Lecture 14: Proof of Bott periodicity (con't)

There are many spaces of operators in the proof, and it is confusing to follow at first. So we'll first try to sort things out a bit.

For a *super* Hilbert space $H_s = H^0 \oplus H^1$ we have a sequence of spaces of skew-adjoint odd Fredholm operators which exhibit just two homeomorphism types, as in (12.52). Letting H denote a (non-super) Hilbert space, and identifying $H = H^0 = H^1$, we can identify $\mathrm{Fred}_0(H_s)$ with $\mathcal{F} = \mathrm{Fred}(H)$ by identifying $\left(\begin{smallmatrix} 0 & -T^* \\ T & 0 \end{smallmatrix}\right)$ with T; see (12.41). Also, the argument after (12.71) shows that we can identify $\mathrm{Fred}_{-1}(H_s)$ with the space $\widehat{\mathcal{F}}$ of skew-adjoint Fredholms (on $(\mathrm{C}\ell^{\mathbb{C}}_{-1} \otimes H_s)^0$) by identifying A with $e_1 A$ restricted to the even subspace. We proved in Corollary 13.35 that $\mathcal{F}$ has the homotopy type $\mathbb{Z} \times BGL_\infty$, which is then the homotopy type of all spaces in the first line of (12.52). Its loop space $\Omega\mathcal{F}$ has the homotopy type GL_∞. Theorem 13.39, whose proof we complete in this lecture, says that that is also the homotopy type of the nontrivial component $\widehat{\mathcal{F}_*}$ of $\widehat{\mathcal{F}}$. The identification with $\mathrm{Fred}_{-1}(H_s)$, which we re-define to denote only this nontrivial component, then determines the homotopy type of the spaces in the second line of (12.52) as GL_∞. This completes the proof of Bott periodicity, which in this form is Corollary 12.56.

Fiber bundles, fibrations, and quasifibrations

If $p\colon E \to B$ is a continuous map with contractible fibers we might like to conclude that p is a homotopy equivalence, but that is not always true. (Counterexample: Take $E = B = \mathbb{R}$ and p the identity map, but topologize E as a discrete set and B with the usual topology.) Not surprisingly, we need control over the fibers. The three classes of maps in the title are successively more general yet retain just such control. Namely, assuming the base is path connected, the fibers are respectively (i) homeomorphic, (ii) homotopy equivalent, (iii) weakly homotopy equivalent.

For convenience assume B is path connected, and always assume that E, B are metrizable.

(14.1) *Fiber bundles.* We already discussed these in **(12.23)**.

Definition 14.2. $p\colon E \to B$ is a *fiber bundle* if for every $b \in B$ there exists an open neighborhood U and a local trivialization

$$\begin{array}{ccc} U \times p^{-1}(b) & \longrightarrow & p^{-1}(U) \\ & \searrow \quad \swarrow & \\ & B & \end{array} \tag{14.3}$$

Many important maps in geometry are fiber bundles.

(14.4) *Fibrations.* Now assume that E, B are pointed spaces with basepoints $e, \pi(e) = b$. A fibration is characterized by the *homotopy lifting property.*

Definition 14.5. $p\colon E \to B$ is a *fibration* if for every pointed space X, continuous map $f\colon [0,1] \times X \to B$ and lift $\tilde{f}_0\colon X \to E$ of f_0 there exists an extension $\tilde{f}\colon [0,1] \times X \to E$ lifting f.

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The lift is encoded in the diagram

$$\begin{array}{ccc} \{0\}\times X & \xrightarrow{\tilde f_0} & E \\ \big\downarrow & \overset{\tilde f}{\nearrow} & \big\downarrow{\scriptstyle p} \\ [0,1]\times X & \xrightarrow{f} & B \end{array} \tag{14.6}$$

Theorem 14.7. *Suppose $p\colon E\to B$ is a fibration.*

(i) *$p_*\colon \pi_n(E,p^{-1}(b),e)\to\pi_n(B,b)$ is an isomorphism for all $n\in\mathbb{Z}^{\geqslant 0}$.*

(ii) *There is a long exact sequence*

$$\cdots\longrightarrow\pi_n(F,e)\longrightarrow\pi_n(E,e)\longrightarrow\pi_n(B,b)\longrightarrow\pi_{n-1}(F,e)\longrightarrow\cdots \tag{14.8}$$

in which $F=p^{-1}(b)$.

