{H})$ of *unitary* automorphisms—automorphisms which fix the inner product). The natural topology for us is the compact-open topology, on which we comment more in subsequent lectures.

Families of vector spaces; vector bundles

(1.6) *Spaces and smooth manifolds.* In this course we move back and forth between topological spaces $X, Y, \dots$ and smooth manifolds $M, N, \dots$ It is important in both cases that partitions of unity exist. Thus we assume all topological spaces are paracompact, and Hausdorff. In particular, we assume all smooth manifolds are paracompact and Hausdorff. These assumptions hold throughout.

Definition 1.7. Let X be a space. A *family of vector spaces parametrized by* X is a space E, a continuous surjection $\pi\colon E \to X$, and a finite dimensional vector space structure on each fiber $\pi^{-1}(x)$, $x \in X$ compatible with the topology of E.

To spell out what this compatibility is, recall that the data[4] for a single vector space consists of a set $\mathbb{E}$, a distinguished element $0 \in \mathbb{E}$, the operation of vector addition $+\colon \mathbb{E} \times \mathbb{E} \to \mathbb{E}$, and the operation of scalar multiplication $m\colon \mathbb{C} \times \mathbb{E} \to \mathbb{E}$. (For definiteness we consider complex vector spaces; an analogous discussion holds in the real case.) The data for the family is a zero section $z\colon X \to E$, vector addition[5] $+\colon E \times_X E \to E$, and scalar multiplication $m\colon \mathbb{C} \times E \to E$. The maps are required to be compatible with π: they preserve fibers. The compatibility in Definition 1.7 is that $z, +, m$ are all continuous maps.

Definition 1.8. Let $\pi\colon E \to X$ be a family of vector spaces parametrized by X. Its *rank* is the function $\operatorname{rank} E\colon X \to \mathbb{Z}^{\geqslant 0}$ defined by $(\operatorname{rank} E)(x) = \dim \pi^{-1}(x)$.

Heuristically (for the moment), we can imagine a parameter space $\mathcal{V}$ of all vector spaces, the components of $\mathcal{V}$ labeled by the dimension of the vector space. Then a family of vector spaces parametrized by X is a map $X \to \mathcal{V}$.

Example 1.9 (constant vector bundle)**.** Let $\mathbb{E}$ be a finite dimensional vector space. The constant (trivial) bundle with fiber $\mathbb{E}$ is $p_1\colon X \times \mathbb{E} \to X$, projection onto the first factor, with the constant vector space structure on fibers. We use the notation $\underline{\mathbb{E}} \to X$ for the trivial bundle with fiber $\mathbb{E}$.

Example 1.10 (tangent bundle)**.** The tangent bundle $\pi\colon TS^2 \to S^2$ is a non-constant family: the tangent spaces to the sphere at different points are not naturally identified with each other. In fact, the *hairy ball theorem* asserts that there does not exist a global nonzero section of π; every vector field on S^2 has a zero. A manifold whose tangent bundle admits a global basis of sections is termed *parallelizable* and such a basis is a global parallelism. The circle S^1 and 3-sphere S^3 are examples of parallelizable manifolds. (The 0-sphere S^0 is also parallelizable, trivially.) A theorem of Kervaire, Bott, and Milnor (independently) states that the only other parallelizable sphere is S^7. These four parallelizable spheres correspond to the four division algebras (reals, complexes, quaternions, octonions).

[4] As opposed to conditions, or axioms, of which there are many.

[5] Here $E \times_X E = \{(e_1, e_2) : \pi(e_1) = \pi(e_2)\}$ denotes the fiber product of π with itself: pairs of points in a fiber.

