NOTES ON LECTURE 13 (RIEMANNIAN GEOMETRY)

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Contents

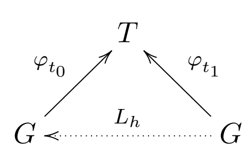
I thought I should write down, at least in telegraphic form, some of the basic definitions and results about principal bundles, etc., so that you have a text to refer to. These are in part adapted from old notes...

1. Principal bundles

Connections on principal bundles and their associated fiber bundles are a basic structure in differential geometry. Geometric structures on a manifold are encoded in a reduction of the frame bundle, and basic features of the structure are computed in terms of a connection, though these notes stop short of defining connections: see your notes from the class lectures.

(1.1) *Torsors and associated spaces.* Let G be a Lie group. Recall that a right G-torsor T is a manifold with a simply transitive right action of G on T. Thus for any $t_0 \in T$ we have a diffeomorphism $\varphi_{t_0}\colon G \to T$ defined by $\varphi_{t_0}(g) = t_0 g$. If $t_1 = t_0 h \in T$ is any other point, then we have the diagram of trivializations

(1.2)



and the change of trivialization map $\varphi_{t_0}^{-1} \circ \varphi_{t_1} : G \to G$ is the left translation L_h. In other words, a right G-torsor T is identified with G (as a right G-torsor) up to a left translation. Thus any left-invariant "notions" on G are defined on a right G-torsor. For example, a left invariant vector field on G determines a vector field on T: it is the infinitesimal G-action. So every tangent space to T is canonically identified with the Lie algebra $\mathfrak{g}$. Dually, there is a canonical 1-form $\theta \in \Omega^1_T(\mathfrak{t})$ induced from the Maurer-Cartan form, and it satisfies the equation $d\theta + \frac{1}{2}[\theta \wedge \theta] = 0$.

Now let F be a manifold with a *left* G-action. Then we can form the *mixing construction* or *associated space*

$$F_T = T \times_G F = (T \times F)/G, \tag{1.3}$$

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where the G-action is the equivalence relation

$$[tg, f] = [t, gf], \qquad t \in T,\ f \in F,\ g \in G. \tag{1.4}$$

Each $t_0 \in T$ gives a diffeomorphism $\psi_{t_0}\colon F \to F_T$ which is defined by $\psi_{t_0}(f) = [t_0, f]$. If $t_1 = t_0 h \in T$, then $\psi_{t_0}^{-1} \circ \psi_{t_1}\colon F \to F$ is the action of h. Thus we have identifications of F_T with F up to the action of G.

Example 1.5. Let V be a real n-dimensional vector space and $\mathcal{B}(V) = \{b\colon \mathbb{R}^n \to V\}$ the right $GL_n(\mathbb{R})$-torsor of bases. Then $GL_n(\mathbb{R})$ acts on $\mathbb{R}^n$ and the associated space is canonically V by the map $[b, \xi] \mapsto b(\xi)$ for $\xi \in \mathbb{R}^n$. Similarly, $GL_n(\mathbb{R})$ acts on the Grassmannian $Gr_k(\mathbb{R}^n)$ of subspaces of dimension k in $\mathbb{R}^n$ with associated space the Grassmannian $Gr_k(V)$. It also acts on the space of metrics on $\mathbb{R}^n$ with fixed signature, and the associated space is the corresponding space of metrics on V; it acts on all tensor spaces built from $\mathbb{R}^n$ with associated spaces of tensors on V, etc.

A positive definite metric on V may be specified by a sub O_n-torsor $\mathcal{B}_O(V) \subset \mathcal{B}(V)$ of orthonormal frames. The space associated to the action of O_n on the unit sphere $S^{n-1}(\mathbb{R}^n)$ is the unit sphere in V.

Heuristically, a G-torsor may be regarded as a space of abstract bases, or "internal states". Working with torsors in this way is democratic: we make no choice of distinguished basis unless it is part of the geometry.

Notice in this example that $\mathbb{R}^n$ is a vector space and the $GL_n(\mathbb{R})$-action is by vector space automorphisms. Therefore, the associated space is a vector space. Quite generally, if F has some structure (vector space, algebra, Lie algebra, group, etc.) preserved by the G-action, then F_T inherits that structure.

Suppose $\rho\colon G \to G'$ is a homomorphism of Lie groups and T a right G-torsor. Then ρ defines a left action of G on G' by multiplication. This preserves the structure of G' as a right G'-torsor, so the associated space $T \times_G G'$ is a right G'-torsor. This motivates the following definition.

Definition 1.6.

(i) Let $\rho\colon G \to G'$ be a homomorphism of Lie groups and T' a G'-torsor. Then a *reduction* of T' to G is a pair (T, θ) consisting of a G-torsor T and an isomorphism $\theta\colon T \times_G G' \to T'$ of G'-torsors.

(ii) If V is a real vector space of dimension n and $\rho\colon G \to GL_n(\mathbb{R})$ a homomorphism, then a *G-structure* on V is a reduction of $\mathcal{B}(V)$ to G.