Proposition 14.9. *Let $p\colon (E,e)\to(B,b)$ be a fibration, $b'\in B$, and $P_e\big(E;p^{-1}(b')\big)$ the space of paths in E which begin at e and terminate on the subspace $p^{-1}(b')$. Then p induces a fibration*

$$P_e\big(E;p^{-1}(b')\big)\longrightarrow P_b(B;b') \tag{14.10}$$

with contractible fibers, so is a weak homotopy equivalence.

The last conclusion follows from the long exact sequence (14.8). We leave the reader to provide a proof of Proposition 14.9 using the homotopy lifting property.

(14.11) *Quasifibrations.* A quasifibration is a map for which the statements in Theorem 14.7 hold, but the homotopy lifting property does not necessarily hold. See Figure 4 for a typical example.

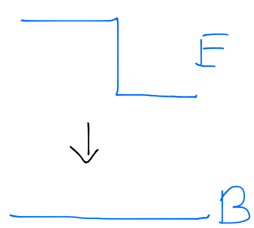

Figure 4. A quasifibration which is not a quasifibration

Definition 14.12. A map $p\colon E\to B$ (of unpointed spaces) is a *quasifibration* if $p_*\colon \pi_n(E,p^{-1}(b),e)\to\pi_n(B,b)$ is an isomorphism for all $b\in B$, $e\in p^{-1}(b)$, and $n\in\mathbb{Z}^{\geqslant 0}$.

The long exact sequence (14.8) follows.

An equivalent condition is that the natural map from each fiber to the *homotopy fiber* is a weak homotopy equivalence.

Quasifibrations are useful in part because of the following criterion to recognize them. This was proved by Dold-Thom [DT], who introduced quasifibrations.

Theorem 14.13. *Suppose $q\colon E \to B$ is a surjective map with B path connected. Let*

$$F_0B \subset F_1B \subset F_2B \subset \cdots \tag{14.14}$$

be an increasing filtration of B with $\bigcup_{n=0}^{\infty} F_nB = B$ such that

(i) *$q\big|_U$ is a quasifibration for all open $U \subset F_nB \backslash F_{n-1}B$, and*
(ii) *For $n \geqslant 1$ there exists an open neighborhood $U_n \subset F_nB$ of $F_{n-1}B$ and deformation retractions*

$$\begin{aligned} U_n &\xrightarrow{h_t} F_{n-1}B \\ q^{-1}U_n &\xrightarrow{H_t} q^{-1}F_{n-1}B \end{aligned} \tag{14.15}$$

such that $H_1\colon q^{-1}(b) \to q^{-1}(h_1b)$ is a weak homotopy equivalence.

Then q is a quasifibration.

There is a nice exposition of quasifibrations in [Ha2, pp. 476–481] based on [Ma]. You will find the proofs of the theorems and much more there.

The basic diagram

We continue where we left off in Lecture 13. Introduce

$$\widehat{F}_* = \{T \in \pi^{-1}(\widehat{G}_*) : \|T\| = 1\}. \tag{14.16}$$

Thus an operator $T\colon H \to H$ in $\widehat{F}_*$ satisfies:

$$\begin{aligned} &T \text{ is Fredholm,} \\ &T^* = -T, \\ &\|T\| = 1, \\ &\operatorname{ess\,spec} T = \{+i, -i\}. \end{aligned} \tag{14.17}$$

Lemma 14.18. *$\widehat{F}_*$ is a deformation retract of $\widehat{\mathcal{F}_*}$.*

