Problem Set # 12

M392C: Riemannian Geometry

1. In this exercise you will construct the Grassmann manifold and tautological vector bundles over it. Let V be a real vector space of dimension $n \in \mathbb{Z}^{\geq 0}$. Define the Grassmannian $Gr_k(V)$, $0 \leq k \leq n$, to be the set of all k-dimensional subspaces of V.

   (a) Introduce a locally Euclidean topology on $Gr_k(V)$. Here is one way to do so: Suppose $W \in Gr_k(V)$ is a k-dimensional subspace and C an $(n-k)$-dimensional subspace such that $W \oplus C = V$. (We say that C is a complement to W in V.) Then define a subset $\mathcal{O}_{W,C} \subset Gr_k(V)$ by

   $$\mathcal{O}_{W,C} = \{W' \subset V : W' \text{ is the graph of a linear map } W \to C\}.$$

   Show that $\mathcal{O}_{W,C}$ is a vector space, so has a natural topology. Prove that it is consistent to define a subset $U \subset Gr_k(V)$ to be open if and only if $U \cap \mathcal{O}_{W,C}$ is open for all W, C. Note that $\{\mathcal{O}_{W,C}\}$ is a cover of $Gr_k(V)$. (For example, show that $W \in \mathcal{O}_{W,C}$.)

   (b) Use the open sets $\mathcal{O}_{W,C}$ to construct an atlas on $Gr_k(V)$. That is, check that the transition functions are smooth. (Hint: You may first want to check it for two charts with the same W but different complements. Then it suffices to check for two different W which are transverse, using the same complement for both.)

   (c) Now construct the complex Grassmannian: take V complex and use only complex subspaces.

   (d) Prove that $G = GL(V)$ acts smoothly and transitively on $Gr_k(V)$ in both the real and complex cases. What is the subgroup H which fixes $W \in Gr_k(V)$?

   (e) Is G/H a symmetric space? Can you express the Grassmannian as a symmetric space G'/H' for another Lie group G' and closed Lie subgroup $H' \subset G'$? Explicitly identify the complement $\mathfrak{m}$ in the Lie algebra.

   (f) Construct a short exact sequence of vector bundles

   $$0 \longrightarrow S \longrightarrow \underline{V} \longrightarrow Q \longrightarrow 0$$

   over $Gr_k(V)$ in which $\operatorname{rank} S = k$, $\operatorname{rank} Q = n - k$, and $\underline{V} = Gr_k(V) \times V$ is the bundle with constant fiber V.

   (g) Construct an isomorphism $TGr_k(V) \to \operatorname{Hom}(S, Q)$.

2. For each of the following pairs $H \subset G$ of Lie groups, explore the geometry of the symmetric space G/H. For example, identify $\mathfrak{m} \subset \mathfrak{g}$ such that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Identify the corresponding involutions on G and $\mathfrak{g}$. Construct the Riemannian metric. Identify the Riemann curvature tensor. Does G/H have constant curvature? Is it an Einstein manifold?

   (a) $SU_n \subset SU_n \times SU_n$ $(n \geq 2)$

   (b) $SO_n \subset SO_{n+1}$ $(n \geq 2)$

   (c) $SO_n \subset SO^0_{1,n}$ ($n \geq 2$; the superscript denotes the identity component)

   (d) $Sp_p \times Sp_q \subset Sp_{p+q}$ $(p, q \geq 1)$