Example 1.11 (family associated to a linear operator)**.** Let $T\colon \mathbb{E} \to \mathbb{E}$ be a linear operator on a finite dimensional vector space. Define the family of vector spaces parametrized by $\mathbb{C}$ whose fiber at $x \in \mathbb{C}$ is $\ker(xI - T)^n$, where $I = \mathrm{id}_{\mathbb{E}}$ and $n = \dim \mathbb{E}$. The *support* of the family—the set of $x \in X$ where the fiber is not the zero vector space—consists of the eigenvalues of T and the corresponding fiber is the generalized eigenspace. The direct sum of the fibers is canonically isomorphic to $\mathbb{E}$. In this example the rank is not a continuous function.

If $\pi\colon E \to X$ and $\pi'\colon E' \to X$ are two families of vector spaces parametrized by X, then a morphism of the families is a continuous map $T\colon E \to E'$ such that $\pi' \circ T = \pi$ and T is linear on each fiber. It is an isomorphism if it is bijective with continuous inverse.

Definition 1.12. A family of vector spaces $\pi\colon E \to X$ parametrized by X is a *vector bundle* if it is locally trivial, i.e., if for each $x \in X$ there exists an open set $U \subset X$ containing x and an isomorphism of $\pi\big|_{\pi^{-1}(U)}\colon \pi^{-1}(U) \to U \times \mathbb{E}$ with a constant vector bundle with fiber some vector space $\mathbb{E}$.

For a vector bundle $\operatorname{rank} E\colon X \to \mathbb{Z}^{\geqslant 0}$ is a locally constant function.

Remark 1.13. We emphasize that local triviality is a condition, not data: the local trivializations are not part of the structure of a vector bundle. Local triviality is a key property of vector bundles, and more generally of *fiber bundles*, as exposed in the influential book of Norman Steenrod [St]. One of the earliest appearances in the literature is perhaps a 1935 paper of Whitney [Wh]. We will see in the next lecture that local triviality leads to homotopy invariance, consequently to topological invariants.

Example 1.10 is a vector bundle whereas Example 1.11 is not. The latter can be given the structure of a *sheaf*.

Remark 1.14. We will consider infinite rank vector bundles as well. The definition is the same, but we need to be careful about the topology on the vector spaces (Remark 1.5).

Definition 1.15. Let M be a smooth manifold. A vector bundle $\pi\colon E \to M$ is *smooth* if E is a smooth manifold, π is a smooth map, and the structure maps $z, +, m$ are smooth.

See the text following Definition 1.7 for the structure maps of a family of vector spaces. The tangent bundle of a smooth manifold (Example 1.10) is a smooth real vector bundle.

Clutching construction of vector bundles

(1.16) *Clutching on S^2.* The homotopy invariance we prove in the next lecture implies that any vector bundle over a contractible space is trivializable. Write the 2-sphere S^2 as the union $S^2 = B_+ \cup B_-$ of two balls $B_+ = S^2 \backslash \{p_+\}$, $B_- = S^2 \backslash \{p_-\}$, where $p_+ \neq p_-$ are distinct points on S^2. Any smooth complex *line*[6] bundle $L \to S^2$ can be trivialized over $B_\pm$. Fix isomorphisms $L\big|_{B_\pm} \xrightarrow{\cong} B_\pm \times \mathbb{C}$, for example by stereographic projection as illustrated in Figure 3. The ratio of the isomorphisms over $B_+ \cap B_-$ is a smooth map to $\mathbb{C}^\times$, the nonzero complex numbers. Fix a diffeomorphism

[6] A line is a vector space of dimension 1, so a line bundle is a vector bundle of constant rank 1.

$B_+ \cap B_- \cong \mathbb{C}^\times$, so that overlap data is identified with a smooth function $\phi\colon \mathbb{C}^\times \to \mathbb{C}^\times$. Another consequence of homotopy invariance is that the isomorphism class of the line bundle $L \to S^2$ depends only on the homotopy class of ϕ. The homotopy class is determined by the winding number of ϕ, and so isomorphism classes of complex line bundles $L \to S^2$ correspond to $\mathbb{Z}$; the integer invariant is called the *degree* of the line bundle. The tangent bundle has degree 2.

