Problem Set # 4

M392C: Riemannian Geometry

I have been posting notes and handouts on the website, so be sure to check often.

Problems

1. Let A be a $\mathbb{Z}$-graded algebra and T_1, T_2 derivations of degree s_1, s_2, respectively. Prove that the commutator $[T_1, T_2]$ is a derivation of degree $s_1 + s_2$. Use the Koszul sign rule throughout.

2. (a) Suppose A is an $n \times n$ real matrix. Define $e^A = \exp(A)$ using a power series. Prove carefully that the series does define a matrix.

   (b) Prove that $e^{A+B} = e^A e^B$ if A and B commute. In particular, show that e^A is invertible. What is the first correction to this formula if A and B do not commute?

   (c) Compute the derivative of e^{tA} with respect to the real variable t.

   (d) What can you say about $\det e^A$? What can you say about e^A if A is skew-symmetric?

3. Let ω be a $(k-1)$-form on a manifold M and $\xi_1, \dots, \xi_k$ vector fields on M. Compute $d\omega(\xi_1, \dots, \xi_k)$. (In lecture we covered the case $k = 2$.)

4. (a) State carefully what it means for a Lie group G to act on a manifold M on the left or on the right.

   (b) If G acts on M, then there is an induced linear map $\mathfrak{g} \to \mathcal{X}(M)$ from the Lie algebra of G to the linear space of vector fields on M. Show that for a right action the map $\mathfrak{g} \to \mathcal{X}(M)$ is a homomorphism of Lie algebras. For a right action it is an antihomomorphism: the bracket of the image is minus the image of the bracket.

5. (a) Let G be a Lie group. The left-invariant forms are closed under d, so form a subcomplex of the de Rham complex of G. As a vector space identify the left-invariant forms as the exterior algebra $\bigwedge^{\bullet}\mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G. Construct the de Rham differential d on $\bigwedge^{\bullet}\mathfrak{g}^*$ in terms of the Lie bracket.

   (b) Suppose G is a *compact* Lie group and α a bi-invariant form. (In other words, α is both left-invariant and right-invariant.) Prove that $d\alpha = 0$.

   (c) Compute the complex of left-invariant forms and bi-invariant forms for the circle group $\mathbb{T}$ (consisting of complex numbers of unit norm) and for the group SU_2. What happens for $SL_2(\mathbb{R})$, the group of 2×2 real matrices of determinant one?

6. Let G be a Lie group. A *torsor* for G is a smooth manifold T on which G acts simply transitively. Thus a *right G-torsor* is a manifold T with a right G action $T \times G \to T$ so that the map $T \times G \to T \times T$ defined by $(t, g) \mapsto (t, t \cdot g)$ is a diffeomorphism.

   (a) Let L be a real inner product space of dimension one. Prove that the elements of unit norm in L form a torsor for $\mathbb{Z}/2\mathbb{Z}$.

   (b) Let V be a real vector space and $\mathcal{B}(V)$ the space of all ordered bases of V. It is convenient to regard a basis of V as an invertible linear map $b\colon \mathbb{R}^n \to V$. Then the group $GL_n(\mathbb{R})$ of invertible linear maps $g\colon \mathbb{R}^n \to \mathbb{R}^n$ acts on the right by composition. Prove that $\mathcal{B}(V)$ is a right $GL_n(\mathbb{R})$-torsor.

   (c) Now endow V with an inner product and show that the space $\mathcal{O}(V)$ of orthonormal bases is a right O_n-torsor. What if we endow V with an orientation instead of a metric? What if we consider an oriented inner product space?

   (d) Let E be a Euclidean space and $\mathcal{O}(E)$ the space of all orthonormal frames at all possible points of E. Here an orthonormal frame is an isometry $f\colon \mathbb{E}^n \to E$ from the standard Euclidean space to E. Construct a right action of the Euclidean group Euc_n and show that $\mathcal{O}(E)$ is a right Euc_n-torsor.

   (e) Verify that the canonical left-invariant 1-form on a Lie group G is well-defined on a right G-torsor (but it is not right-invariant). Show that it satisfies the Maurer-Cartan equation.

7. Example or proof of nonexistence: A codimension 1 foliation on the sphere S^4.