NOTES ON LECTURE 10 (RIEMANNIAN GEOMETRY)

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Contents

Some brief comments on signs and another issue from today's lecture.

1. On the definition of the Lie derivative

Let X be a smooth manifold and ξ a vector field on X. Suppose ξ generates a flow φ_t. (In general there is only a local flow, which is fine since my comments are local.) Suppose we want to Lie differentiate some tensor field T, which you can think is either a function f or a vector field η. We want to define the Lie derivative $\mathcal{L}_\xi T$ evaluated at a point $p \in X$. Ostensibly there are two choices: (i) we can stand at p and transport T by the flow and differentiate, or (ii) we can move ourselves along the flow and sample the fixed T as we pass p to measure the rate of change. This is an active vs. passive dichotomy. While in general I am in favor of being active, in this case we choose to be passive. The criterion is to be compatible with the directional derivative $\xi_p f$. In that case we take a parametrized curve $\gamma\colon (-\epsilon, \epsilon) \to X$ such that $\gamma(0) = p$, $\dot\gamma(0) = \xi_p$, and differentiate $t \mapsto f\big(\gamma(t)\big)$ at $t = 0$. So you see we sample the fixed function f as we move along the curve $\gamma(t)$. Therefore, when we have a flow we should do the same. The only issue is that we have to compare tensor fields at different points, and so we use the flow to move the value of T at $\varphi_t(p)$ to the point p: the result is $\big((\varphi_{-t})_*T\big)(p) = (\varphi_{-t})_*(T_{\varphi_t(p)})$. That is what we differentiate in the definition.

2. On the Lie algebra of a Lie group

Let G be a Lie group. I defined the Lie algebra $\mathfrak{g}$ as the space of left-invariant vector fields. Evaluation at the identity gives an isomorphism $\mathfrak{g} \to T_eG$, but usually it is best to think of left-invariant vector fields. (A notable exception is for matrix groups, that is for a group embedded in $GL_N\mathbb{R}$ for some N.) Thus the proof I gave in class of the following was a bit muddled.

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Theorem 2.1. *Let $\theta \in \Omega^1_G(\mathfrak{g})$ be the Maurer-Cartan form. Then for any $g \in G$ we have*

$$R_g^*\theta = \operatorname{Ad}_{g^{-1}}\theta, \tag{2.2}$$

where $R_g\colon G \to G$ is right multiplication by g.

Proof. Evaluate both sides of (2.2) on a vector $\xi_x \in T_xG$. Since $\theta_x(\xi_x)$ is the left-invariant extension ξ of ξ_p, we are reduced to proving that

$$(R_g)_*\xi = \operatorname{Ad}_{g^{-1}}\xi \tag{2.3}$$

for any left-invariant vector field ξ. Since left and right multiplication commute, the left hand side of (2.3) is $(L_{g^{-1}})_*(R_g)_*\xi = (A_{g^{-1}})_*\xi$, where $A_g(x) = gxg^{-1}$ is conjugation. The differential of conjugation is the adjoint action, which implies (2.3). □

3. Interpretation of Θ^i_j

First, consider the group $GL_n\mathbb{R}$ with Lie algebra identified with the vector space $M_n\mathbb{R}$ of $n \times n$ matrices, the tangent space to $GL_n\mathbb{R}$ at the identity matrix. The Maurer-Cartan form is the matrix of 1-forms (Θ^i_j). Suppose $A \in M_n\mathbb{R}$; then $\Theta^i_j(A) = A^i_j$ is the (i,j)-entry of the matrix A. The parametrized curve $t \mapsto I + tA$ consists of invertible matrices for t small and has tangent A at $t = 0$. Applied to the basis vector $e_j \in \mathbb{R}^n$ we obtain the parametrized curve of vectors $t \mapsto e_j + tA^i_je_i$. Differentiating at $t = 0$ we obtain $A^i_je_i$. So for fixed i,j, we see that $\Theta^i_j(A)$ is the infinitesimal rate at which e_j is "turning" towards e_i. (I may have had the indices backwards in lecture.)

I find it more geometric to consider the right $GL_n\mathbb{R}$-torsor of bases $\mathcal{B}(V)$ of an n-dimensional vector space V. We transport Θ^i_j to $\mathcal{B}(V)$, as in lecture, and then need to interpret it on a parametrized curve $\{e_i(t)\}$ of bases of V. Write the curve as bg_t, where g_t is a curve in $GL_n\mathbb{R}$. Then you see the interpretation is the same: Θ^i_j is the infinitesimal rate at which e_j is turning towards e_i.

If we restrict to the orthogonal group, then 'turning' is more apt, since orthogonal transformations do turn: the Lie algebra consists of skew-symmetric matrices, and if we consider the matrix A whose entries are zero except for $A^i_j = 1$ and $A^j_i = -1$, then e^{tA} is a curve of rotations in the (i,j)-plane which rotates e_j towards e_i. (Compute the exponential!) That interpretation applies both on the group O_n and on the right torsor of orthonormal bases of an n-dimensional vector space, as well as on the Euclidean group and its right torsor of orthonormal frames on a Euclidean n-dimensional space.