Lecture 2: Homotopy invariance

We give two proofs of the following basic fact, which allows us to do topology with vector bundles. The basic input is local triviality of vector bundles (Definition 1.12).

Theorem 2.1. *Let $E \to [0,1] \times X$ be a vector bundle. Denote by $j_t\colon X \to [0,1]\times X$ the inclusion $j_t(x) = (t,x)$. Then there exists an isomorphism*

$$j_0^*E \xrightarrow{\;\cong\;} j_1^*E. \tag{2.2}$$

The idea of both proofs is to construct parametrized trivializations along the axes $[0,1]\times\{x\}$, $x \in X$, of the cylinder $[0,1]\times X$. For the first proof we assume that X is a smooth manifold and that the vector bundle is smooth. Then we write a differential equation (parallel transport via a covariant derivative) which gives infinitesimal trivializations. The solution to the differential equation gives the global isomorphism. For the second proof we only assume continuity, so X is a (paracompact, Hausdorff) space, and use the local triviality of vector bundles in place of an (infinitesimal) differential equation. Then a patching argument constructs the global isomorphism. Partitions of unity are used as a technical tool in both situations.

Differential equations are used throughout differential geometry to prove global theorems. In this case we use an *ordinary* differential equation, for which there is a robust general theory. For *partial* differential equations the global questions are more delicate. (Think, for example, of the Ricci flow equations which "straighten out" the metric on a Riemannian 3-manifold to one of constant curvature.)

(2.3) *Partitions of unity.* The definition of 'paracompact' varies in the literature. Sometimes it includes the Hausdorff condition. The usual definition is that that every open cover of X has a locally finite refinement, or one can take the following theorem as a definition. Recall that a *partition of unity* is a set A of continuous functions $\rho_\alpha\colon X \to [0,1]$, $\alpha \in A$, with locally finite supports such that $\sum\limits_\alpha \rho_\alpha = 1$. It is *subordinate to an open cover* $\{U_i\}_{i\in I}$ if there exists a map $i\colon A \to I$ such that $\operatorname{supp}\rho_\alpha \subset U_{i(\alpha)}$.

Theorem 2.4. *Let X be a paracompact Hausdorff space and $\{U_i\}_{i\in I}$ an open cover.*

(*i*) *There exists a partition of unity $\{\rho_i\}_{i\in I}$ subordinate to $\{U_i\}_{i\in I}$ such that at most countably many ρ_i are not identically zero.*
(*ii*) *There exists a partition of unity $\{\sigma_\alpha\}_{\alpha\in A}$ subordinate to $\{U_i\}_{i\in I}$ such that each σ_α is compactly supported.*
(*iii*) *If X is a smooth manifold, then we can take the functions ρ_i, σ_α to be smooth.*

For a proof, see [War, §1].

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Covariant derivatives

(2.5) *Differentiation of vector-valued functions.* Let M be a smooth manifold and $\mathbb{E}$ a complex[1] vector space. The differential of smooth $\mathbb{E}$-valued functions is a linear map

$$d\colon \Omega^0_M(\mathbb{E}) \longrightarrow \Omega^1_M(\mathbb{E}) \tag{2.6}$$

which satisfies the Leibniz rule

$$d(f\cdot e) = df\cdot e + f\cdot de, \qquad f\in\Omega^0_M(\mathbb{C}),\quad e\in\Omega^0_M(\mathbb{E}), \tag{2.7}$$

where '$\cdot$' is pointwise scalar multiplication. However, it is not the unique map with those properties. Any other has the form

$$d + A, \qquad A\in\Omega^1_M(\operatorname{End}\mathbb{E}). \tag{2.8}$$

It acts as the first order differential operator

$$\begin{aligned} d+A\colon \Omega^0_M(\mathbb{E}) &\longrightarrow \Omega^1_M(\mathbb{E}) \\ e &\longmapsto de + A(e) \end{aligned} \tag{2.9}$$

The last evaluation is the pairing $\operatorname{End}\mathbb{E}\otimes\mathbb{E}\to\mathbb{E}$. The directional derivative in a direction $\xi\in T_pM$ at some point $p\in M$ is

$$de_p(\xi) + A_p(\xi)(e). \tag{2.10}$$

Observe that the space of differentiations of $\mathbb{E}$-valued functions is the infinite dimensional vector space $\Omega^1_M(\operatorname{End}\mathbb{E})$.

