Problem Set # 9

M392C: Riemannian Geometry

1. Let X be a Riemannian manifold of dimension ≥ 2, $x \in X$, and $\pi \subset X$ a 2-dimensional subspace. Choose $\epsilon > 0$ so that the geodesic $\gamma\colon (-\epsilon, \epsilon) \to X$ with initial position x and initial velocity a unit vector $\xi \in T_xX$ exists for all ξ. Show that the union of those geodesics is a smooth 2-dimensional submanifold $\Sigma_\pi \subset X$. It inherits a Riemannian metric from X. Prove that the Gauss curvature of Σ_π at x is the sectional curvature $K_X(\pi)$ of X at x evaluated on the 2-plane π.

2. Fix a positive integer m and let $S \subset \mathbb{C}^{m+1}$ be the unit sphere with respect to the standard Hermitian metric. It inherits a Riemannian metric.

   (a) Verify that the group U_{m+1} of unitary transformations of $\mathbb{C}^{m+1}$ acts by isometries on S. In particular, the diagonal subgroup $\mathbb{T}$ acts freely by isometries. Verify that the quotient is the complex projective space $\mathbb{CP}^n$ and the quotient map $\pi\colon S \to \mathbb{CP}^n$ is a principal $\mathbb{T}$-bundle.

   (b) Let $H \subset TS$ be the orthogonal complement to the tangents to the orbits of the $\mathbb{T}$-action. Verify that H is a connection on π.

   (c) Compute the curvature of π.

   (d) The vector bundle $H \to S$ inherits an inner product from the Riemannian metric on S. Verify that it is $\mathbb{T}$-invariant. Induce a Riemannian metric on $\mathbb{CP}^n$.

   (e) The tangent bundle $T\mathbb{CP}^n \to \mathbb{CP}^n$ has a complex structure $I \in \mathrm{End}(T\mathbb{CP}^n)$, i.e., and endomorphism with $I^2 = -\mathrm{id}_{T\mathbb{CP}^n}$. What is the compatibility between the Riemannian metric and I?

   (f) Compute the curvature of the Riemannian metric on $\mathbb{CP}^n$. What compatibilities can you detect with I?

   (g) Specialize to $n = 1$. Identify $\mathbb{CP}^1$ with the 2-sphere S^2. What Riemannian metric is obtained? What is the integral of the curvature over $\mathbb{CP}^1$?

3. Let X be a smooth manifold, ξ, η vector fields on X, and $x \in X$. For sufficiently small $\epsilon > 0$ define $\gamma\colon (-\epsilon, \epsilon) \to X$ as follows. To compute $\gamma(t)$, start at x; follow the integral curve of ξ for time $\sqrt{t}$; then follow the integral curve of η for time $\sqrt{t}$; then follow the integral curve of $-\xi$ for time $\sqrt{t}$; then follow the integral curve of $-\eta$ for time $\sqrt{t}$. Prove that $\gamma'(0)$ is the Lie bracket $[\xi, \eta]$ at x.

4. (a) We saw in lecture that the canonical vector fields on the orthonormal frame bundle $\mathcal{B}_O(X)$ of a constant curvature K Riemannian n-manifold form a $n(n+1)/2$-dimensional Lie algebra under Lie bracket. Identify that Lie algebra explicitly in terms of K.

(b) Let $x^0, x^1, \dots, x^n$ be standard coordinates in affine space $\mathbb{A}^{n+1}$, and define X_K, $K \neq 0$, to be the component of the space of solutions to

$$\sum_{i=1}^{n} (x^i)^2 = \operatorname{sign} K \left[\left(\frac{1}{K}\right)^2 - \left(x^0 - \frac{1}{K}\right)^2 \right].$$

which contains the origin. Show that $\lim_{K \to 0} X_K$ of these quadrics is the affine subspace $X_0 = \{x^0 = 0\}$. Be sure to picture this family of hypersurfaces, perhaps on the computer, for $n = 2$. Let $\mathbb{A}^{n+1}$ have the translationally invariant indefinite possibly degenerate metric

$$(\operatorname{sign} K)(dx^0)^2 + (dx^1)^2 + \cdots + (dx^n)^2.$$

Prove that the metric inherited by X_K has constant sectional curvature K.

5. Let $\pi \colon P \to X$ be a principal G-bundle with connection Θ and curvature Ω.
   (a) Differentiate the structure equation $\Omega = d\Theta + \frac{1}{2}[\Theta \wedge \Theta]$. Interpret the result as an equation for the covariant derivative of Ω.
   (b) Specialize to the orthonormal frame bundle of a Riemannian manifold and its Levi-Civita connection. What do you learn about the Riemann curvature tensor?
   (c) Write the result relative to a local orthonormal framing about $x \in X$ assuming that the covariant derivative of the frame vanishes at x. Equivalently, this is a section $s \colon U \to \mathcal{B}_O(X)\big|_U$ over an open neighborhood U of x such that $s_*(T_xX)$ is the horizontal subspace at $s(x)$.

6. In class I mentioned a theorem of Yamabe et al that a pathwise connected subgroup of a Lie group is a Lie subgroup. Construct a connected but not pathwise connected subgroup $H \subset (\mathbb{R}/\mathbb{Z})^{\times 2}$. Show that H is not a Lie subgroup.

7. Let G be a Lie group and $H \subset G$ a closed Lie subgroup. Suppose $\pi \colon P \to X$ is a principal G-bundle.
   (a) Prove that reductions of π to H are in 1:1 correspondence with sections of $P/H \to X$.
   (b) Suppose that Θ is a connection on π. Prove that reductions of π to H such that Θ is induced from a connection on the reduction are in 1:1 correspondence with *flat* sections of $P/H \to X$.