Lecture 11: Clifford algebras

In this lecture we introduce Clifford algebras, which will play an important role in the rest of the class. The link with K-theory is the *Atiyah-Bott-Shapiro construction* [ABS], which implements the K-theory of suspensions via Clifford modules. We will begin the next lecture with this ABS construction.

An algebra from the orthogonal group

The orthogonal group O_n is a subset of an algebra: the algebra $M_n\mathbb{R}$ of $n \times n$ matrices. The *Clifford algebra* plays a similar role for a double cover group of the orthogonal group.

(11.1) *Heuristic motivation.* Orthogonal transformations are products of reflections. For a unit norm vector $\xi \in \mathbb{R}^n$ define

$$\rho_\xi(\eta) = \eta - 2\langle \eta, \xi \rangle \xi, \tag{11.2}$$

where $\langle -, - \rangle$ is the standard inner product.

Theorem 11.3 (Sylvester)**.** *Any $g \in O_n$ is the composition of $\leqslant n$ reflections.*

Proof. The statement is trivial for $n = 1$. Proceed by induction: if $g \in O_n$ fixes a unit norm vector ξ then it fixes the orthogonal complement $(\mathbb{R} \cdot \xi)^\perp$, and we are reduced to the theorem for O_{n-1}. If there are no fixed unit norm vectors, then for any unit norm vector ζ set $\xi = \frac{g(\zeta)-\zeta}{|g(\zeta)-\zeta|}$. The composition $\rho_\xi \circ g$ fixes ζ and again we reduce to the $(n-1)$-dimensional orthogonal complement. □

Now generate an algebra from the unit norm vectors, with relations inspired by those of reflections. Note immediately that the vectors $\pm\xi$ both correspond to the same reflection $\rho_\xi = \rho_{-\xi}$. Therefore, we expect from the beginning that the Clifford algebra "double counts" orthogonal transformations. Now since the square of a reflection is the identity, we impose the relation

$$\xi^2 = \pm 1, \qquad |\xi| = 1. \tag{11.4}$$

The sign ambiguity is that described above, and we choose a sign independent of ξ. It follows that

$$\xi^2 = \pm|\xi|^2 \tag{11.5}$$

for any $\xi \in \mathbb{R}^n$. Now if $\langle \xi_1, \xi_2 \rangle = 0$, then $(\xi_1 + \xi_2)/\sqrt{2}$ has unit norm and from

$$\pm 1 = \left(\frac{\xi_1 + \xi_2}{\sqrt{2}}\right)^2 = \frac{\xi_1^2 + \xi_2^2 + (\xi_1\xi_2 + \xi_2\xi_1)}{2} = \frac{\pm 2 + (\xi_1\xi_2 + \xi_2\xi_1)}{2} \tag{11.6}$$

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$$\xi_1\xi_2 + \xi_2\xi_1 = 0, \qquad \langle \xi_1, \xi_2 \rangle = 0. \tag{11.7}$$

Equations (11.4) and (11.7) are the defining relations for the Clifford algebra. Check that the reflection (11.2) is given by

$$\rho_\xi(\eta) = -\xi\eta\xi^{-1} \tag{11.8}$$

in the Clifford algebra. By composition using Theorem 11.3 we obtain the action of any orthogonal transformation on $\eta \in \mathbb{R}^n$.

Definition 11.9. For $n \in \mathbb{Z}$ define the *real Clifford algebra* $\mathrm{C}\ell_n$ as the unital associative real algebra generated by $e_1, \dots, e_{|n|}$ subject to the relations

$$\begin{aligned} e_i^2 &= \pm 1, \qquad && i = 1, \dots, n \\ e_ie_j + e_je_i &= 0, && i \neq j. \end{aligned} \tag{11.10}$$

The complex Clifford algebra $\mathrm{C}\ell_n^{\mathbb{C}}$ is the complex algebra with the same generators and same relations.

