# Problem Set # 10

M392C: Riemannian Geometry

1. Let $X$ be a complex manifold and $P \to X$ a holomorphic principal $GL_m\mathbb{C}$-bundle. Show that a $GL_m\mathbb{C}$-invariant horizontal distribution $H \subset TP$ is invariant under the complex structure $I$ if and only if the corresponding connection form $\Theta \in \Omega^1(\mathfrak{gl}_m\mathbb{C})$ is of type $(1,0)$.

2. Let $X$ be a smooth manifold with an almost complex structure, i.e., a section $I$ of $\operatorname{End}(TX) \to X$ which satisfies $I^2 = -\operatorname{id}_{TX}$. The $+i$-eigenspace of $I$ is a complex distribution $T_{(1,0)}X \subset TX \otimes \mathbb{C}$. Its Frobenius tensor $\Phi$ is a $(2,0)$-form with values in $T_{(0,1)}X = \overline{T_{(1,0)}X}$. Assume that $\Phi = 0$. Let $\nabla$ be a torsionfree covariant derivative on $TX \to X$. (Why does one exist?) For (real) vector fields $\xi, \eta$ set
$$\nabla'_\xi \eta = \nabla_\xi \eta - (\nabla_{I\eta} I)\xi - I\big((\nabla_\eta I)\xi\big) - 2I\big((\nabla_\xi I)\eta\big).$$
Prove that $\nabla'$ is a torsionfree covariant derivative which satisfies $\nabla' I = 0$.

3. Let $X^{2m}$ be a complex manifold with complex structure $I$ and suppose $\langle -, - \rangle$ is a Riemannian metric on $X$. Assume $I$ is orthogonal. Let $\omega$ be the associated 2-form. Here are two more proofs that $d\omega = 0$ implies $X$ is Kähler.
   (a) Show that $\omega$ has type $(1,1)$.
   (b) Let $\nabla$ denote the Levi-Civita covariant derivative on $TX \to X$. Show that for vector fields $\xi, \eta, \zeta$ we have
   $$2\langle (\nabla_\xi I)\eta, \zeta \rangle = 3d\omega(\xi, I\eta, I\zeta) - 3d\omega(\xi, \eta, \zeta).$$
   Conclude that $X$ is Kähler if $d\omega = 0$.
   (c) Let $\theta^1, \dots, \theta^{2m}$ be a local orthonormal coframing adapted to $I$: the dual framing $\xi_1, \dots, \xi_{2m}$ satisfies $\xi_2 = I\xi_1$, $\xi_4 = I\xi_3$, etc. Show that
   $$\omega = \theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4 + \cdots .$$
   Let $\Theta^i_j$ denote the Levi-Civita connection forms, characterized by the equations
   $$\begin{aligned} d\theta^i + \Theta^i_j \wedge \theta^j &= 0 \\ \Theta^i_j + \Theta^j_i &= 0 \end{aligned}$$
   Show that $I$ is parallel if and only if $\Theta I = I\Theta$ if and only if
   $$\begin{aligned} \Theta^3_2 + \Theta^4_1 &= 0 \\ \Theta^3_1 - \Theta^4_2 &= 0 \\ &\cdots \end{aligned}$$
   Prove that this is satisfied if and only if $d\omega = 0$.

4. (a) Let $V$ be an $m$-dimensional complex vector space with a Hermitian metric. Its automorphism group is denoted $U(V)$. Let $SU(V)$ denote the closed Lie subgroup of automorphisms which act trivially on $\bigwedge^m V^*$. Show that these are precisely the automorphisms of determinant one. An element of $\bigwedge^m V^*$ is a *complex volume form* on $V$: it attaches a complex number to the complex parallelepiped spanned by $m$ (ordered) vectors in $V$.

   (b) Let $X^{2m}$ be a Riemannian manifold whose holonomy group is a subgroup of $SU_m$. Construct a nonzero parallel complex volume form $\Omega \in \Omega_X^{m,0}$. Prove that $d\Omega = 0$.