Remark 1.17. This generalizes to higher dimensional spheres and higher rank bundles. Rank N bundles on S^n are classified by homotopy classes of maps $S^{n-1} \to GL_N\mathbb{C}$. So the topology of this Lie group is fundamental for K-theory, and we will see shortly that it is the *stable* topology—the topology as $N \to \infty$—which is relevant. The *Bott periodicity theorem* determines the stable homotopy groups; see Theorem 1.31 below. It is the cornerstone of topological K-theory, and so we may end up giving 3 independent proofs in the course.

(1.18) *More general clutching; groupoids.* This gluing construction has a vast generalization. First, if X is a space and $\{U_i\}_{i\in I}$ an open cover, then we can imagine X as constructed from the disjoint union $\coprod_{i\in I} U_i$ by identifying $p_i \in U_i$ and $p_j \in U_j$ if they correspond to the same point of X. The situation is depicted in Figure 1. Double-headed arrows connect points which are glued. The clutching data for a vector bundle on X, then, is a vector bundle on the disjoint union together with isomorphisms of the fibers for each double-headed arrow. Furthermore, the isomorphisms must satisfy a consistency condition for pairs of arrows which share a vertex, as in the figure.

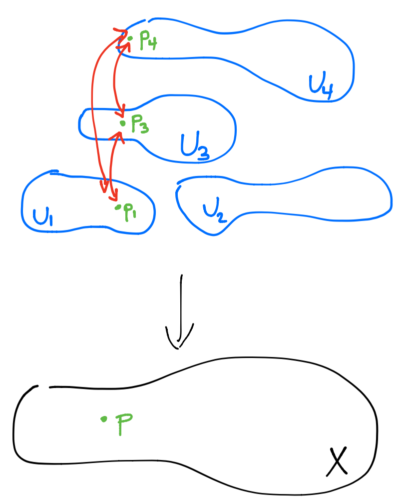

FIGURE 1. The groupoid of an open cover

The geometric structure of points with arrows is a *groupoid*, and it is a more general notion of space which we will use. Above we used a groupoid to present a topological space, but not every groupoid represents a space. At another extreme we can consider a groupoid with a single point equipped with a *group* G of self-arrows, as in Figure 2. A vector bundle over that groupoid is a single vector space $\mathbb{E}$ over the point, and an automorphism of $\mathbb{E}$ for each $g \in \mathbb{E}$. These automorphisms compose according to the group law (which is the "consistency condition" of the previous paragraph in this context), and so we simply have a linear representation of G on $\mathbb{E}$. There is a groupoid representing the action of a group G on a space X, and a vector bundle over it is a G-*equivariant* vector bundle over X. More general groupoids need not come from actions.

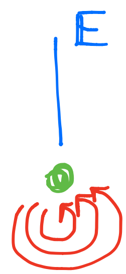

FIGURE 2. A vector bundle over a groupoid with one point

Variations

(1.19) *Families of linear maps.* The tangent space T_pM of a smooth manifold is the best linear approximation at p to the nonlinear space M. Similarly, if $f\colon M\to N$ is a smooth map of manifolds, at each p its differential

$$df_p\colon T_pM \longrightarrow T_{f(p)}N \tag{1.20}$$

is the best linear approximation at p to the nonlinear map f. The differential df is a family of linear maps parametrized by X, mapping between the families of vector spaces in the domain and codomain of (1.20). As a map of vector bundles

$$df\colon TM \longrightarrow f^*TN, \tag{1.21}$$

where f^*TN is the *pullback* of the vector bundle $TN\to N$ via the map $f\colon M\to N$.

The infinite dimensional case is particularly interesting. It happens in interesting circumstances that the differential (1.20), which is say a linear map between infinite dimensional Hilbert spaces, is almost an isomorphism in the sense that the kernel and cokernel have finite dimension. Such an operator is termed *Fredholm*, and in that case the differential (1.21) is a family of Fredholm operators. Families of Fredholms, which may or may not arise as linear approximations to a nonlinear Fredholm map, carry interesting topological information which is measured in K-theory.

