Lecture 3: Group completion and the definition of K-theory

The goal of this lecture is to give the basic definition of K-theory. The process of *group completion*, which "completes" a commutative monoid M to an abelian group $K(M)$, loses information in general. In our topological setting what is retained is the *stable* equivalence class of a vector bundle. The notion of stability occurs in many guises, and they are all different facets of K-theory.

We begin with a basic proposition which allows us to replace the noncompact general linear groups with compact groups of isometries. This is convenient in many arguments. After describing group completion, we define K-theory and prove one of the basic theorems about the existence of inverses (Proposition 3.15). The basic building blocks of any cohomology theory is the value of that theory on spheres, and we prove in Proposition 3.26 that for K-theory those values are homotopy groups of the stable unitary group (stable orthogonal group in the real case).

A deformation retraction from Gram-Schmidt

Proposition 3.1. *There are deformation retractions*

$$\begin{aligned} GL_n\mathbb{C} &\longrightarrow U_n \\ GL_n\mathbb{R} &\longrightarrow O_n \end{aligned} \tag{3.2}$$

Here $U_n \subset GL_n\mathbb{C}$ is the subgroup of unitary matrices and $O_n \subset GL_n\mathbb{R}$ is the subgroup of orthogonal matrices. The reader should supply pictures for the case $n = 1$; the deformation retractions in that case is the first step in the general proof.

Proof. The proof is the same in both cases; for convenience, we use the notation of the complex version. Identify $GL_n\mathbb{C}$ with the space of bases of $\mathbb{C}^n$: the columns of an invertible $n\times n$ matrix form a basis. Then U_n is the subspace of orthonormal bases. The Gram-Schmidt process, which converts an arbitrary basis into an orthonormal basis, is a composition of deformation retractions. The first takes a basis $e_1,\dots,e_n$ and constructs one in which $|e_1| = 1$. The deformation fixes $e_2,\dots,e_n$ and at time $t \in [0,1]$ has first vector $\bigl((1-t) + t/|e_1|\bigr)e_1$. The second step we move e_2 only and make it orthogonal to e_1 via the path $e_2 - t\langle e_2, e_1\rangle e_1$. Now repeat. Move e_2 to have unit norm and then move e_3 to be orthogonal to both e_1 and e_2. After $2n-1$ steps we are done. ∎

Group completion and universal properties

(3.3) *The group completion of a commutative monoid.* Recall that a commutative monoid M is a set with a commutative, associative composition law $M \times M \to M$ and a unit $0 \in M$.

Definition 3.4. Let M be a commutative monoid. A *group completion* (A, i) of M is an abelian group A and a homomorphism $i\colon M \to A$ of commutative monoids which satisfies the following

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K-Theory (M392C, Fall '15), Dan Freed, September 8, 2015 universal property: If B is an abelian group and $f\colon M \to B$ a homomorphism of commutative monoids, then there exists a unique group homomorphism $\tilde{f}\colon A \to B$ which makes the diagram

$$\begin{tikzcd} M \arrow[rr, "i"] \arrow[dr, "f"'] & & A \arrow[dl, dashed, "\tilde{f}"] \\ & B & \end{tikzcd} \tag{3.5}$$

commute.

The definition does not prove the existence of the group completion—we must provide a proof—but the universal property does imply a strong uniqueness property. Namely, if (A, i) and (A', i') are group completions of M, then there is a unique isomorphism $\phi\colon A \to A'$ of groups which makes the diagram

$$\begin{tikzcd} M \arrow[rr, "i"] \arrow[dr, "i'"'] & & A \arrow[dl, "\phi"] \\ & A' & \end{tikzcd} \tag{3.6}$$

commute. The proof uses four applications of the universal property (to $f = i$ and $f = i'$ to construct the isomorphism and its inverse, and then two more to prove the compositions are identity maps). To construct an explicit group completion, define A as the quotient of $M \times M$ in which (m_1, m_2) is identified with $(m_1 + n, m_2 + n)$ for all $m_1, m_2, n \in M$. Addition in A is defined component-wise in $M \times M$, the unit is $[0, 0]$, and $-[m_1, m_2] = [m_2, m_1]$. (The square brackets denote the equivalence class.)