(2.11) *Differentiation of vector bundle-valued functions.* Let $\pi\colon E\to M$ be a smooth vector bundle. The local triviality (Definition 1.12) implies the existence of an open cover $\{U_i\}_{i\in I}$ of M and vector bundle isomorphisms $\varphi_i\colon U_i\times\mathbb{E}_i\to\pi^{-1}(U_i)$ for some vector spaces $\mathbb{E}_i$. Using φ_i we identify sections of E over U_i with $\mathbb{E}_i$-valued functions on U_i, and so transport the differentiation operator (2.6) to a differentiation operator[2]

$$\nabla_i\colon \Omega^0_{U_i}(E) \longrightarrow \Omega^1_{U_i}(E), \tag{2.12}$$

that is, a linear map satisfying the Leibniz rule (2.7), where now $e\in\Omega^0_{U_i}(E)$. Let $\{\rho_i\}_{i\in I}$ be a partition of unity satisfying Theorem 2.4(i,iii). Let $j_i\colon U_i\hookrightarrow M$ denote the inclusion. Then

$$\begin{aligned} \nabla\colon \Omega^0_M(E) &\longrightarrow \Omega^1_M(E) \\ e &\longmapsto \sum_i \rho_i\nabla_i(j_i^*e) \end{aligned} \tag{2.13}$$

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[1] The discussion applies without change to real vector spaces and, below, real vector bundles.

[2] '∇' is pronounced 'nabla'.

defines a global differentiation on sections of E. The first-order differential operator (2.13) is called a *covariant derivative*, and the argument given proves their existence on any smooth vector bundle.

If ∇, ∇' are covariant derivatives on E, then the difference $\nabla' - \nabla$ is linear over functions, as follows immediately from the difference of their Leibniz rules, and so is a tensor $A \in \Omega^1_M(\operatorname{End} E)$. Therefore, the set of covariant derivatives is an affine space over the vector space $\Omega^1_M(\operatorname{End} E)$.

Remark 2.14. The averaging argument with partitions of unity works to average geometric objects which live in a convex space, or better sections of a bundle whose fibers are convex subsets of affine spaces. For example, it is used to prove the existence of hermitian metrics on complex vector bundles, and also the existence of splittings of short exact sequences of vector bundles (**2.28**).

(2.15) *Parallel transport.* Let $\gamma\colon [0,1] \to M$ be a smooth parametrized path. The covariant derivative pulls back to a covariant derivative on the pullback bundle $F := \gamma^*E \to [0,1]$. We use the covariant derivative to construct an isomorphism

$$\rho\colon F_0 \longrightarrow F_1 \tag{2.16}$$

from the fiber over 0 to the fiber over 1, called parallel transport. A section $s\colon [0,1] \to F$ of $F \to [0,1]$ is *parallel* if $\nabla_{\partial/\partial t} s = 0$.

Lemma 2.17. *Let P denote the vector space of parallel sections. Then the restriction map $P \to F_0$ which evaluates a parallel section at $0 \in [0,1]$ is an isomorphism.*

Proof. Assume first that $F \to [0,1]$ is trivializable and fix a basis of sections $e_1, \dots, e_n$, where $n = \operatorname{rank} F$. Define functions $A^i_j\colon [0,1] \to \mathbb{C}$ by

$$\nabla_{d/dt} e_j = A^i_j e_i. \tag{2.18}$$

(Here and forever we use the summation convention to sum over indices repeated once upstairs and once downstairs.) Then the section $f^j e_j$ is parallel if and only if

$$\frac{df^i}{dt} + A^i_j f^j = 0, \qquad i = 1, \dots, n. \tag{2.19}$$

The fundamental theorem of ordinary differential equations asserts that there is a unique solution f^j with given initial values $f^j(0)$, which is equivalent to the assertion in the lemma.

In general, by the local triviality of vector bundles and the compactness of $[0,1]$, we can find $0 = t_0 < t_1 < t_2 < \dots < t_{N-1} < t_N = 1$ such that $F\big|_{[t_i-1,t_i]} \to [t_{i-1}, t_i]$ is trivializable. Make the argument in the preceding paragraph on each interval and compose the resulting parallel transports to construct (2.16). ∎

(2.20) *Parametrized parallel transport.* We turn now to Theorem 2.1.