Note $\mathrm{C}\ell_0 = \mathbb{R}$ and $\mathrm{C}\ell_0^{\mathbb{C}} = \mathbb{C}$.

Example 11.11. There is an isomorphism $\mathrm{C}\ell_{-n}^{\mathbb{C}} \cong \mathrm{C}\ell_n^{\mathbb{C}}$ obtained by multiplying each generator e_i by $\sqrt{-1}$.

Example 11.12. $\mathrm{C}\ell_{-1}$ can be embedded in the matrix algebra $M_2\mathbb{R}$ by setting

$$e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{11.13}$$

The same equation embeds $\mathrm{C}\ell_{-1}^{\mathbb{C}}$ in $M_2\mathbb{C}$.

Example 11.14. We identify $\mathrm{C}\ell_{-2}^{\mathbb{C}}$ with $\mathrm{End}(\mathbb{C}^2)$ by setting

$$e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{11.15}$$

where $i = \sqrt{-1}$. This does *not* work over the reals. The product

$$e_1e_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{11.16}$$

is $-i$ times a *grading operator* on $\mathbb{C}^2$.

Example 11.17. The real Clifford algebras $\mathrm{C}\ell_1$ and $\mathrm{C}\ell_{-1}$ are not isomorphic. We embed in $M_2\mathbb{R}$ in the former case by setting

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{11.18}$$

and in the latter using (11.13). Note that the doubled orthogonal group $\{\pm 1, \pm e_1\}$ is different in the two cases: in $\mathrm{C}\ell_1$ it is isomorphic to the Klein group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ whereas in $\mathrm{C}\ell_{-1}$ it is cyclic of order four.

(11.19) *Spin and Pin.* For $n>0$ let $S(\mathbb{R}^n)\subset\mathbb{R}^n$ denote the sphere of unit norm vectors. Since $\mathbb{R}^n$ embeds in $\mathrm{C}\ell_{\pm n}$, so too does $S(\mathbb{R}^n)$. We assert without proof that the group it generates is a Lie group $\mathrm{Pin}_{\pm n}\subset\mathrm{C}\ell_{\pm n}$. It follows from Theorem 11.3 that there is a surjection $\mathrm{Pin}_{\pm n}\to O_n$ defined by composing the reflections (11.8). The inverse image $\mathrm{Spin}_{\pm n}$ of the special orthogonal group SO_n consists of products of an even number of elements in $S(\mathbb{R}^n)$. There is an isomorphism $\mathrm{Spin}_n\cong\mathrm{Spin}_{-n}$, but as we saw in Example 11.14 this is not true in general for Pin.

(11.20) *The Dirac operator.* The Clifford algebra arises from the following question, posed by Dirac: Find a square root of the Laplace operator. We work on flat Euclidean space $\mathbb{E}^n$, which is the affine space $\mathbb{A}^n$ endowed with the translation-invariant metric constructed from the standard inner product on the underlying vector space $\mathbb{R}^n$ of translations. Let $x^1,\dots,x^n$ be the standard affine coordinates on $\mathbb{E}^n$. The Laplace operator is

$$\Delta=-\sum_{i=1}^{n}\frac{\partial^2}{(\partial x^i)^2}. \tag{11.21}$$

A first-order operator

$$D=\gamma^i\frac{\partial}{\partial x^i} \tag{11.22}$$

satisfies $D^2=\Delta$ if and only if γ^i satisfy the Clifford relation

$$\gamma^i\gamma^j+\gamma^j\gamma^i=-2\delta^{ij},\qquad 1\leqslant i,j\leqslant n, \tag{11.23}$$

as in (11.10). If we let (11.21), (11.22) act on the space $C^\infty(\mathbb{E}^n;\mathbb{S})$ of functions with values in a vector space $\mathbb{S}$, then we conclude that $\mathbb{S}$ is a $\mathrm{C}\ell_{-n}$*-module.*

(11.24) *$\mathbb{Z}/2\mathbb{Z}$-gradings.* So far we have not emphasized the $\mathbb{Z}/2\mathbb{Z}$-grading evident in the examples: odd products of generators such as (11.13), (11.15), (11.18) are represented by block off-diagonal matrices whereas even products of generators (11.16) are represented by block diagonal matrices.

