Problem Set # 7

M392C: Riemannian Geometry

Throughout G is a Lie group with Lie algebra $\mathfrak{g}$.

Problems

1. You've already seen this, but just in case ...

   (a) Let V be an n-dimensional real vector space and $\mathcal{B}(V)$ the right $GL_n(\mathbb{R})$-torsor of bases. Let Θ^i_j be the Maurer-Cartan forms in the standard basis of the Lie algebra of $GL_n(\mathbb{R})$. Suppose $b(t)$ is a smooth curve in $\mathcal{B}(V)$. Write the basis $b(t)$ as $\{e_1(t), \dots, e_n(t)\}$ and the dual basis as $\{e^1(t), \dots, e^n(t)\}$. Prove that

   $$\Theta^i_j(\dot{b}) = \langle e^i(0), \dot{e}_j(0)\rangle.$$

   The pairing is duality $\langle -, -\rangle\colon V^* \times V \to \mathbb{R}$.

   (b) Let A be an n-dimensional real affine space and $\mathcal{B}(A)$ the right $\mathrm{Aff}_n(\mathbb{R})$-torsor of bases of the underlying vector space at all points of A. Let θ^i, Θ^i_j be the Maurer-Cartan forms in the standard basis of the Lie algebra of $\mathrm{Aff}_n(\mathbb{R})$. (Define this!) Suppose $b(t)$ is a smooth curve in $\mathcal{B}(A)$ which projects to the curve $x(t)$ in A, and write the underlying basis of V as in (a). Prove that

   $$\theta^i(\dot{b}) = \langle e^i(0), \dot{x}(0)\rangle.$$

2. Let G be a Lie group.

   (a) Show that the automorphism group $\mathrm{Aut}(T)$ of a G-torsor is the Lie group associated to T (mixing construction) by the conjugation action of G on itself. Why is that associated space a Lie group?

   (b) Show that the Lie algebra of the group in (a) is the Lie algebra associated to T by the adjoint action $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$. Why is that associated space a Lie algebra?

   (c) Suppose $\pi\colon P \to M$ is a principal G-bundle. Construct associated bundles $G_P \to M$, $\mathfrak{g}_P \to M$ of Lie groups and Lie algebras. The latter is the *adjoint bundle*. Show that the group $\mathrm{Aut}_0 P$ of *gauge transformations*—automorphisms of $P \to M$ which cover id_M—is the group of sections of $G_P \to M$ and its Lie algebra is $\Omega^0_M(\mathfrak{g}_P)$, the group of sections of the adjoint bundle.

3. Let $\pi\colon P \to M$ be a principal G-bundle and $\Theta_0, \Theta_1 \in \Omega^1_P(\mathfrak{g})$ connections. The difference $\Theta_1 - \Theta_0$ lives in a vector subspace of $\Theta^1_P(\mathfrak{g})$. Which subspace? Identify the space of connection forms as an affine subspace of $\Theta^1_P(\mathfrak{g})$ over that vector space. Write the affine space of connections as the space of sections of a fiber bundle of affine spaces over X? Use that to prove the existence of connections.

4. Let $\pi\colon P \to M$ be a principal G-bundle and $\rho\colon G' \to G$ a homomorphism of Lie groups. A *reduction of P to G'* is a pair consisting of a principal G'-bundle $Q \to M$ and a map $\varphi\colon Q \to P$ which covers id_M and intertwines ρ: $\varphi(qg') = \varphi(q)\rho(g')$ for all $q \in Q$, $g' \in G'$.
   (a) Prove that if ρ is injective, then a reduction is equivalent to a section of the associated bundle with fiber G/G'. (First define this associated bundle.) What does this say if $G' = \{e\}$?
   (b) Use covering space theory to analyze the reduction problem when G', G are discrete groups. When is a lift defined? Can you determine an obstruction which measures whether a lift exist? What are the equivalence classes of lifts? You may want to try the case when ρ is injective with abelian cokernel, or the case when ρ is surjective with abelian kernel.

5. (a) If G is a Lie group and $H \subset G$ a closed subgroup, verify that $\pi\colon G \to G/H$ is a principal H-bundle. (You can consult Warner's book for some details.) Prove that the tangent bundle of G/H is the associated bundle via the adjoint representation $\mathrm{Ad}\colon H \to \mathrm{Aut}(\mathfrak{g}/\mathfrak{h})$ on the quotient of Lie algebras.
   (b) Show that there is a left G-action by automorphisms on the principal bundle in (a). This is called a *homogeneous* bundle.
   (c) Write the Hopf bundle, which is a principal $\mathbb{T}$-bundle over S^2, as a homogeneous bundle. What is the total space?
   (d) Construct a homogeneous SU_2-bundle over S^4. Construct a homogeneous SO_4-bundle over S^4. What is the total space of each? Can you find a 3-sphere bundle over S^4 whose total space is the 7-sphere?

6. Let $\pi\colon P \to M$ be a principal G-bundle.
   (a) Suppose $\alpha \in \Omega^1_M$. Show that $\beta = \pi^*\alpha$ satisfies $\iota_\eta\beta = 0$ for all vertical vectors β and $R_g^*\beta = \beta$ for all $g \in G$. Conversely, show that if a 1-form $\beta \in \Omega^1_P$ satisfies these two properties, then $\beta = \pi^*\alpha$ for a unique $\alpha \in \Omega^1_M$.
   (b) Repeat the exercise for a q-form for arbitrary $q \geq 0$. Is the same assertion true?

7. Fix $n \in \mathbb{Z}^{\geq 1}$. Let $\mathbb{R}^{n,1}$ be the vector space $\mathbb{R}^{n+1}$ with inner product

$$\langle(\xi^0, \dots, \xi^n), \langle\eta^0, \dots, \eta^n\rangle\rangle = -\xi^0\eta^0 + \xi^1\eta^1 + \cdots + \xi^n\eta^n.$$

   Consider the quadric $Q \subset \mathbb{R}^{n,1}$ of vectors of square norm 1. Prove that Q has two components. Let Q^+ be the component where $\xi^0 > 0$. Show that the inner product induces a positive definite metric on Q^+. Show that the subgroup of $O_{n,1}$ which preserves $\operatorname{sign}\xi^0$ has two components and contains the identity component. Show that the orthonormal frame bundle $\mathcal{B}_O(Q^+)$ is a right torsor over that subgroup. Compute that Q^+ has constant curvature -1. (Hint: Think about the structure equations of $O_{n,1}$.)

8. Let X be a smooth manifold and $E \subset TX$ a distribution of rank k. Let $\mathcal{B}_E(X) \subset \mathcal{B}(X)$ be the subbundle of frames for which the first k vectors form a basis of E. Prove that $\mathcal{B}(X) \to X$ admits a torsionfree connection if and only if E is involutive.