Proof of Theorem 2.1—smooth case. Let ∇ be a covariant derivative on $E \to [0,1] \times M$. Use parallel transport along the family of paths $[0,1] \times \{x\}$, $x \in M$, to construct an isomorphism (2.2). ∎

The ordinary differential equation of parallel transport (2.19) is now a family of equations: the coefficient functions A^i_j vary smoothly with x. Therefore, we need a parametrized version of the fundamental theorem of ODEs: we need to know that the solution varies smoothly with parameters. One reference for a proof is [La, §IV.1].

Proof for continuous bundles

Now we turn to the case when X is a space, and we follow [Ha, §1.2] closely; we defer to that reference for details. Choices of local trivializations replace choices of local covariant derivatives in this proof.

Proof of Theorem 2.1—continuous case. Observe first that if $\varphi\colon U\times \mathbb{E}\xrightarrow{\;\cong\;}E$ is a trivialization of a vector bundle $E\to U$, then for any $p,q\in U$ the trivialization gives an isomorphism $E_p\to E_q$ of the fibers which varies continuously in p,q. Thus a trivialization of a vector bundle $E\to[a,b]$ over an interval in $\mathbb{R}$ gives an isomorphism $E_a\to E_b$.

Now if $E\to[0,1]\times X$ is a vector bundle, we can find an open cover of $[0,1]\times X$ such that the bundle is trivializable on each open set; then by compactness of $[0,1]$ an open cover $\{U_i\}_{i\in I}$ of X such that the bundle is trivializable on each $[0,1]\times U_i$; and, choosing trivializations, continuous isomorphism $E\big|_{\{a\}\times U_i}\to E\big|_{\{b\}\times U_i}$ for any $0\leqslant a\leqslant b\leqslant 1$. (We use the observation in the previous paragraph.) Choose a partition of unity $\{\rho_i\}_{i\in I}$ subordinate to $\{U_i\}_{i\in I}$, and order the countable set of functions which are not identically zero: $\rho_1,\rho_2,\dots$. Define $\psi_n=\rho_1+\cdots+\rho_n$, $n=1,2,\dots$, and set $\psi_0\equiv 0$. Let $\Gamma_n\subset[0,1]\times X$ be the graph of ψ_n. The trivialization on $[0,1]\times U_n$ gives an isomorphism $\tilde\psi_n\colon E\big|_{\Gamma_{n-1}}\xrightarrow{\;\cong\;}E\big|_{\Gamma_n}$. The composition $\cdots\circ\tilde\psi_2\circ\tilde\psi_1$ is well-defined by the local finiteness of $\{\rho_i\}$ and gives the desired isomorphism (2.2). ☐

Remark 2.21. If X is a smooth manifold, then we choose ρ_i to be smooth and this proof produces a smooth isomorphism.

Consequences

We prove some standard corollaries of Theorem 2.1.

Corollary 2.22. *Let $f\colon[0,1]\times X\to Y$ be a continuous map between topological spaces, $f_t\colon X\to Y$ its restriction to $\{t\}\times X$, and let $E\to Y$ be a vector bundle. Then $f_0^*E\cong f_1^*E$.*

Proof. Apply Theorem 2.1 to $f^*E\to[0,1]\times X$. ☐

Corollary 2.23. *Let X be a contractible space and $E\to X$ a vector bundle. Then $E\to X$ is trivializable.*

Proof. The identity map id_X is homotopic to a constant map $c\colon X\to X$, and the pullback $c^*E\to X$ is a constant vector bundle with fiber E_c. Now apply Corollary 2.22. ☐

Corollary 2.24. *Let $X=U_1\cup U_2$ be the union of two open sets, $E_i\to U_i$ vector bundles, and $\alpha\colon[0,1]\times U_1\cup U_2\to\mathrm{Iso}\big(E_1\big|_{U_1\cap U_2},E_2\big|_{U_1\cap U_2}\big)$ a homotopy of clutching data. Then the vector bundles $\mathcal{E}_0\to X$ and $\mathcal{E}_1\to X$ obtained by clutching with α_0, α_1 are isomorphic.*

Here $\mathrm{Iso}(-,-)$ is the set of isomorphisms between the indicated vector bundles.