Example 1.22. Nonlinear Fredholm maps are a key ingredient in Donaldson's gauge-theoretic approach to the topology of 4-manifolds [Do] and in the various flavors of Floer theory. A specific example of the latter is the Chern-Simons-Dirac functional [KM, §4.1].

Example 1.23. Let Σ denote a closed[7] oriented 2-manifold and $\mathcal{M}$ the space of conformal structures on Σ. Each conformal structure defines a linear operator

$$\bar\partial\colon \Omega^{0,0}(\Sigma) \longrightarrow \Omega^{0,1}(\Sigma) \tag{1.24}$$

[7] A manifold is *closed* if it is compact without boundary.

on complex functions. It is a *first-order elliptic differential operator* and as such extends to a Fredholm operator on various Hilbert space completions.

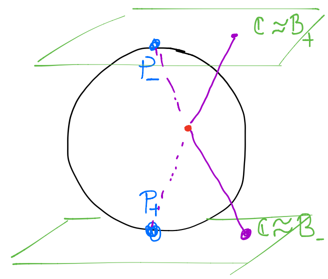

FIGURE 3. Overlap of two coordinate charts on the 2-sphere

(1.25) *$\mathbb{Z}$-gradings and complexes.* Another variation of the notion of a vector space is a *complex* of vector spaces. If we use cohomological grading conventions, then it is a sequence of linear maps

$$\cdots \longrightarrow E^{-2} \xrightarrow{d} E^{-1} \xrightarrow{d} E^{0} \xrightarrow{d} E^{1} \xrightarrow{d} E^{2} \longrightarrow \cdots \tag{1.26}$$

such that $d \circ d = 0$. Such complexes arise naturally in geometry. Examples include the de Rham complex of a smooth manifold and the Dolbeault complex of a complex manifold, a particular example of which is (1.24).[8] If the *differential* d vanishes, then the complex is a *$\mathbb{Z}$-graded vector space*.

Remark 1.27. The marriage of complexes of vector bundles and Fredholm operators gives *Fredholm complexes* [Se1].

Example 1.28. We have already seen a family of operators on a finite dimensional vector space in Example 1.11, namely the family $x \mapsto xI - T$ parametrized by $x \in \mathbb{C}$. The kernels jump in dimension as x varies, and they do not form a vector bundle. However, the family of operators, which can be viewed as a family of complexes (1.26) of vector spaces with $E^n = 0$, $n \neq 0, 1$, does represent an element of K-theory.

K-theory

Let X be a compact[9] space. Denote by $\mathrm{Vect}^{\cong}(X)$ the set of equivalence classes of finite rank complex vector bundles over X. The operation of direct sum passes to $\mathrm{Vect}^{\cong}(X)$, where it is a commutative, associative composition law with identity element represented by the zero vector bundle. In short, $\mathrm{Vect}^{\cong}(X)$ is a commutative monoid. The universal abelian group associated to $\mathrm{Vect}^{\cong}(X)$ is the *K-theory group* $K(X)$, the eponym of this course. It is a homotopy invariant of the space X, which is one face of a cohomological invariant. The other is invariance under suspension, after shifting degree, and for that we will proceed formally at first, roughly defining

[8] In that example $E^i = 0$ for $i \neq 0, 1$.

[9] We will give more general constructions later, which include noncompact spaces.

$K^{-q}(X)$ as the K-theory of the q^{th} suspension of X. In this way we will construct a generalized cohomology theory. All the apparatuses of algebraic topology—Mayer-Vietoris, spectral sequences, etc.—can be brought to bear on computations.

(1.29) *Variant geometric models.* The geometric variants of vector bundles listed above—Fredholm operators, complexes of vector bundles, Fredholm complexes—define K-theory classes. It is very important to have flexible geometric models for topological objects, since this is how they appear in geometry and physics.

