Problem Set # 3

M392C: Riemannian Geometry

I have been posting notes and handouts on the website, so be sure to check often.

Problems

1. The geodesic equation

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\big(x(t)\big)\frac{dx^j}{dt}\frac{dx^k}{dt} = 0$$

makes sense in a coordinate system on an arbitrary Riemannian manifold of any dimension.

   (a) Show that a parametrized curve $x^i = x^i(t)$ which satisfies this equation satisfies the corresponding equation in any coordinate system.

   (b) Show that there is a unique parametrized geodesic with given initial position and initial velocity.

   (c) How can you use geodesics to define local coordinates about any point p on a Riemannian manifold?

2. (a) Construct a vector field $\xi = \xi(x)\frac{\partial}{\partial x}$ on the line $\mathbb{A}^1$ which is not complete.

   (b) Show that any vector field on a compact manifold is complete. (Assume the local existence of integral curves.)

3. (a) Suppose M is a manifold and $\varphi\colon \mathbb{R} \times M \to M$ a smooth map such that $\varphi_t\colon M \to M$ is a diffeomorphism for each $t \in \mathbb{R}$. (As usual, $\varphi_t(x) = \varphi(t,x)$.) Differentiate in t to construct a time-varying vector field $\xi(t)$ on M. What condition on φ guarantees that $\xi(t)$ is static, that is, independent of t?

   (b) Conversely, given a time-varying vector field $\xi(t)$ construct a curve φ_t of diffeomorphisms. You should assume that for each t the vector field $\xi(t)$ is complete. (You need to assume uniformity in t as well; consider M compact as a special case.) In lecture I sketched how to construct integral curves for $\xi(t)$; you'll need to review that argument.

4. Let M be a Riemannian manifold with metric g.

   (a) Let ξ, η, ζ smooth vector fields. Prove the following formula for the Lie derivative of the metric:

$$(\mathcal{L}_\zeta g)(\xi,\eta) = \zeta \cdot g(\xi,\eta) - g\big([\zeta,\xi],\eta\big) - g\big(\xi,[\zeta,\eta]\big).$$

(b) The Lie derivative is a map

$$\mathcal{X}(M) \longrightarrow \Gamma(\mathrm{Sym}^2 TM \to M)$$

from vector fields on M to symmetric tensors on M. Write this map in a local coordinate system $x^1, \dots, x^n$.

5. Let $\Sigma \subset \mathbb{E}^3$ be the unit sphere centered at the origin. With respect to standard Euclidean coordinates x^1, x^2, x^3 let ξ_i be the vector field on Σ which is the orthogonal projection of the constant vector field $\partial/\partial x^i$ on $\mathbb{E}^3$ to Σ. Compute $[\xi_1, \xi_2]$.

6. (a) Let M be a 3-manifold and α a nonzero 1-form. Prove that the 2-dimensional distribution determined by α is integrable if and only if $\alpha \wedge d\alpha = 0$.

   (b) The Hopf fibration $\pi\colon S^3 \to S^2$ may be constructed by identifying S^3 as the unit sphere in $\mathbb{C}^2$ and S^2 as $\mathbb{CP}^1$; then the map is $\pi(z^1, z^2) = [z^1, z^2]$, where $(z^1)^2 + (z^2)^2 = 1$ and $[z^1, z^2]$ is the equivalence class in the projective line. The kernel of the differential π_* is an (integrable) one-dimensional distribution on S^3. Let $E \subset TS^3$ be the 2-dimensional distribution whose fiber at $p \in S^3$ is the orthogonal complement of $\ker \pi_*$ relative to the standard round metric. Is E integrable? Find a nonzero 1-form α which generates the ideal $\mathcal{I}(E)$ associated to E. Compute $\alpha \wedge d\alpha$.