Proof. First use the deformation retraction $\big((1-t) + t\|\pi(T)^{-1}\|\big)T$ onto the subspace of $S \in \widehat{\mathcal{F}_*}$ with $\|\pi(S)^{-1}\| = 1$. Then deformation retract $i\mathbb{R}$ symmetrically onto $[-i, +i]$ and use the spectral theorem. (The symmetry ensures we stay in the space of skew-adjoint operators.) □

Corollary 14.19. *$\hat{\pi}\colon \widehat{F}_* \to \widehat{G}_*$ is a homotopy equivalence*

Now we have the basic diagram

$$\begin{array}{ccc} \widehat{F}_* & \xrightarrow{\ \delta\ } & P_1(U, -U^{\mathrm{cpt}}) \\ {\scriptstyle\hat{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle\rho} \\ \widehat{G}_* & \xrightarrow{\ \epsilon\ } & P_1(G, -1) \end{array} \tag{14.20}$$

Both δ and ϵ are given by the formula

$$x \longmapsto \exp \pi t x, \qquad 0 \leqslant t \leqslant 1. \tag{14.21}$$

In (14.20) we know that $\hat{\pi}$ is a homotopy equivalence and we need to prove that ϵ is a homotopy equivalence (Theorem 13.47). We will do so by proving that δ, ρ are homotopy equivalences.

That ρ is a weak homotopy equivalence follows directly from Proposition 14.9 once we observe (see (13.31)) that $U \to G$ is a principal fiber bundle (hence fibration) with fiber U^{cpt}. All spaces in the game have the homotopy type of CW complexes, so weak homotopy equivalences are homotopy equivalences.

Proposition 14.22. *Evaluation at the endpoint is a homotopy equivalence*

$$P_1(U, -U^{\mathrm{cpt}}) \longrightarrow -U^{\mathrm{cpt}} \tag{14.23}$$

Proof. The map (14.23) is a fibration with fiber ΩU, and the latter is contractible by Kuiper's Theorem 12.1. ☐

From the basic diagram (14.20) we are reduced to proving the following.

Theorem 14.24. *The map*

$$\begin{aligned} q\colon \widehat{F}_* &\longrightarrow -U^{\mathrm{cpt}} \\ T &\longmapsto \exp(\pi T) \end{aligned} \tag{14.25}$$

is a homotopy equivalence.

A dense quasifibration

To gain some intuition, let's look at a few fibers of the map q in (14.25). We write $P \in -U^{\mathrm{cpt}}$ as $P = -\mathrm{id}_H + \ell$ where $\ell \in \mathrm{cpt}(H)$ is a compact operator.

Example 14.26. Suppose that ℓ has finite rank. Then $K = \ker(\ell)$ is a closed subspace of finite codimension and $H = K \oplus K^\perp$; the dimension of $K^\perp$ is finite. Suppose $T \in q^{-1}(P)$ so $\exp \pi T = P$ and T satisfies the conditions in (14.17). The first observation is that $T\big|_{K^\perp}$ is determined by $P\big|_{K^\perp}$. For on this finite dimensional space we can diagonalize the operators and we are studying the map $\exp(\pi -)\colon [-i, +i] \to \mathbb{T}$, which is an isomorphism except at the endpoints, both of which map to $-i \in \mathbb{T}$. On $K^\perp$ the operator P does not have eigenvalue $-i$ so the logarithm (inverse image under q) is unique. On the other hand, the operator $T\big|_K$ has spectrum contained in $\{+i, -i\}$, and by the last condition in (14.17) both $+i$ and $-i$ are in the spectrum with "infinite multiplicity". It follows that there is a decomposition

$$K = K_+ \oplus K_- \tag{14.27}$$

with $T\big|_{K_\pm} = \pm i$ and $\dim K_+ = \dim K_- = \infty$. The fiber $q^{-1}(P)$ is then identified with the Grassmannian of all splittings (14.27). This Grassmannian is diffeomorphic to the homogeneous space $U(K) \,/\, U(K_+) \times U(K_-)$. All three unitary groups are contractible by Kuiper, hence so is the fiber.