Proof. Clutch over $[0,1]\times X$ and apply Theorem 2.1. ∎

Let $\mathrm{Vect}^{\cong}(X)$ denote the set of isomorphism classes of vector bundles over X. It is a *commutative monoid*: the sum operation is defined by direct sum

$$[E]+[E']=[E\oplus E'] \tag{2.25}$$

and the zero element is represented by the constant vector bundle with fiber the zero vector space. In fact, $\mathrm{Vect}^{\cong}(X)$ is a *semiring*, with multiplication defined by

$$[E]\times[E']=[E\otimes E'] \tag{2.26}$$

Corollary 2.27. *Let $f\colon X\to Y$ be a continuous map. Then the induced pullback $f^*\colon \mathrm{Vect}^{\cong}(Y)\to \mathrm{Vect}^{\cong}(X)$ depends only on the homotopy class of f.*

We write $\mathrm{Vect}^{\cong}_{\mathbb{R}}(X)$ and $\mathrm{Vect}^{\cong}_{\mathbb{C}}(X)$ to indicate the ground field explicitly.

Further applications of partitions of unity

(2.28) *Short exact sequences of vector bundles.* Let

$$0\longrightarrow E'\xrightarrow{\;i\;}E\xrightarrow{\;j\;}E''\longrightarrow 0 \tag{2.29}$$

be a short exact sequence of vector bundles over a space X.[3] A *splitting* of (2.29) is a linear map $E''\xrightarrow{\;s\;}E$ such that $j\circ s=\mathrm{id}_{E''}$. A splitting determines an isomorphism

$$E''\oplus E'\xrightarrow{\;s\oplus i\;}E. \tag{2.30}$$

Lemma 2.31. *The space of splittings is a nonempty affine space over the vector space* $\mathrm{Hom}(E'',E')$.

Let's deconstruct that statement, and in the process prove parts of it. First, if s_0,s_1 are splittings, then the difference $\phi=s_1-s_0$ is a linear map $E''\to E$ such that $j\circ\phi=0$. The exactness of (2.29) implies that ϕ factors through a map $\tilde\phi\colon E''\to E'$: in other words, $\phi=i\circ\tilde\phi$. This, then, is the affine structure. But we must prove that the space of splittings is nonempty. First, we observe that any short exact sequence of vector spaces splits, and so using local trivializations we deduce that splittings of (2.29) exist locally on X. Now we use a partition of unity argument. Remember that partitions of unity can be used to average sections of a fiber bundle whose fibers are convex subsets of affine spaces. Of course, an affine space is a convex subset of itself. I leave the details to the reader.

[3] These can be real, complex, or quaternionic.

(2.32) *Inner products on vector bundles.* Recall that if $\mathbb{E}$ is a complex vector space, then an inner product is a bilinear map

$$\langle -,-\rangle\colon \overline{\mathbb{E}}\times\mathbb{E}\longrightarrow\mathbb{C} \tag{2.33}$$

which satisfies

$$\langle\bar{\xi}_1,\xi_2\rangle=\overline{\langle\bar{\xi}_2,\xi_1\rangle},\qquad \xi_1,\xi_2\in\mathbb{E}, \tag{2.34}$$

$$\langle\xi,\xi\rangle\in\mathbb{R}^{>0},\qquad \xi\in\mathbb{E},\quad \xi\neq 0 \tag{2.35}$$

Here $\overline{\mathbb{E}}$ denotes the conjugate vector space, which is the same abelian group as $\mathbb{E}$ but with scalar multiplication conjugated. The space of inner products on $\mathbb{E}$ is a subset of the vector space of bilinear maps (2.33) which are symmetric in the sense of (2.34); it is the convex cone of elements which satisfy the positivity condition (2.35).

Lemma 2.36. *A complex vector bundle $E\to X$ admits a positive definite hermitian inner product. The space of inner products is contractible.*

The proof is similar to that of Lemma 2.31 and is left to the reader. We emphasize the importance of the convexity of the set of inner products on a single vector space.

References

[Ha] Allen Hatcher, *Vector Bundles and K-Theory*, 2009. Version 2.1.

[La] Serge Lang, *Introduction to differentiable manifolds*, second ed., Universitext, Springer-Verlag, New York, 2002.

[War] Frank W Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer-Verlag, 1983.