Problem Set # 1

M392C: Riemannian Geometry

As I wrote in the First Day Handout, the problem sets are for your benefit. You should be making your own exercises from the lectures; I'm stunting your growth by doing it for you. It will not surprise me, then, if the problem sets taper off a bit as we go along.

Some of these problems are routine. Some are hard. Don't get discouraged.

The problems are not to hand in. Rather, form study groups, discuss them with your friends, then come to office hours and discuss them with me. Write some of them up—for yourself and for each other. Present solutions to each other at a blackboard. If there is demand, I will offer problem sessions to answer questions about the problems (and lectures).

Problems

1. This problem gives practice with the index notation and summation convention that I use and recommend to you. Note carefully the placement (superscript vs. subscript) of the indices in what follows. The actual name of the index (i or j or α) is arbitrary, though as always a judicious choice of notation helps you and your readers.

   Let V be an n dimensional (real) vector space. Suppose $\{e_j\}$ and $\{f_i\}$ are two bases for V which are related by the equation

$$e_j = P^i_j f_i,$$

   where P is an invertible matrix. When we view P^i_j as the entry in a matrix, then i is the row index and j the column index

   (a) Suppose $\xi \in V$ is a vector. Then we can find real numbers ξ^j and $\tilde{\xi}^i$ such that $\xi = \xi^j e_j = \tilde{\xi}^i f_i$. Express $\tilde{\xi}^i$ in terms of the ξ^j.

   (b) Suppose $T\colon V \to V$ is a linear transformation. Relative to the basis $\{e_j\}$ it is expressed as the matrix A defined by $Te_j = A^i_j e_i$, and relative to the basis $\{f_i\}$ it is expressed as the matrix B defined by $Tf_i = B^j_i f_j$. What is the relationship between A and B?

   (c) The *dual space* V^* is the vector space of all linear functionals $V \to \mathbb{R}$; it is also n dimensional. Every basis of V gives rise to a *dual basis* of V^*. For the basis $\{e_j\}$ of V the dual basis $\{e^i\}$ of V^* is defined by the equation

$$e^i(e_j) = \delta^i_j = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

   (This equation defines the symbol δ^i_j.) The dual basis $\{f^j\}$ is defined similarly. Express f^j in terms of the e^i.

(d) Suppose $\omega \in V^*$. Then we define its components relative to the basis $\{e^i\}$ by the equation $\omega = \omega_i e^i$ and its components relative to the basis $\{f^j\}$ by the equation $\omega = \tilde{\omega}_j f^j$. Express the ω_i in terms of the $\tilde{\omega}_j$.

(e) The tensor and exterior powers of V have natural induced bases. For example, what is the basis for $\otimes^2 V^* = V^* \otimes V^*$, the space of bilinear forms on V? What about $\bigwedge^2 V^*$, the space of alternating bilinear forms? What about $\mathrm{Sym}^2 V^*$, the space of symmetric bilinear forms?

2. Let $\langle \cdot, \cdot \rangle$ be an inner product (metric) on a finite dimensional real vector space V. Suppose we fix a basis $e_1, \dots, e_n$ for V.

(a) Relative to this basis we describe the metric by the set of numbers $g_{ij} = \langle e_i, e_j \rangle$. Show that $g_{ij} = g_{ji}$.

(b) If a second basis $f_1, \dots, f_n$ is given, related to the first by

$$e_j = P^i_j f_i,$$

then what is the metric relative to this new basis? Write the equation relating the metric in the two bases as a matrix equation, thinking of g_{ij} as the $(i,j)^{\text{th}}$-entry of the matrix.

(c) The metric defines a map $V \to V^*$ which associates to a vector ξ the linear functional $\eta \mapsto \langle \xi, \eta \rangle$. Show that this map $V \to V^*$ is an isomorphism. Use it to define a metric on V^*. Define $g^{ij} = \langle e^i, e^j \rangle$, where $e^1, \dots, e^n$ is the dual basis of V^* and the metric is the induced metric on the dual space. What is the relationship of g_{ij} and g^{ij}?

(d) Define induced metrics on tensor powers and exterior powers of V and V^*.

3. (a) Derive the transformation law for the Riemannian metric in local coordinates. That is, if x^i, y^a are two coordinate systems, and the local form of the metric is given by functions g_{ij}, h_{ab} respectively, then

$$g_{ij} = h_{ab} \frac{\partial y^a}{\partial x^i} \frac{\partial y^b}{\partial x^j}.$$

Be sure you understand what this equation means. What are the domains and codomains of the functions? Can you multiply the functions on the right hand side as written? What is really meant?

(b) Make sense of the equation

$$g = g_{ij}\, dx^i \otimes dx^j$$

for the metric. Rederive part (a) using this expression. What is the equation for the inverse metric g^{-1}, that is, for the induced metric on the cotangent bundle?

(c) Check all steps in the derivation of the Riemann tensor carefully.

(d) Using the formulas

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\ell} \left(\frac{\partial g_{j\ell}}{\partial x^k} + \frac{\partial g_{k\ell}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell} \right)$$

$$R^i_{jk\ell} = \frac{\partial \Gamma^i_{j\ell}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^\ell} + \Gamma^m_{j\ell}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{m\ell},$$

check that

$$R = R^i_{jk\ell} \frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^\ell$$

is independent of the coordinate system.

4. A theorem in classical Euclidean geometry, named after the great Napolean, goes as follows. Suppose A, B, C are points in the Euclidean plane $\mathbb{E}^2$. Construct a point C' external to the triangle ABC so that the triangle ABC' is equilateral. Similarly, construct equilateral triangles $A'BC$ and $AB'C$. Let A'', B'', C'' be the centers of the triangles BCA', CAB', ABC'. The theorem states that $A''B''C''$ is equilateral.

   Prove this using group theory. Let $R(P)$ denote the rotation through angle $2\pi/3$ clockwise centered about the point P. Then if A'', B'', C'' are arranged clockwise, consider the composition $\rho = R(C'') \circ R(B'') \circ R(A'')$. Now for any points P, Q compute the composition of rotations with centers P and Q. Also notice that ρ has certain fixed points.

5. (a) Compute the Riemann tensor $R^i_{jk\ell}$ for the round 2-sphere. Use standard spherical coordinates $x^1 = \phi,\ x^2 = \theta$. So first compute g_{ij} and then apply the formula. How many nonzero components are there?

   (b) More generally, suppose we have local coordinates x^1, x^2 on a Riemannian 2-manifold such that $g_{11} = 1,\ g_{12} = 0,\ g_{22} = G = G(x^1, x^2)$. Compute the Riemann curvature tensor.

6. Define

$$R_{ijk\ell} = g_{i\lambda} R^\lambda_{jk\ell}.$$

   Use the formula for $R^\lambda_{jk\ell}$ to find some symmetries of $R_{ijk\ell}$. How many independent components are there in two dimensions? In three dimensions? In n dimensions?

7. Prove that S^4 does not admit an almost complex structure. (That is, prove that there is not a complex structure on the real rank 4 vector bundle $TS^4 \to S^4$. You will probably need to use some algebraic topology well beyond a first course ...)

8. Let $C \subset E$ be a co-oriented 1-dimensional submanifold of a Euclidean plane. Formulate precisely and prove the statement that the signed curvature is the rate of turning of the oriented unit normal vector with respect to arclength. (It's the arclength part we did not discuss in lecture.)