Example 14.28. Suppose $e_1, e_2, \dots$ is an orthonormal basis of the Hilbert space H. Consider the following two operators in $-U^{\mathrm{cpt}}$:

$$\begin{aligned} P_1(e_n) &= \exp\bigl(\pi i(1-\frac{1}{n})\bigr) \\ P_2(e_n) &= \exp\bigl(\pi i(1+\frac{(-1)^n}{n})\bigr) \end{aligned} \tag{14.29}$$

There is a unique operator $T_i\colon H \to H$ which exponentiates to P_i under $\exp(\pi -)$, but as the essential spectrum of T_1 is $\{+i\}$ it is not an element of $\widehat{F}_*$. Thus $q^{-1}(P_1)$ is empty whereas $q^{-1}(P_2)$ has a single point.

Since not all fibers of q are weakly homotopy equivalent, q is not a quasifibration. However, it is still a homotopy equivalence. Atiyah-Singer prove this by proving that q is a quasifibration over the dense subspace of operators of the form treated in Example 14.26, and in turn the inclusion of the subspaces of both base and total space are homotopy equivalences.

Definition 14.30. Let $n \in \mathbb{Z}^{>0}$. Define

(i) $-U^{\mathrm{cpt}}(n) \subset -U^{\mathrm{cpt}}$ as the subset $\{P = -\mathrm{id}_H + \ell : \operatorname{rank} \ell \leqslant n\}$,
(ii) $\widehat{F}_*(n) \subset \widehat{F}_*$ as the subset $q^{-1}\bigl(-U^{\mathrm{cpt}}(n)\bigr)$.

In each case we have an increasing filtration of the unions

$$\begin{aligned} -U^{\mathrm{cpt}}(\infty) &= \bigcup_{n=1}^{\infty} -U^{\mathrm{cpt}}(n) \\ \widehat{F}_*(\infty) &= \bigcup_{n=1}^{\infty} \widehat{F}_*(n) \end{aligned} \tag{14.31}$$

The first union is the space of all unitaries which differ from $-\mathrm{id}_H$ by a finite rank operator. That resembles the union of the groups (13.26) in which we fix the subspaces on which the operator deviates from $-\mathrm{id}_H$. In any case the homotopy type of the unions are the same.

Proposition 14.32. *The inclusion maps*

$$\begin{aligned} i\colon -U^{\mathrm{cpt}}(\infty) &\longrightarrow -U^{\mathrm{cpt}} \\ i\colon \widehat{F}_*(\infty) &\longrightarrow \widehat{F}_* \end{aligned} \tag{14.33}$$

are homotopy equivalences.

Proposition 14.34. $q\big|_{\widehat{F}_*(\infty)}$ *is a quasifibration with contractible fibers.*

Theorem 14.24 follows immediately from these propositions.

We sketch the proofs (literally) and defer to [AS3] for details. For Proposition 14.32 we must show that any compact $X \subset \widehat{F}_*$ can be deformed to a subset of $\widehat{F}_*(n)$ for some n. We do that by a spectral deformation, illustrated in Figure 5 in which $0 < \alpha < 1$. The key observation is that

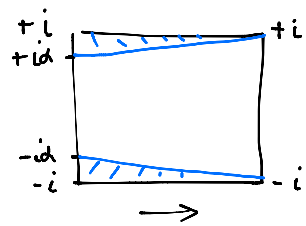

FIGURE 5. Spectral deformation for Proposition 14.32

any operator in $\widehat{F}_*$ has only a finite spectrum in the interval $[-i\alpha, i\alpha]$ as the essential spectrum is $\{-i, +i\}$. The argument for U^{cpt} is similar.

