Problem Set # 5

M392C: Riemannian Geometry

I have been posting notes and handouts on the website, so be sure to check often.

Problems

1. Let G be a smooth manifold whose underlying set is equipped with a group structure, and assume that multiplication $m\colon G \times G \to G$ is smooth. Prove that the inverse map $i\colon G \to G$ is also smooth, and hence G is a Lie group.

2. Let G be a Lie group. Recall that $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ is defined by differentiating conjugation. Namely, if for $g \in G$ we define $A_g\colon G \to G$ by $A_g(x) = gxg^{-1}$, then $\mathrm{Ad}_g = d(A_g)_e$.

   (a) Prove that Ad_g is an automorphism of the Lie algebra $\mathfrak{g}$, i.e., it preserves the Lie bracket:

   $$\mathrm{Ad}_g[\xi, \eta] = [\mathrm{Ad}_g \xi, \mathrm{Ad}_g \eta], \qquad \xi, \eta \in \mathfrak{g}.$$

   (b) Compute $d(\mathrm{Ad}_g)_e$ in terms of the Lie bracket.

3. In the last problem set you learned that the set of bases of a vector space V is a right torsor for $GL_n(\mathbb{R})$. Here are other geometric examples of torsors—verify that they are indeed torsors.

   (a) If M is an orientable manifold, then the set of orientations is a torsor for $H^0(M; \mathbb{Z}/2\mathbb{Z})$, the group of locally constant functions $M \to \mathbb{Z}/2\mathbb{Z}$.

   (b) (This requires that you know about spin structures.) If M is a spinable manifold (with a fixed orientation), then the set of equivalence classes of spin structures compatible with the given orientation is a torsor for $H^1(M; \mathbb{Z}/2\mathbb{Z})$.

   (c) If $\bar{a} \in \mathbb{R}/\mathbb{Z}$, then $\{x \in \mathbb{R} : x \equiv \bar{a} \pmod 1\}$ is a $\mathbb{Z}$-torsor.

   (d) The fiber of a regular covering space $\tilde{X} \to X$ is a torsor for the group of deck transformations.

   (e) An affine space A is a torsor for its underlying vector space of translations V.

4. Let G be a Lie group and θ the (left-invariant) Maurer-Cartan form.

   (a) Compute $m^*\theta$, where $m\colon G \times G \to G$ is the multiplication map.

   (b) Compute $i^*\theta$, where $i\colon G \to T$ is inversion.

5. (a) What is the space of left-invariant metrics on a Lie group G? What is the space of bi-invariant metrics?

   (b) Let SU_2 be the group of all 2×2 hermitian matrices with determinant one. (The entries are complex numbers.) Show that SU_2 is a Lie group. What is its dimension? Is it compact? Find all bi-invariant metrics on SU_2.

   (c) Let $SL_2(\mathbb{R})$ be the group of 2×2 real matrices with determinant one. Prove that $SL_2(\mathbb{R})$ is a Lie group. Find all bi-invariant metrics on SU_2.

6. Suppose N is a manifold and G a Lie group. Let $\theta^i,\ i = 1, \dots, n$ be a basis of left-invariant 1-forms on G and suppose

$$d\theta^i + \frac{1}{2} c^i_{jk} \theta^j \wedge \theta^k = 0$$

for constants c^i_{jk}. Let $\theta^i_N,\ i = 1, \dots, n$ be 1-forms on N. Consider the ideal of differential forms on $N \times G$ generated by $\pi_2^* \theta^i - \pi_1^* \theta^i_N$, where $\pi_1 \colon N \times G \to N$ and $\pi_2 \colon N \times G \to G$ are projections. Prove that this ideal is closed under d if and only if

$$d\theta^i_N + \frac{1}{2} c^i_{jk} \theta^j_N \wedge \theta^k_N = 0$$

7. Let E be a 3-dimensional Euclidean space and $\gamma \colon (a, b) \to E$ a smooth map such that $|\dot\gamma| = 1$. In other words, γ is the unit speed parametrization of a curve. Assume further that the acceleration $\ddot\gamma$ is nowhere zero. Finally, assume the normal bundle to (the image of) γ is oriented.

   (a) Construct a canonical lift $\tilde\gamma \colon (a, b) \to \mathcal{B}_O(E)$ to the orthonormal frame bundle of E. This is called the *Frenet* frame. In other words, construct a curve $\big(e_1(t), e_2(t), e_3(t)\big)$ of orthonormal frames of the vector space of translations of E. Make this construction so that e_1 is tangent to the curve, and e_2 is determined by the acceleration.

   (b) Compute the pullbacks of the Maurer-Cartan forms θ^i, Θ^i_j on $\mathcal{B}_O(E)$. You will meet two functions of t called *curvature* and *torsion*.

   (c) Given curvature and torsion functions on an interval (a, b), prove that there exists a curve in E, unique up to a Euclidean motion, with the given curvature and torsion. Are there any restrictions on the curvature and torsion?

   (d) Compute curvature and torsion for a plane curve. For a helix.

8. Let X be a Riemannian manifold and $\xi \in \mathcal{X}(X)$ a vector field on X. The goal is to define an operator

$$\nabla_\xi \colon \mathcal{X}(X) \longrightarrow \mathcal{X}(X)$$

which satisfies the following two properties for all $\eta, \zeta \in \mathcal{X}(X)$:

$$\begin{aligned} \xi\langle\eta, \zeta\rangle &= \langle\nabla_\xi \eta, \zeta\rangle + \langle\eta, \nabla_\xi \zeta\rangle \\ \nabla_\xi \eta - \nabla_\eta \xi &= [\xi, \eta]. \end{aligned}$$

(a) Use these properties to derive a formula for $\langle\nabla_\xi \eta, \zeta\rangle$.

(b) Prove that $\nabla_\xi \eta$ is linear over functions (tensorial) in ξ and satisfies a Leibniz rule in η.

(c) Let $x^1, \ldots, x^n$ be local coordinates. Compute $\nabla_{\partial/\partial x^j} \partial/\partial x^k$. Have you seen that formula before?