Example 3.7. If $M = \mathbb{Z}^{\geqslant 0}$ under addition, then the group completion is $\mathbb{Z}$ under addition. If $M = \mathbb{Z}^{>0}$ under multiplication, then the group completion is $\mathbb{Q}^{>0}$ under multiplication.

Example 3.8. If $M = \mathbb{Z}^{\geqslant 0}$ under multiplication, then the group completion $(K(M), i)$ is the trivial group. For there exists $x \in K(M)$ such that $x \cdot i(0) = 1$, and so for any $n \in M$ we have

$$i(n) = \big(x \cdot i(0)\big) \cdot i(n) = x \cdot \big(i(0) \cdot i(n)\big) = x \cdot i(0 \cdot n) = x \cdot i(0) = 1. \tag{3.9}$$

Now apply *uniqueness* of the factorization.

This example is the first illustration of how information may be lost in passing to the group completion.

The abelian group $K(X)$ for X compact

Let X be a compact[1] Hausdorff space. Let $\mathrm{Vect}^{\cong}(X)$ denote the set of isomorphism classes of complex vector bundles $E \to X$. Then the operation of direct sum on vector bundles induces a commutative, associative composition law on $\mathrm{Vect}^{\cong}(X)$; the equivalence class of the zero vector bundle is a unit.

[1]The definitions work for any paracompact Hausdorff X, but for noncompact spaces may give the "wrong" group. We give a more general definition later.

Definition 3.10. $K(X)$ is the group completion of the commutative monoid $\mathrm{Vect}^{\cong}(X)$.

Remark 3.11. For the empty set we have $\mathrm{Vect}^{\cong}(\varnothing) = K(\varnothing) = 0$, since there is a unique vector bundle over $\varnothing$.

(3.12) *Functorial property.* Let Top_c denote the category of compact Hausdorff spaces and continuous maps. Then $X \mapsto K(X)$ is a contravariant functor from Top_c to the category of abelian groups. For a continuous map $f\colon X \to X'$ induces a pullback on bundles. Furthermore, it follows immediately from Corollary 2.22 that K is a *homotopy* functor: homotopic maps $f_0 \simeq f_1$ induce equal maps on K-groups.

Remark 3.13. The functor $X \mapsto \mathrm{Vect}^{\cong}(X)$ to commutative monoids is also a homotopy functor. However, it is more difficult to compute, which is why we pass to the group completion. The group completion loses information in principle, but experience shows that the trade-off for increased computability is a good deal.

(3.14) *Real K-theory.* The proofs work equally for real vector bundles. The group completion of $\mathrm{Vect}^{\cong}_{\mathbb{R}}(X)$, the commutative monoid of real vector bundles, is denoted $KO(X)$.

Proposition 3.15. *Let X be a compact Hausdorff space and $\pi\colon E \to X$ a vector bundle. Then there exists a vector bundle $E' \to X$ such that $E \oplus E' \to X$ is trivializable.*

Observe that if $X \neq \varnothing$, then $\mathbb{Z} \hookrightarrow K(X)$ as the group completion of the trivial bundles. Prove this by choosing applying K to the unique map $X \to \mathrm{pt}$ and a section $\mathrm{pt} \to X$ obtained by choosing a point of X. Proposition 3.15 implies that for any $\alpha \in K(X)$ there exists $\alpha' \in K(X)$ such that $\alpha + \alpha' = N$ for some $N \in \mathbb{Z}$.

Proof. Let $\{U_1, \dots, U_K\}$ be a finite cover for which the restriction of E to each U_i is trivializable. Let $\{\rho_1, \dots, \rho_K\}$ be a partition of unity subordinate to $\{U_1, \dots, U_K\}$. For each open set U_j choose a basis of sections $e_j^1, \dots, e_j^n$ of $E^*\big|_{U_i} \to U_i$. Then $S = \{\rho_1 e_1^1, \rho_1 e_1^2, \dots, \rho_1 e_1^n, \rho_2 e_2^1, \rho_2 e_2^2, \dots\}$ is a set of nK sections of $E^* \to X$. Let $V = (\mathbb{C}S)^*$ be dual to the vector space with basis S. Evaluation defines a map of vector bundles $E \to \underline{V}$ of E to the bundle with constant fiber V, and this is an injective map of vector bundles. Let $E' = \underline{V}/E$ be the quotient bundle, so that we have a short exact sequence

$$0 \longrightarrow E \longrightarrow \underline{V} \longrightarrow E' \longrightarrow 0 \tag{3.16}$$

of vector bundles over X. Any short exact sequence of vector bundles splits (**2.28**), and a splitting determines a trivialization of $E \oplus E'$. □