For Proposition 14.34 we use the Dold-Thom criterion Theorem 14.13. To verify condition (i) we sup up the argument in Example 14.26 to show that over the subspace where rank $\ell = n$ is constant the kernels K form a vector bundle, as do their orthogonal complements. Thus the restriction of q over this subspace is a fiber bundle with contractible fibers, so in particular is a quasifibration on any open subset. For (ii) we observe that an operator in $-U^{\mathrm{cpt}}(n)$ has at most n eigenvalues not equal to -1. We need to deform a neighborhood of $-U^{\mathrm{cpt}}(n-1)$ in $-U^{\mathrm{cpt}}(n)$ to $-U^{\mathrm{cpt}}(n-1)$. Let U_n be the subset where there is such an eigenvalue with negative real part. Make a spectral deformation as illustrated in Figure 6. It is easy to check that the induced map on fibers is a homotopy equivalence.

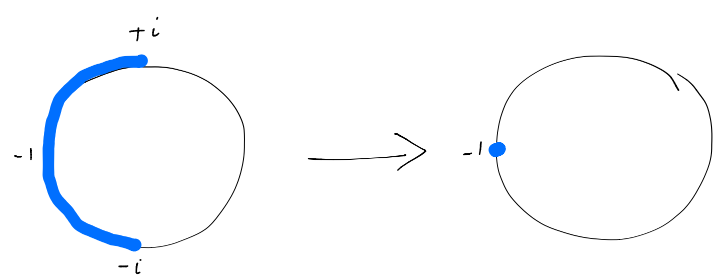

FIGURE 6. Spectral deformation for Proposition 14.34

McDuff's proof of Bott periodicity (Bonus material)

We begin with the observation of an exchange between finite and infinite dimensions. Let V be a finite dimensional complex vector space and H an infinite dimensional Hilbert space. Then whereas $GL(H)$ is contractible (Kuiper), $GL(V)$ definitely has interesting topology: the colimit for $\dim V$ large is the homotopy type GL_∞. On the other hand, the space $\mathrm{Fred}(H)$ of Fredholm operators is interesting—it has the homotopy type $\mathbb{Z} \times BGL_\infty$—whereas the space $\mathrm{End}(V)$ is contractible.

McDuff [McD] gave a proof of Bott periodicity by constructing a variation of (14.25) from finite dimensional spaces which exchanges the spaces of interest: the total space in her quasifibration is contractible, whereas it is the fibers of (14.25) which are contractible. Thus her quasifibration has the form

$$\begin{array}{ccc} \mathbb{Z}\times BGL_\infty & \longrightarrow & \mathrm{pt} \\ & & \downarrow \\ & & GL_\infty \end{array} \tag{14.35}$$

This shows that $\Omega GL_\infty \simeq \mathbb{Z}\times BGL_\infty$. It is trivial that $\Omega(\mathbb{Z}\times BGL_\infty)\simeq GL_\infty$, and the two statements together immediately imply Bott periodicity.

For each finite dimensional vector space V we let $\mathfrak{u}(V)_{\leqslant 1}$ be the subspace of skew-adjoint operators with operator norm $\leqslant 1$, and as in (14.25) consider the map

$$\begin{aligned} q\colon \mathfrak{u}(V)_{\leqslant 1} &\longrightarrow U(V) \\ T &\longmapsto \exp(\pi T) \end{aligned} \tag{14.36}$$

This is a quasifibrations with fibers the Grassmannian of the $(-i)$-eigenspace, exactly as in the analysis of Example 14.26. The idea is to replace V by the colimit $\mathbb{C}^\infty$ of finite dimensional vector spaces and work with a "restricted Grassmannian" and "restricted general linear group". Details of the argument are worked out in the series of papers [AP], [Beh1], [Beh2]; the latter also works out real Bott periodicity.

We want not only Bott periodicity but also a geometric model of K-theory. As we meet Fredholm operators in geometry this alternative proof, while very beautiful and elegant, does not suffice for our purposes.

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