Problem Set # 11

M392C: Riemannian Geometry

1. Let X be a Kähler manifold, I the complex structure on TX, and R its Riemann curvature tensor.

   (a) Show that for real tangent vecturs ξ, η at some $x \in X$ we have $R(I\xi, I\eta) = R(\xi, \eta)$. Conclude that R is of type $(1,1)$.

   (b) Prove that $R(\xi, \eta)$ commutes with I. Show that the resulting complex endomorphism of the $(1,0)$ tangent space is skew-Hermitian.

   (c) Choose a *complex* basis $e_1, \dots, e_m$ of the $(1,0)$ tangent space and let $\overline{e_1}, \dots, \overline{e_m}$ be the complex conjugate basis of the $(0,1)$ tangent space. Write the curvature tensor $R^a_{b,c,d}$ in that basis, where the indices a, b, c, d each take on the $2m$ values $\mu, \bar{\mu} = 1, \dots, m$. Which R^a_{bcd} are possibly nonzero? (For example, $R^a_{b\mu\nu} = 0$.)

2. Let $P \to X$ be a principal G-bundle with connection Θ and curvature Ω.

   (a) Define the *adjoint bundle* $\mathfrak{g}_P \to X$ as the vector bundle associated to the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G. Show that $\mathfrak{g}_P \to X$ is a bundle of Lie algebras.

   (b) Interpret Ω as a 2-form on X with values in the adjoint bundle, i.e., an element of $\Omega^2_X(\mathfrak{g}_P)$.

   (c) The adjoint bundle has a covariant derivative ∇ induced from the connection Θ. Use it to construct an operator
   $$d_\nabla \colon \Omega^q_X(\mathfrak{g}_P) \longrightarrow \Omega^{q+1}_X(\mathfrak{g}_P)$$
   which agrees with ∇ for $q = 0$. (Hint: Write an element of $\Omega^q_X(\mathfrak{g}_P)$ as a sum of terms ωs, where $\omega \in \Omega^q_X$ and s is a section of $\mathfrak{g}_P \to X$.)

   (d) Prove the *Bianchi identity* $d_\nabla \Omega = 0$.

   (e) Now regard Ω as an element of $\Omega^2_P(\mathfrak{g})$. Compute $d\Omega$. How is the equation you get related to (d).

   (f) What does the Bianchi identity say if G is abelian? What is the adjoint bundle in that case? Try the specific cases $G = \mathbb{T}$ and $G = \mathbb{C}^\times$.

   (g) Prove that the Ricci form of a Kähler manifold (defined how?) is closed.

3. Let X be a complex and $z^1, \dots, z^m$ a local holomorphic coordinate system.

   (a) $K\colon X \to \mathbb{R}$ a real-valued function. Define $\omega = 2i\partial\bar{\partial}K$. Write it in the local coordinates as $\omega = g_{\mu\bar{\nu}} dz^\mu \overline{dz^\nu}$. Show that $(g_{\mu\bar{\nu}})$ defines a hermitian form. If it is positive definite, then show that it is a Kähler metric. In this case K is called a *Kähler potential*. It is a theorem that on any Kähler manifold there exist local Kähler potentials.

(b) Show that the Ricci form is $\rho = -i\partial\bar{\partial}\log\det(g_{\mu\bar{\nu}})$. Deduce again $d\rho = 0$.

(c) Another Kähler form in the same Kähler class has the form $\omega' = \omega + 2i\partial\bar{\partial}\phi$ for some (global) real function ϕ. Similarly, another possible Ricci form is $\rho' = \rho - i\partial\bar{\partial}f$ for a real function f. Write

$$(\omega')^m = F\omega^m$$

for a positive function F. Deduce that ρ' is the Ricci form of ω' if and only if $A = F/e^f$ is constant (assuming X is connected.)

(d) Show that the displayed equation above is

$$\det\left(g_{\mu\bar{\nu}} + \frac{\partial^2\phi}{\partial z^\mu \partial \overline{z^\nu}}\right) = Ae^f \det(g_{\mu\bar{\nu}}).$$

The Calabi conjecture involves solving this complex *Monge-Ampère* equation.

4. Does the homogeneous manifold $GL_{2m}\mathbb{R}/GL_m\mathbb{C}$ admit a $GL_{2m}\mathbb{R}$-invariant Riemannian metric?