Superalgebra

For a more systematic treatment, see [DM, §1]. We use 'super' synonymously with '$\mathbb{Z}/2\mathbb{Z}$-graded'.

(11.25) *Super vector spaces.* Let k be a field, which in our application will always be $\mathbb{R}$ or $\mathbb{C}$. A super vector space $\mathbb{S}=\mathbb{S}^0\oplus\mathbb{S}^1$ is a pair $(\mathbb{S},\epsilon)$ of a vector space over k and an operator ϵ with $\epsilon^2=\mathrm{id}_{\mathbb{S}}$. The subspaces $\mathbb{S}^0,\mathbb{S}^1$ are the $+1,-1$-eigenspaces, respectively. Eigenvectors are called even, odd. The tensor product $\mathbb{S}'\otimes\mathbb{S}''$ of super vector spaces carries the grading $\epsilon'\otimes\epsilon''$. The main new point is the *Koszul sign rule*, which is the symmetry of the tensor product:

$$\begin{aligned}\mathbb{S}'\otimes\mathbb{S}''&\longrightarrow\mathbb{S}''\otimes\mathbb{S}'\\ s'\otimes s''&\longmapsto(-1)^{|s'||s''|}s''\otimes s',\end{aligned} \tag{11.26}$$

(11.27) *Superalgebras.* Let $A = A^0 \oplus A^1$ be a super algebra, an algebra with a compatible grading: $A^i \cdot A^j \subset A^{i+j}$, where the degree is taken in $\mathbb{Z}/2\mathbb{Z}$. A homogeneous element z in its *center* satisfies $za = (-1)^{|z||a|}az$ for all homogeneous $a \in A$. The center is itself a super algebra, which is of course commutative (in the $\mathbb{Z}/2\mathbb{Z}$-graded sense). The *opposite* super algebra A^{op} to a super algebra A is the same underlying vector space with product $a_1 \cdot a_2 = (-1)^{|a_1||a_2|}a_2a_1$ on homogeneous elements. All algebras are assumed unital. Tensor products of super algebras are taken in the graded sense: the multiplication in $A' \otimes A''$ is

$$(a_1' \otimes a_1'')(a_2' \otimes a_2'') = (-1)^{|a_1''||a_2'|}a_1'a_2' \otimes a_1''a_2''. \tag{11.28}$$

Undecorated tensor products are over the ground field. Unless otherwise stated a module is a left module. An ideal $I \subset A$ in a super algebra is *graded* if $I = (I \cap A^0) \oplus (I \cap A^1)$.

(11.29) *Super matrix algebras.* Let $S = S^0 \oplus S^1$ be a finite dimensional super vector space over k. Then $\operatorname{End} S$ is a central simple super algebra. Endomorphisms which preserve the grading on S are even, those which reverse it are odd. A super algebra isomorphic to $\operatorname{End} S$ is called a *super matrix algebra.*

Clifford algebras

For more details see [ABS, Part I], [De1, §2].

A quadratic form on a vector space V is a function $Q\colon V \to k$ such that

$$B(\xi_1, \xi_2) = Q(\xi_1 + \xi_2) - Q(\xi_1) - Q(\xi_2), \qquad \xi_1, \xi_2 \in V, \tag{11.30}$$

is bilinear and $Q(n\xi) = n^2Q(\xi)$.

Definition 11.31. The *Clifford algebra* $C\ell(V, Q) = C\ell(V)$ of a quadratic vector space is an algebra equipped with a linear map $i\colon V \to C\ell(V, Q)$ which satisfies the following universal property: If $\varphi\colon V \to A$ is a linear map to an algebra A such that

$$\varphi(\xi)^2 = Q(\xi) \cdot 1_A, \qquad \xi \in V, \tag{11.32}$$

then there exists a unique algebra homomorphism $\tilde\varphi\colon C\ell(V, Q) \to A$ such that $\varphi = \tilde\varphi \circ i$.