(1.30) *Bott periodicity.* The basic theorem in the subject is the periodicity of the K-theory groups.

Theorem 1.31 (Bott). *There is a natural isomorphism $K^{q+2}(X) \cong K^q(X)$.*

The theorem Bott actually proved is that the stable homotopy groups of the unitary groups are periodic, which is related to Theorem 1.31 by the clutching construction **(1.16)**, extended to spheres of arbitrary dimension as in Remark 1.17. We will prove Theorem 1.31 in a few different ways, for example using Fredholm operators and the periodicity of Clifford algebras, following Atiyah-Singer [AS3].

Twistings and twisted vector bundles (Bonus material)

The modern incarnations of K-theory often occur in twisted form, as mentioned at the beginning of the lecture. We briefly catalog twisted notions of a complex vector space and linear representation of a group; there are corresponding twistings over a space (or groupoid) and twisted vector bundles. They are all "1-dimensional" in the sense that ordinary vector spaces and linear representations act by tensor product and, in some vague sense, the twisted versions for a fixed twisting are generated by a single object. From these twisted geometric objects we will extract twisted K-theory groups.

(1.32) *$\mathbb{Z}/2\mathbb{Z}$-gradings.* A *$\mathbb{Z}/2\mathbb{Z}$-graded vector space* $\mathbb{E} = \mathbb{E}^0 \oplus \mathbb{E}^1$ is simply a direct sum of two vector spaces. Homogeneous elements of $\mathbb{E}^0$ are termed *even*, homogeneous elements of $\mathbb{E}^1$ are termed *odd*. The word 'super' is used synonymously with '$\mathbb{Z}/2\mathbb{Z}$-graded', an inheritance from supersymmetry in quantum field theory. While the replacement the $\mathbb{Z}$-gradings of **(1.25)** by $\mathbb{Z}/2\mathbb{Z}$-gradings is not strictly a twisting, it does open the way for more twistings than are possible in the $\mathbb{Z}$-graded world. In the super situation the differential in a complex (1.26) is replaced by an odd endomorphism

$$T = \begin{pmatrix} 0 & T'' \\ T' & 0 \end{pmatrix} \tag{1.33}$$

of the vector space $\mathbb{E}$. We do not require $T^2 = 0$, so this is already a twisted version of a differential.[10]

[10] It is called a *curved differential* in the context of A^∞ modules; T^2 is the *curving*.

(1.34) *$\mathbb{Z}/2\mathbb{Z}$-graded groups.* Let G be a Lie group or topological group. A continuous homomorphism $\epsilon\colon G \to \{\pm 1\}$ is a *$\mathbb{Z}/2\mathbb{Z}$-grading* of the group G. It twists the notion of a linear representation on a super vector space $\mathbb{E} = \mathbb{E}^0 \oplus \mathbb{E}^1$. Namely, in an *$\epsilon$-twisted representation* an element $g \in G$ with $\epsilon(g) = +1$ acts by an even automorphism and an element $g \in G$ with $\epsilon(g) = -1$ acts by an odd automorphism.

(1.35) *Central extensions and projective representations.* Let G be a Lie group and suppose

$$1 \longrightarrow \mathbb{T} \longrightarrow G^\tau \longrightarrow G \longrightarrow 1 \tag{1.36}$$

is a group extension with $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ central. A *τ-twisted representation* of G is a representation of G^τ on a complex vector space $\mathbb{E}$ such that $\lambda \in \mathbb{T}$ acts as scalar multiplication by λ. This induces an action of G on the projective space $\mathbb{P}\mathbb{E}$ of lines (1-dimensional subspaces) in $\mathbb{E}$, a projective representation.

