Problem Set # 6

M392C: Riemannian Geometry

I have been posting notes and handouts on the website, so be sure to check often.

Problems

1. Suppose G is a connected, simply connected Lie group, and H any Lie group. Let $\dot\varphi\colon \mathfrak{g} \to \mathfrak{h}$ be a homomorphism of the Lie algebra of G to the Lie algebra of H. Prove that there exists a unique homomorphism of Lie groups $\varphi\colon G \to H$ whose differential (at the identity or on left-invariant vector fields) is $\dot\varphi$.

2. Let G be a Lie group and M a manifold. Then a *principal G-bundle* over M is a manifold P with a free G-action and a smooth map $\pi\colon P \to M$ whose fibers are the G-orbits. Furthermore, π admits local sections; that is, about every point of M there is an open neighborhood U and a smooth map $s\colon U \to P$ such that $\pi \circ s$ is the identity map on U.
   (a) Show that a local section gives an isomorphism of $\pi^{-1}U$ with the trivial principal bundle $U \times G$. (The isomorphism should commute with the G-actions, hence also with the projections to U.)
   (b) Give an example of a principal bundle which is not isomorphic to a trivial bundle.
   (c) You have seen this notion before in case G is a countable discrete group. What is it?
   (d) Prove carefully that the orthonormal frame bundle $O(M) \to M$ of a Riemannian manifold is a principal O_n-bundle. You may want to use the Gram-Schmidt process to construct local sections.
   (e) Construct a principal $\mathbb{T}$-bundle $S^{2n+1} \to \mathbb{CP}^n$. (Here $\mathbb{T}$ is the circle group of unit norm elements in the complex line.)

3. Let $A = (A^i_j)$ be an $n \times n$ real matrix and define the vector field

$$\xi = A^i_j x^j \frac{\partial}{\partial x^i}$$

   on $\mathbb{R}^n$. This is called a *linear vector field* for obvious reasons.
   (a) Articulate those reasons.
   (b) What is the flow generated by ξ?
   (c) What can you say about the flow if $G \subset GL_n\mathbb{R}$ is a subgroup and the matrix A lies in its Lie algebra? Investigate $G = O_2 \subset GL_2\mathbb{R}$.

4. (a) Suppose the metric on a surface has the form $Edx^2 + Gdy^2$, where E, G are functions of x, y. (This means that the coordinate lines are orthogonal.) Compute a formula for the Gauss curvature.

   (b) Recall that for a surface immersed in a 3-dimensional Euclidean space, if we choose a local coframe θ^1, θ^2 and an orientation of the normal bundle, then we obtain 1-forms $\Theta_i^3 = h_{ij}\theta^j$ which give the second fundamental form of the surface. The Codazzi equation is the equation for $d\Theta_i^3$. Compute it for a local orthogonal parametrization of the surface. That is, if x, y, z are Euclidean coordinates on Euclidean space, parametrize the surface by writing x, y, z as functions of u, v and assume that the vector fields $\partial/\partial u$ and $\partial/\partial v$ are orthogonal.

   (c) How can you recognize umbilic points in terms of θ^i, Θ_j^i?

5. You may use moving frames, parametrizations, or coordinates to make computations in this problem.

   (a) Let E be a 3-dimensional Euclidean space and $c\colon (a, b) \to E$ a smooth curve parametrized by arc length. The surface M which is the union of the tangent lines to the image C of c is called the *tangent developable* of C. Find conditions on c so that M is a smooth manifold. Compute its Gauss curvature.

   (b) More generally, a *ruled surface* is a union of lines. One example is a Möbius strip. Find a parametrization of the Möbius strip. Compute the Gauss curvature. Another ruled surface is the *helicoid*, obtained by fixing perpendicular lines $\ell, \ell' \subset E$ and taking the surface swept out by ℓ' as it moves along ℓ, remaining perpendicular to ℓ and rotating at a constant speed as it moves. Find a parametrization and compute the Gauss curvature and mean curvature.

   (c) Parametrize a *surface of revolution*. Compute a formula for the Gauss curvature. What curves revolve to give surfaces of constant Gauss curvature? Analyze this question both locally and globally.

6. Let M be a Riemannian surface and $c\colon (a, b) \to M$ a smooth parametrized curve with oriented normal bundle. Construct a canonical orthonormal frame along c. Use the structure equations to define the *geodesic curvature* of the curve. Now suppose we have a parametrized *closed* curve $c\colon S^1 \to M$ which is the boundary of some region in M. Use the fundamental surface equations and Stokes' theorem to get an equation involving the geodesic curvature, the Gauss curvature, and perhaps other geometric quantities.