We leave the reader to prove that i is injective and that $C\ell(V, Q)$ is unique up to unique isomorphism. Furthermore, there is a surjection

$$\otimes V \longrightarrow C\ell(V, Q) \tag{11.33}$$

from the tensor algebra, as follows from its universal property. This gives an explicit construction of $C\ell(V, Q)$ as the quotient of $\otimes V$ by the 2-sided ideal generated by $\xi^2 - Q(\xi) \cdot 1_{\otimes V}$, $\xi \in V$. The tensor algebra is $\mathbb{Z}$-graded, and since the ideal sits in even degree the quotient Clifford algebra is

$\mathbb{Z}/2\mathbb{Z}$-graded. The increasing filtration $\otimes^0 V \subset \otimes^{\leqslant 1} V \subset \otimes^{\leqslant 2} V \subset \cdots$ induces an increasing filtration on $\mathrm{C}\ell(V,Q)$ whose associated graded is isomorphic to the ($\mathbb{Z}$-graded) exterior algebra $\bigwedge^\bullet V$. There is a canonical isomorphism

$$\mathrm{C}\ell(V' \oplus V'', Q' \oplus Q'') \cong \mathrm{C}\ell(V', Q') \otimes \mathrm{C}\ell(V'', Q''), \tag{11.34}$$

deduced from the universal property. The standard Clifford algebras in Definition 11.9 have the form $\mathrm{C}\ell(V,Q)$ for $V = \mathbb{R}^n, \mathbb{C}^n$ and Q the positive or negative definite standard quadratic form on V.

The Clifford algebras are *central simple* as $\mathbb{Z}/2\mathbb{Z}$-graded algebras. I will leave the simplicity (there are no nontrivial 2-sided homogeneous ideals) as an exercise and here prove the centrality.

Proposition 11.35. *$\mathrm{C}\ell(V,Q)$ has center k.*

Proof. Suppose $x = x^0 + x^1$ is a central element. Fix an orthonormal basis $e_1, \dots, e_n$ of V. Then for every $i = 1, \dots, n$ we have

$$\begin{aligned} x^0 e_i &= e_i x^0 \\ x^1 e_i &= -e_i x^1 \end{aligned} \tag{11.36}$$

There is a unique decomposition $x^0 = a^0 + e_i b^1$ where a^0, b^1 belong to the Clifford algebra generated by the basis elements excluding e_i. Then

$$\begin{aligned} x^0 e_i &= a^0 e_i + e_i b^1 e_i = e_i a^0 - (e_i)^2 b^1 \\ e_i x^0 &= e_i a^0 + (e_i)^2 b^1. \end{aligned} \tag{11.37}$$

Since x^0 is central we have $x^0 e_i = e_i x^0$, and so (11.37) implies that $b^1 = 0$. Since this holds for every i, we conclude that x^0 is a scalar. Similarly, write $x^1 = a^1 + e_i b^0$ so that

$$\begin{aligned} x^1 e_i &= a^1 e_i + e_i b^0 e_i = -e_i a^1 + (e_i)^2 b^0 \\ -e_i x^1 &= -e_i a^1 - (e_i)^2 b^0 \end{aligned} \tag{11.38}$$

from which $x^1 = 0$. ∎

For a vector space L and $\theta \in L^*$ let ϵ_θ denote exterior multiplication by θ, which is an endomorphism of the exterior algebra $\bigwedge^\bullet L^*$. For $\ell \in L$ the adjoint of exterior multiplication by ℓ is contraction ι_ℓ, an endomorphism of $\bigwedge^\bullet L^*$ of degree -1.