(3.17) *Reduced K-theory.* Any space X has a unique map $X \to \mathrm{pt}$ to the 1-point space, and the induced map $\mathbb{Z} = K(\mathrm{pt}) \to K(X)$ is injective if X is nonempty, in which case it is the injection mentioned after the statement of Proposition 3.15. (If X is nonempty the map $X \to \mathrm{pt}$ admits sections.)

Definition 3.18. The *reduced K-theory group* is the quotient $\widetilde{K}(X) = K(X)/K(\mathrm{pt})$.

Proposition 3.19. *Let X be a compact Hausdorff space, $E, E' \to X$ vector bundles. Then $[E] = [E'] \in \widetilde{K}(X)$ if and only if $E \oplus \underline{\mathbb{C}^r} \cong E' \oplus \underline{\mathbb{C}^{r'}}$ for some r, r'.*

Proof. If $[E] = [E']$ then there exists $s \in \mathbb{Z}^{\geqslant 0}$ and a vector bundle $F \to X$ such that $E \oplus F \cong E' \oplus F \oplus \underline{\mathbb{C}^s}$. By Proposition 3.15 there exists $F' \to X$ such that $F \oplus F' \cong \underline{\mathbb{C}^r}$. The forward implication follows; the backward implication is immediate from Definition 3.18. ☐

Bundles E, E' which satisfy the hypothesis of Proposition 3.19 are said to be *stably equivalent*, and the reduced K-theory is the group of stable equivalence classes of vector bundles.

Fiber bundles

The definition of a fiber bundle is simpler than that (Definition 1.12) of a vector bundle: the fibers of a fiber bundle are topological spaces with no additional structure. Thus a vector bundle is a special case of a fiber bundle.

Definition 3.20. A *fiber bundle* is a surjective continuous map $\pi\colon \mathcal{E} \to X$ of topological spaces which admits local trivializations: every point $x \in X$ has an open neighborhood U containing x and a topological space F such that there exists a homeomorphism $\varphi\colon U \times F \to \mathcal{E}\big|_U$ which makes the diagram

$$\begin{tikzcd} U \times F \arrow[rr, "\varphi"] \arrow[dr, "pr_1"'] & & \mathcal{E}\big|_U \arrow[dl, "\pi"] \\ & U & \end{tikzcd} \tag{3.21}$$

commute.

Example 3.22. Let $\pi\colon E \to X$ be a complex vector bundle. There are many associated fiber bundles; we indicate the total space. $\mathbb{P}E$ is a fiber bundle whose fibers are the projectivizations of the fibers of E. More generally, for $k \geqslant 0$ we have the bundle of Grassmannians Gr_kE; the projectivization is $k = 1$. If E has a metric then we can form the sphere bundle $S(E)$. There is a bundle of groups $\mathrm{Aut}(E)$. If $\mathrm{rank}\colon E \to \mathbb{Z}$ is the constant function r, then $\mathrm{Iso}(\mathbb{C}^n, E)$ is the bundle of frames (bases) of E. It is a *principal fiber bundle*, or *principal bundle* for short: its fibers are right torsors over the group $GL_n\mathbb{C}$.

Fiber bundles satisfy the *homotopy lifting property*—they are fibrations. Assume that $\mathcal{E}, X$ are pointed spaces with basepoints $e, \pi(e) = b$. A fibration is characterized by the *homotopy lifting property.*

Definition 3.23. $p\colon \mathcal{E} \to X$ is a *fibration* if for every pointed space S, continuous map $f\colon [0,1] \times S \to X$ and lift $\tilde{f}_0\colon S \to \mathcal{E}$ of f_0 there exists an extension $\tilde{f}\colon [0,1] \times S \to \mathcal{E}$ lifting f.