Equivalently, θ is a map which intertwines ρ. The notion of a G-structure on a vector space formalizes Felix Klein's *Erlangen program.*

(1.7) *Principal bundles and associated fiber bundles.* A principal G-bundle over a space is a locally trivial family of G-torsors. Any left G-space F then induces, by the mixing construction, a fiber bundle with structure group G in the sense of Steenrod. We spell this out in the smooth context.

Definition 1.8. Let M be a smooth manifold and G a Lie group. A *principal G-bundle over M* is a smooth map $\pi\colon P \to M$, where the manifold P is equipped with a free right G-action, π is a quotient map for the G-action, and π admits local sections: about each point $m \in M$ is an open neighborhood $U \subset M$ and a smooth section $s\colon U \to P$ of π.

The freeness of the action means the fibers of π are right G-torsors. Part of the definition is that the set of equivalence classes of the G-action on P is the smooth manifold M. (For noncompact G this set need not be a manifold in general, e.g. for the irrational action of $\mathbb{R}$ on the 2-torus.) A local section s induces a local trivialization analogous to (**??**):

$$\begin{array}{ccc} U \times G & \xrightarrow{\varphi_s} & P\big|_U \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & U & \end{array} \tag{1.9}$$

Here π_1 is projection onto the first factor and the diagram commutes: φ_s is an isomorphism of G-torsors point by point on M.

Definition 1.10. Let $\pi\colon P \to M$ be a principal G-bundle and F a smooth manifold with a left G-action. Then the *associated fiber bundle* $\pi\colon F_P \to M$ is the quotient $F_P = P \times_G F = (P \times F)/G$ defined in (**??**).

A local section of $\pi\colon P \to M$ induces a local trivialization $\psi_s\colon U \times F \to F_P\big|_U$ by the formula $\psi_s(m, f) = [s(m), f]$. We make the important observation that a section f of $\pi\colon F_P \to M$ is equivalently a G-equivariant map $\tilde{f}\colon P \to F$; the equivariance is

$$\tilde{f}(pg) = g^{-1}\tilde{f}(p), \qquad p \in P,\ g \in G. \tag{1.11}$$

The most important first example of a principal bundle is the bundle of frames $\pi\colon \mathcal{B}(M) \to M$, which is a parametrized version of Example **??**. It encodes the *intrinsic* geometry of a manifold M. The tangent bundle and all tensor bundles are associated to linear representations of $GL_n(\mathbb{R})$, assuming that M has a fixed dimension n. For example, a vector field on M is a smooth map $\xi\colon \mathcal{B}(M) \to \mathbb{R}^n$ such that for all $g \in GL_n(\mathbb{R})$ we have $\xi(bg) = g^{-1}\xi(b)$. That is, we can specify a vector field as a vector-valued function on the collection of all bases.

Definition **??** has a straightforward parametrized generalization to the notion of reduction of structure group for principal bundles. In particular, we have the notion of a G-structure on a manifold as a reduction of the frame bundle. Any geometry associated to a G-structure may be considered *intrinsic*.

Example 1.12. Let G be a Lie group and $H \subset G$ a closed subgroup. Then $\pi\colon G \to G/H$ is a principal H-bundle over the homogeneous space G/H. The tangent bundle $T(G/H)$ is associated to the linear representation of H on the quotient $\mathfrak{g}/\mathfrak{h}$ of the Lie algebras. More precisely, a frame $\mathbb{R}^N \to \mathfrak{g}/\mathfrak{h}$ at the basepoint induces an H-structure on G/H. Typically H is much "smaller" than $GL_N(\mathbb{R})$, where $N = \dim G/H$. For example, the sphere S^{4n} can be presented as a homogeneous space in at least three different ways: $S^{4n} \simeq O_{4n+1}/O_{4n} \simeq U_{2n+1}/U_{2n} \simeq Sp_{n+1}/Sp_n$. For $n = 1$ the frame bundle of S^4 has 16-dimensional structure group $GL_4(\mathbb{R})$, and the reductions to O_4, U_2, and Sp_1 have dimensions 6, 4, 3, respectively.

Example 1.13. Let V be a real vector space, possibly infinite dimensional. (We can take V complex by changing $\mathbb{R} \to \mathbb{C}$ in what follows.) For $k \leqslant \dim V$ define the *Stiefel manifold*

$$St_k(V) = \{b\colon \mathbb{R}^k \to V : b \text{ is injective}\}. \tag{1.14}$$

It has a free right action of $GL_k(\mathbb{R})$ whose quotient $Gr_k(V)$ is a manifold, the *Grassmannian* of k-dimensional subspaces of V. For $k = 1$ we obtain the projective space $\mathbb{P}(V)$ of lines in V. The *tautological vector bundle* of rank k is associated to the standard representation of $GL_k(\mathbb{R})$ on $\mathbb{R}^k$; the fiber of the associated bundle at a k-plane $W \in Gr_k(V)$ is canonically identified with W. The tangent bundle to $Gr_k(V)$ is *not* associated to the Stiefel bundle $St_k(V) \to Gr_k(V)$.