Proposition 11.39. *Suppose $V = L \oplus L^*$ with the split quadratic form $Q(\ell + \theta) = \theta(\ell)$, $\ell \in L$, $\theta \in L^*$. Set $\mathbb{S} = \bigwedge^\bullet L^*$ with its $\mathbb{Z}/2\mathbb{Z}$-grading by the parity of the degree. Then the map $V \to \operatorname{End}\mathbb{S}$*

$$\begin{aligned} \ell &\longmapsto \iota_\ell \\ \theta &\longmapsto \epsilon_\theta \end{aligned} \tag{11.40}$$

extends to an isomorphism $\mathrm{C}\ell(V) \xrightarrow{\cong} \operatorname{End}\mathbb{S}$ of the Clifford algebra with a super matrix algebra.

Proof. Using (11.34) we reduce to the case $\dim L = 1$ which can be checked by hand; it is essentially Example 11.14. ∎

(11.41) *Algebraic Bott periodicity.* We may in the future discuss basic *Morita theory*, in which we will see that super matrix algebras are in some sense trivial. That is the spirit of the following theorem. We say the dimension of a finite dimensional super vector space $\mathbb{S} = \mathbb{S}^0 \oplus \mathbb{S}^1$ is $d^0|d^1$ if $\dim \mathbb{S}^i = d^i$.

Theorem 11.42. *There are isomorphisms of superalgebras*

$$\begin{aligned} C\ell^{\mathbb{C}}_{-2} &\xrightarrow{\;\cong\;} \mathrm{End}(\mathbb{S}), &\qquad \dim \mathbb{S} &= 1|1, \\ C\ell_{-8} &\xrightarrow{\;\cong\;} \mathrm{End}(\mathbb{S}_{\mathbb{R}}), &\qquad \dim \mathbb{S}_{\mathbb{R}} &= 8|8. \end{aligned} \tag{11.43}$$

Proof. The complex case is Example 11.14. For the real case we let $C\ell_{-2}$ act on $\mathbb{W} = \mathbb{C}^{1|1}$ via the formulas in (11.15). This action commutes (in the graded sense) with the *odd* real structure

$$J(z^0, z^1) = (\overline{z^1}, \overline{z^0}). \tag{11.44}$$

That is, $J\colon \mathbb{W} \to \mathbb{W}$ is antilinear, odd, and squares to $-\mathrm{id}_{\mathbb{W}}$. Set $\mathbb{S} = W^{\otimes 4}$. It carries an action of $C\ell_{-2}^{\otimes 4} \cong C\ell_{-8}$ which commutes with $J^{\otimes 4}$. The latter is antilinear, even, and squares to $\mathrm{id}_{\mathbb{S}}$, so is a real structure. □

As stated in the proof, $\mathbb{W}^{\otimes 2}$ carries a *quaternionic* structure $J^{\otimes 2}$: the Koszul sign rule (11.26) implies that $J^{\otimes 2}$ squares to *minus* the identity. (Check that sign! It will test your understanding of the sign rule.)

(11.45) *Spin and Pin redux.* Sitting inside the Clifford algebra $C\ell(V,Q)$ is the pin group $\mathrm{Pin}(V,Q)$ generated by $S(V)$ and its even subgroup $\mathrm{Spin}(V,Q) = \mathrm{Pin}(V,Q) \cap C\ell(V,Q)^0$. When V is real and Q is definite these are compact Lie groups. In that case we can average a metric over a real or complex Clifford module $\mathbb{S} = \mathbb{S}^0 \oplus \mathbb{S}^1$ so that $\mathrm{Pin}(V,Q)$ acts orthogonally (unitarily in the complex case). It follows that $e \in S(V)$ is self- or skew-adjoint, according as Q is positive or negative definite.

Remark 11.46. There is a tricky sign in the proper definition of 'self-adjoint' and 'skew-adjoint' in the super world. There is a way around that sign to a more standard convention, which is the one we use; see [DM, §4.4], [De2, §4].

References

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