The lift is encoded in the diagram

$$\begin{tikzcd} \{0\} \times S \arrow[r, "\tilde{f}_0"] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ {[0,1] \times S} \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & X \end{tikzcd} \tag{3.24}$$

For a proof that fiber bundles are fibrations, see [Ha2, p. 375].

Recall that a topological space F is *m-connected* if every continuous map $\phi\colon S \to F$ with domain a CW complex S of dimension $\leqslant m$ is null homotopic.

Proposition 3.25. *Let X be a CW complex of dimension $\leqslant n$, $\pi\colon \mathcal{E} \to X$ be a fiber bundle, and suppose the fibers of π are $(n-1)$-connected. Then π admits a section.*

Proof. We give the proof for a finite CW complex by inducting over the skeleta $X_k \subset X$. Since X_0 is a discrete set of points, there is a section of $\mathcal{E}\big|_{X_0} \to X_0$. Now suppose we have a section over the $(k-1)$-skeleton and consider a k-cell with characteristic map $\Phi\colon D^k \to X_k$. Since D^k is contractible there is a trivialization[2] $\underline{F} \xrightarrow{\cong} \Phi^*\mathcal{E}$. The section on the $(k-1)$-skeleton then pulls back via $\partial\Phi$ to a map $S^{k-1} \to F$. By hypothesis this map is null homotopic, so extends over D^k and, unwinding with the trivialization and Φ, extends the section over this k-cell. □

Bott periodicity

We express the reduced K-theory of spheres as homotopy groups of unitary groups.

Proposition 3.26. *Let n be a nonnegative integer and $N \geqslant n/2$. Then there is an isomorphism*

$$\pi_{n-1}U_N \longrightarrow \widetilde{K}(S^n). \tag{3.27}$$

Proof. We construct (3.27) as a composition of isomorphisms of *sets*

$$\pi_{n-1}U_N \xrightarrow{\;i\;} [S^{n-1}, U_N] \xrightarrow{\;j\;} \mathrm{Vect}^{\cong}_N(S^n) \xrightarrow{\;k\;} \widetilde{K}(S^n) \tag{3.28}$$

Here $\mathrm{Vect}^{\cong}_N(S^n)$ is the set of isomorphism classes of complex vector bundles of rank N over S^n.

The homotopy group is the set of homotopy classes of pointed maps $S^{n-1} \to U_N$ which send a basepoint $* \in S^{n-1}$ to the identity $e \in U_N$. The first map i forgets basepoints; its inverse sends a map $f\colon S^{n-1} \to U_N$ to $f(*)^{-1}f$.

The second map j is the clutching construction (**1.16**), where we write the sphere as the union $S^n = D^n \cup_{S^{n-1}} D^n$ of two closed hemispheres along the equator. The proof that j is an isomorphism has three ingredients. First, any vector bundle admits a hermitian metric, so the clutching map can be assumed an isometry. Second, to show j is well-defined we apply Corollary 2.24. We need to prove that homotopic clutching maps lead to isomorphic bundles. A homotopy of clutching maps leads to a bundle over $[0,1] \times S^n$, and then Theorem 2.1 applies. Third, distinct homotopy classes of clutching maps construct non-isomorphic bundles, which follows from the fact that there is a unique homotopy class of trivializations on D^n.

To show that k is an isomorphism we need to show that a complex vector bundle $E \to S^n$ of rank $> N$ is stably equivalent to a bundle of rank N (surjectivity of k) and that stably isomorphic bundles of rank N are isomorphic (injectivity of k). To prove the first statement it suffices to construct a *nonzero* section of $E \to S^n$. For such a section spans a *trivial* line subbundle $L \subset E$,

[2] The homotopy invariance arguments in Lecture 2 apply to general fiber bundles, not just vector bundles.

