Problem Set # 8

M392C: Riemannian Geometry

1. You may use covariant derivatives or moving frames, whichever you find more convenient.

   (a) Suppose (M, g) and (M', g') are Riemannian manifolds. Define a Riemannian manifold $(M \times M', g \times g')$ and compute its curvature in terms of the curvatures of (M, g) and (M', g').

   (b) Continuing, let $f\colon M \to \mathbb{R}$ be a positive smooth function. Define the *warped product* $(M \times M', g \times f \cdot g')$ and compute its curvature. You have already seen such metrics when M and M' are each 1-dimensional.

   (c) Let (M, g) be a Riemannian manifold and $\varphi\colon M \to \mathbb{R}$ any smooth function. Compute the curvature of $(M, e^{2\varphi} g)$. (The metric $e^{2\varphi} g$ is *conformally related* to the metric g.)

2. Let G be any Lie group.

   (a) Use left translation to trivialize TG. This defines a global parallelism on G, so by differentiation a (left-invariant) covariant derivative on TG. What is the curvature and torsion of this covariant derivative?

   (b) Repeat with right translation replacing left translation.

   (c) Recall that the Leibniz rule which defines a covariant derivative is an affine equation, so the average of two connections is a connection. Compute the curvature and torsion of the average of the connections in parts (a) and (b).

   (d) You now have three connections on G. Are any of these Levi-Civita connections for a metric on G?

3. Is it possible for a geodesic to intersect itself? Example or counter-proof.

4. Let $X \subset E$ be a submanifold of a Euclidean space. The dimensions of X and E are not fixed.

   (a) Use the global parallelism of E to induce a parallelism—a covariant derivative—on X. So if $\xi \in T_xX$ is a tangent vector at some point $x \in X$ and η a vector field on X defined in a neighborhood of X, use the natural covariant derivative on E to define the covariant derivative $\nabla_\xi \eta$ on X.

   (b) Prove that ∇ preserves the induced Riemannian metric on X.

   (c) Consider the example of a unit 2-sphere X in a 3-dimensional Euclidean space. Let C be the circle obtained by intersecting X with a plane whose distance from the nearest parallel tangent plane is $d < 1$. The holonomy of the parallel transport around C is rotation through some angle θ. Compute θ as a function of d. Make clear your orientations.

5. Let X be a smooth manifold and $\pi\colon E \to X$ a vector bundle equipped with a covariant derivative ∇.
   (a) Interpret ∇ as a linear map $\Omega^0_X(E) \to \Omega^1_X(E)$.
   (b) Use the Leibniz rule to extend to a sequence of linear maps

$$0 \longrightarrow \Omega^0_X(E) \xrightarrow{\nabla} \Omega^1_X(E) \xrightarrow{d_\nabla} \Omega^2_X(E) \xrightarrow{d_\nabla} \Omega^3_X(E) \longrightarrow \cdots$$

   This should reduce to the de Rham complex in case E is the trivial line bundle with trivial covariant derivative.
   (c) Compute d_∇^2.
   (d) Compute d_∇^3.

6. Let X be a smooth n-manifold equipped with a connection on its frame bundle $\mathcal{B}(X) \to X$. (Recall that this is a principal $GL_n\mathbb{R}$-bundle.) Let ∂_k, $k = 1, \dots, n$, be the canonical horizontal vector fields on $\mathcal{B}(X)$. Set $\partial = \partial_k e^k$ to be the horizontal $(\mathbb{R}^n)^*$-valued vector field.
   (a) A differential form on X lifts to a function

$$\omega\colon \mathcal{B}(X) \longrightarrow {\textstyle\bigwedge}^\bullet(\mathbb{R}^n)^*$$

   Compute $R_g^*\omega$, $g \in GL_n\mathbb{R}$.
   (b) Compute $(R_g)_*(\partial)$.
   (c) Let $\epsilon(\alpha)\colon \bigwedge^\bullet(\mathbb{R}^n)^* \to \bigwedge^{\bullet+1}(\mathbb{R}^n)^*$ be exterior multiplication by $\alpha \in (\mathbb{R}^n)^*$. Compute

$$\epsilon(\partial)\omega = \epsilon(e^k)\partial_k\omega.$$

   How does it transform under R_g? How does it compare to $d\omega$? In other words, compare the operators $\epsilon(\partial)$ and d. (You may want to consider functions and 1-forms first. A q-form is a sum of products of these locally. For a 1-form, lift from X to a 1-form on $\mathcal{B}(X)$, which we want to write as a function $\omega_\ell e^\ell$. How can you compute ω_ℓ from the lifted 1-form and the vector field ∂_ℓ?)
   (d) Rewrite this exercise in terms of the covariant derivative

$$\nabla\colon \Omega^q(X) \longrightarrow \Gamma(X; {\textstyle\bigwedge}^q T^*X \otimes T^*X),$$

   where the codomain is the vector space of smooth sections of the indicated vector bundle over X.
   (e) Did this exercise shed light on why we might want a torsionfree connection?