Problem Set # 2

M392C: Riemannian Geometry

I have been posting notes and handouts on the website, so be sure to check often.

Problems

1. Prove the following lemma used in lecture. Let V be a real inner product space, $e_1, \dots, e_r$ linearly independent vectors which span a subspace W, set $g_{ij} = \langle e_i, e_j \rangle$, and let $v \in V$ be an arbitrary vector. Let $e^1, \dots, e^r$ be the dual basis of W^*, and $g^{ij} = \langle e^i, e^j \rangle$. Then the orthogonal projection of v onto W is $g^{i\lambda}\langle v, e_\lambda \rangle e_i$.

2. Let A be an affine space with the associated vector space V of displacements. Suppose $f \colon U \to \mathbb{R}$ is a smooth function defined on an open set $U \subset A$. Fix $p \in U$.

   (a) Recall that the differential $df_p \colon V \to \mathbb{R}$ is a linear functional defined as follows. Each $\xi \in V$ determines a constant vector field on A, and $df_p(\xi)$ is the directional derivative $(\xi f)(p)$.

   (b) Define the second differential, or Hessian, as a bilinear form $d^2 f_p \colon V \times V \to \mathbb{R}$ by setting its value on vectors $\xi_1, \xi_2 \in V$ to be the iterated directional derivative $\big(\xi_1(\xi_2 f)\big)(p)$. Prove that it is a symmetric bilinear form.

   (c) Choose affine coordinates $x^1, \dots, x^n$ on A with corresponding bases $\partial/\partial x^1, \dots, \partial/\partial x^n$ of V and $dx^1, \dots, dx^n$ for V^*. Express the second differential in this basis.

   (d) What can you say about higher derivatives?

   (e) Now suppose that ξ_1, ξ_2 are arbitrary vector fields on A, not necessarily constant. Compute the difference $\xi_1(\xi_2 f) - \xi_2(\xi_1 f)$.

3. On the last problem set I asked you to compute the transformation law for the Riemann tensor $R^i_{jk\ell}$ under an arbitrary change of coordinates. You should first compute the transformation law for Γ^i_{jk}. As a reminder the formulas are

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\lambda} \left\{ \frac{\partial g_{\lambda j}}{\partial x^k} + \frac{\partial g_{\lambda k}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\lambda} \right\}$$

and

$$R^i_{jk\ell} = \frac{\partial \Gamma^i_{j\ell}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^\ell} + \Gamma^\lambda_{j\ell}\Gamma^i_{\lambda k} - \Gamma^\lambda_{jk}\Gamma^i_{\lambda \ell},$$

4. Suppose V is a real inner product space of dimension n.

   (a) In the last homework set you were supposed to construct induced inner products on all tensor powers of V and V^*. Suppose $e_1, \dots, e_n$ is an orthonormal basis of V. Then the dual basis $e^1, \dots, e^n$ of V^* is orthonormal, as is the basis $\{e^{i_1} \wedge \cdots \wedge e^{i_r} : 1 \le i_1 < \cdots < i_r \le n\}$ of $\bigwedge^r V^*$. This defines the induced metrics if you check that the statements hold independent of the original choice of orthonormal basis.

   (b) Construct a notion of n-dimensional volume on V. That is, if $\{v_1, \dots, v_n\}$ is a set of n vectors, show you can compute the volume of the parallelopiped $\{t^i v_i : 0 \le t^i \le 1\}$ they span. Consider first the case $n = 1$. Then reduce to that case by looking at $\operatorname{Det} V = \bigwedge^n V$. By considering the induced metric on subspaces, show that there is a notion of lower dimensional volume as well.

   (c) Review my Notes on Lecture 2 (1/19). In particular, in (2.5) and (2.7) I define lines $\operatorname{Det} V$ and $\mathfrak{o}(V)$ associated to a finite dimensional real vector space V. Now define a third line

$$|\operatorname{Det} V| = \{\mu \colon \mathcal{B}(V) \to \mathbb{R} : \mu(b \circ g) = |\det(g)|^{-1}\mu(b) \text{ for all } b \in \mathcal{B}(V),\ g \in GL_n(\mathbb{R})\}$$

   and prove that there is a canonical isomorphism of it with the tensor product of the other two. Show that $|\operatorname{Det} V|$ has a canonical orientation. Interpret the volume constructed in (b) as a positive element $\mu \in |\operatorname{Det} V^*|$.

   (d) Review the notion of orientation. Briefly, an orientation is a choice of component of $\operatorname{Det} V \setminus \{0\}$. Show that an orientation of V determines a notion of *signed* n-dimensional volume. Equivalently, construct a distinguished nonzero element of $\operatorname{Det} V^* = \bigwedge^n V^*$, the *volume form*.

   (e) Now think through each part of this exercise on a Riemannian manifold X. In particular, a positive section of $|\operatorname{Det} T^*X| \to X$ is a *smooth measure* on X, and a Riemannian manifold has a canonical smooth measure.

5. (a) Let E be a 3-dimensional Euclidean space with Euclidean coordinates q^1, q^2, q^3 and consider an embedding $q^a = q^a(x^1, x^2)$ of an open set in the (x^1, x^2)-plane. We consider x^1, x^2 as local coordinates on the image surface in E. Compute the metric, Γ^i_{jk}, second fundamental form, and Gauss curvature in terms of the functions $q^a(x^1, x^2)$ and their derivatives.

   (b) What is the formula for Gauss curvature in the special case that the parametrization is orthogonal, i.e., $g_{12} = 0$? Simplify further by writing the formula in case $g_{11} = 1$ and $g_{12} = 0$.

   (c) Compute the Gauss curvature and principal curvatures for a torus, obtained by revolving the circle $(q^1 - R)^2 + (q^3)^2 = r^2$ about the q^3-axis, where $R > r > 0$.

   (d) Compute the Gauss curvature for a surface of revolution, obtained by revolving the graph of the positive function $q^1 = q^1(q^3)$ about the q^3-axis.

6. The *helicoid* is the image of the function

$$(u, v) \longmapsto (a \sinh v \cos u, a \sinh v \sin u, au)$$

   from the (u, v)-plane to Euclidean 3-space. Compute the mean curvature of the helicoid.

7. Prove that a connected surface in Euclidean 3-space all of whose points are umbilic (the second fundamental form is a multiple of the first fundamental form) is a subset of a sphere or a plane.