and by splitting the short exact sequence $0 \to L \to E \to E/L \to 0$ we see that E is stably equivalent to E/L, which has rank one less and we can iterate. To construct a nonzero section, fix a hermitian metric on $E \to S^n$ and consider the sphere bundle $S(E) \to S^n$, a fiber bundle with fiber S^{2N-1}. If $2N-1 \geqslant n$ it follows from Proposition 3.25 that $S(E) \to S^n$ admits a section. For the second statement suppose $E_0, E_1 \to S^n$ have rank N and for some $r > 0$ there exists an isomorphism $\varphi\colon E_0 \oplus \underline{\mathbb{C}^r} \xrightarrow{\cong} E_1 \oplus \underline{\mathbb{C}^r}$. Choose metrics (Lemma 2.36) and homotop φ to an isometry (Proposition 3.1). Consider the fiber bundle[3]

$$p\colon \operatorname{Isom}(E_0 \oplus \underline{\mathbb{C}^r}, E_1 \oplus \underline{\mathbb{C}^r}) \longrightarrow S(E_1 \oplus \underline{\mathbb{C}^r}) \tag{3.29}$$

where p maps an isometry to the image of $(0, \dots, 0, 1) \in \mathbb{C}^r$, which lies in the unit sphere bundle. The isometry φ defines a section of the bundle $\operatorname{Isom}(E_0 \oplus \underline{\mathbb{C}^r}, E_1 \oplus \underline{\mathbb{C}^r}) \to S^n$; its composition with p is a section s of the bundle

$$S(E_1 \oplus \underline{\mathbb{C}^r}) \to S^n. \tag{3.30}$$

Now we apply a relative version of Proposition 3.25 to homotop s to a constant section with value $(0, \dots, 0, 1) \in \mathbb{C}^r$: pull (3.30) back over $[0,1] \times S^n$ and extend the section which at $\{0\} \times S^n$ is φ and at $\{1\} \times S^n$ is the constant. Here the base is $(n+1)$-dimensional and the fiber $(2(N+r)-2)$-connected. Finally, use the homotopy lifting property of (3.29) (see (3.24)) to construct a homotopy of φ to a family of isomorphisms which is the identity on the last copy of $\mathbb{C}$, and so restricts to an isomorphism $E_0 \oplus \underline{\mathbb{C}^{r-1}} \xrightarrow{\cong} E_1 \oplus \underline{\mathbb{C}^{r-1}}$. □

Corollary 3.31. *The inclusion $U_N \hookrightarrow U_{N+1}$ induces an isomorphism $\pi_{n-1}U_N \xrightarrow{\cong} \pi_{n-1}U_{N+1}$ if $N \geqslant n/2$.*

Remark 3.32. The common value of $\pi_{n-1}U_N$ for N large is the *stable homotopy group* of the unitary group. It is the homotopy group of a topological group $U = U_\infty$ which can be constructed as the colimit of $U_1 \hookrightarrow U_2 \hookrightarrow U_3 \hookrightarrow \cdots$. There are prettier geometric models for the same homotopy type, even Banach Lie group models.

Theorem 3.33 (Bott)**.** *There are isomorphisms*

$$\pi_{n-1}U \cong \widetilde{K}(S^n) \cong \begin{cases} \mathbb{Z}, & n \text{ even}; \\ 0, & n \text{ odd}. \end{cases} \tag{3.34}$$

We will give several proofs of Theorem 3.33 as well as stronger forms of Bott periodicity.

For complex vector bundles and unitary groups the periodicity has period 2. There is an analogous 8-periodic statement in the real case; the stable unitary group U is replaced by the stable orthogonal group O.

[3]The map p is also a map of fiber bundles over S^n.

Theorem 3.35 (Bott). *There are isomorphisms*

$$\pi_{n-1}O \cong \widetilde{KO}(S^n) \cong \begin{cases} \mathbb{Z}, & n \equiv 0 \pmod 8; \\ \mathbb{Z}/2\mathbb{Z}, & n \equiv 1 \pmod 8; \\ \mathbb{Z}/2\mathbb{Z}, & n \equiv 2 \pmod 8; \\ 0, & n \equiv 3 \pmod 8; \\ \mathbb{Z}, & n \equiv 4 \pmod 8; \\ 0, & n \equiv 5 \pmod 8; \\ 0, & n \equiv 6 \pmod 8; \\ 0, & n \equiv 7 \pmod 8; \end{cases} \tag{3.36}$$

A vocal rendition of $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, 0 , $\mathbb{Z}$, 0 , 0 , 0 , $\mathbb{Z}$ is known as the *Bott song*.

References

[Ha2] Allen Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.