Two examples: Let $G = SO_n$ be the special orthogonal group, and set $G^\tau = \mathrm{Spin}^c_n$ with its spin representation. This occurs in Riemannian geometry. The second example is typically infinite-dimensional and occurs in quantum physics. Namely, the space of pure states of a quantum system is the projective space $\mathbb{P}\mathcal{H}$ of a complex Hilbert space, so the symmetries of a quantum system are projective. A fundamental theorem of Wigner asserts that they lift to be linear or antilinear symmetries of $\mathcal{H}$, determined up to multiplication by a phase, so a group of quantum symmetries gives rise to an extension[11] (1.36).

(1.37) *Antilinearity.* Let G be a Lie group and $\phi\colon G \to \{\pm 1\}$ a $\mathbb{Z}/2\mathbb{Z}$-grading. Then a twisted form of a linear action on a (super) vector space $\mathbb{E}$ has elements $g \in G$ with $\phi(g) = +1$ acting linearly and elements $g \in G$ with $\phi(g) = -1$ acting antilinearly. A particular example is $G = \mathbb{Z}/2\mathbb{Z}$ with the nontrivial grading, in which case a twisted representation on $\mathbb{E}$ is a *real structure*. In this way real vector spaces appear as "twisted" forms of complex vector spaces. Combining with (**1.35**) we can similarly fit a quaternionic structure on a vector space in this framework.

We remark that time-reversing symmetries of a spacetime typically act antilinearly on a quantum mechanical system.

(1.38) *Central simple algebras.* A vector space is a module over the algebra $\mathbb{C}$ of complex numbers, and another form of twisting is to replace $\mathbb{C}$ with a more elaborate algebra A. To obtain a "1-dimensional" notion we require that the algebra A be invertible in the Morita sense. This is equivalent to requiring that A be central simple, and in our current context we use $\mathbb{Z}/2\mathbb{Z}$-graded algebras. These were studied by Wall [Wa]. Each Morita class is represented by a Clifford algebra. We will see that this sort of twisting induces a degree shift in K-theory.

In geometry Clifford modules [ABS] occur in connection with the spin representation and Dirac operator. For example, on an n-dimensional Riemannian spin manifold the natural Dirac operator [LM] has the form (1.33) and acts on a module over the Clifford algebra with n generators. In quantum physics there are algebras of observables, and the quantum Hilbert space is naturally a module over that algebra. In systems with fermions this leads to Clifford modules.

[11] It is not necessarily central due to possible antilinear symmetries. Antilinearity as a twist is discussed next.

(1.39) *Twistings over groupoids.* Each form of twisting can be defined on a groupoid, not just on a group, and they can appear in combination. We will develop a general model of twistings which is sufficiently flexible to develop a general theory and which covers most appearances in geometry and physics. We end this lecture with a specific example related to (**1.34**).

(1.40) *The twisting of a double cover.* Let X be a space and $\widetilde{X} \to X$ a double cover with deck transformation $\sigma\colon \widetilde{X} \to \widetilde{X}$. Descent data for a vector bundle $E \to \widetilde{X}$ is an isomorphism $\sigma^*E \to E$ which squares to the identity. Let $\Pi E = E^1 \oplus E^0$ denote the oppositely graded bundle; it represents the negative of E in K-theory.[12] Then one form of twisted bundle on X is a $\mathbb{Z}/2\mathbb{Z}$-graded bundle $E \to \widetilde{X}$ together with an isomorphism $\sigma^*E \to \Pi E$ which squares to the identity.

Remark 1.41. Double covers twist any cohomology theory since -1 always acts as an automorphism. Here is its incarnation in de Rham cohomology. Let $\widetilde{M} \to M$ be a double cover of a smooth manifold with deck transformation σ. Differential forms $\omega \in \Omega^\bullet(\widetilde{M})$ which satisfy $\sigma^*\omega = \omega$ descend to differential forms on M. On the other hand differential forms $\omega \in \Omega^\bullet(\widetilde{M})$ which satisfy $\sigma^*\omega = -\omega$ are twisted differential forms on M. If $\widetilde{M} \to M$ is the orientation double cover, and ω has top degree, then a twisted form is a *density* on M, the natural objects one can integrate.

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