NOTES ON LECTURE 15 (RIEMANNIAN GEOMETRY)

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Today in class I was asked about problem 2(c) on Problem Set #7, which asks about the *infinite dimensional* group of gauge transformations of a principal bundle and its Lie algebra. I responded that one can treat this as an infinite dimensional Lie group and that there is a choice. If we use smooth gauge transformations, then the manifold underlying the Lie group is modeled on a Fréchet space, whereas if one uses Sobolev completions then one can use a Hilbert manifold, which is technically simpler. There is another possibility, keeping with smooth gauge transformations, and I will outline a bit about it here. It sometimes goes by the name *diffeology*, but I will present it in the more standard context of presheaves, following ideas of Grothendieck.

1. Presheaves on Man

Let **Man** denote the category whose objects are smooth finite dimensional manifolds and whose morphisms are smooth maps between manifolds.

Definition 1.1. A *presheaf on manifolds* is a functor $\mathbf{Man}^{op} \to \mathbf{Set}$.

In this context we view M as a "test manifold" on which we evaluate the presheaf. The presheaf itself is to be considered as a new geometric object which generalizes a manifold. To justify that point of view we must first see that manifolds may be regarded as presheaves. Let X be a smooth finite dimensional manifold, and define the associated presheaf $\mathcal{F}_X$

$$\begin{aligned} \mathcal{F}_X \colon \mathbf{Man}^{op} &\longrightarrow \mathbf{Set} \\ M &\longmapsto \mathbf{Man}(M, X) \end{aligned} \tag{1.2}$$

To a test manifold M this presheaf assigns the set of all smooth maps $M \to X$, the set of maps from M to X in the category **Man**. Throughout we use standard constructions and notations in categories, for example '$C(X,Y)$' for the set of morphisms $X \to Y$ in the category C.

Date: March 7, 2017.

Remark 1.3. The notion of a presheaf is more familiar over a fixed manifold X. A presheaf over X assigns a set to each open set in X and there are coherent restriction maps, so it may be viewed as a functor

$$\mathrm{Open}(X)^{op} \longrightarrow \mathbf{Set} \tag{1.4}$$

on the category whose objects are open subsets of X and whose morphisms are inclusions of open sets.

If presheaves on manifolds are meant to generalize manifolds, then we must be able to do geometry with presheaves, and to begin we define maps between presheaves, so a category **Pre** of presheaves.

Definition 1.5. Let $\mathcal{F}', \mathcal{F}$ be presheaves on manifolds. Then a *map* $\varphi\colon \mathcal{F}' \to \mathcal{F}$ is a natural transformation of functors. Thus for each test manifold M there is a map $\mathcal{F}'(M) \xrightarrow{\varphi(M)} \mathcal{F}(M)$ of sets such that for every smooth map $M' \xrightarrow{f} M$ of test manifolds the diagram

$$\begin{array}{ccc} \mathcal{F}'(M') & \xleftarrow{\mathcal{F}'(f)} & \mathcal{F}'(M) \\ {\scriptstyle \varphi(M')}\downarrow & & \downarrow{\scriptstyle \varphi(M)} \\ \mathcal{F}(M') & \xleftarrow{\mathcal{F}(f)} & \mathcal{F}(M) \end{array} \tag{1.6}$$

commutes.

This definition has the nice feature that if the domain presheaf $\mathcal{F}'$ is that of a smooth manifold X, then we use $M = X$ as a test manifold and so determine $\varphi\colon \mathcal{F}_X \to \mathcal{F}$ by its value on $\mathrm{id}_X \in \mathcal{F}_X(X)$, which is an element $\varphi(\mathrm{id}_X)$ of the set $\mathcal{F}(X)$. More formally, we have the following.

Lemma 1.7 (Yoneda). *For any presheaf $\mathcal{F}$, evaluation on X determines an isomorphism*

$$\mathbf{Pre}(\mathcal{F}_X, \mathcal{F}) \cong \mathcal{F}(X). \tag{1.8}$$

Here '$\mathbf{Pre}(\mathcal{F}_X, \mathcal{F})$' denotes the set of maps in the category of presheaves introduced in Definition 1.5. Because of Lemma 1.7 for any presheaf $\mathcal{F}$ we sometimes write an element of $\mathcal{F}(X)$ as a map $X \to \mathcal{F}$.

Remark 1.9. It is important to observe that smoothness is encoded in the presheaf $\mathcal{F}_X$, even though the values of $\mathcal{F}_X$ are sets with no additional structure. For example, a special case of Lemma 1.7 is that for any smooth manifolds X, Y

$$\mathbf{Pre}(\mathcal{F}_X, \mathcal{F}_Y) \cong \mathcal{F}_Y(X) = \mathbf{Man}(X, Y). \tag{1.10}$$

In other words, maps $\mathcal{F}_X \to \mathcal{F}_Y$ of presheaves are precisely *smooth* maps $X \to Y$ of manifolds.

Remark 1.11. What appears in (1.10) are discrete sets, but the construction actually remembers much more. For if S is any smooth manifold, then the set of smooth maps from S into the function space of maps $X \to Y$ is $\mathbf{Man}(S \times X, Y)$; see Example 1.13 below.

Remark 1.12. The map $X \mapsto \mathcal{F}_X$ defines a functor from **Man** into the category of presheaves on manifolds. Then (1.10) asserts that this functor induces an isomorphism on Hom-sets, i.e., is *fully faithful*. So **Man** is a full subcategory of presheaves, which expresses precisely the sense in which presheaves are generalized manifolds.

Example 1.13 (function spaces)**.** Let X, Y be smooth manifolds. The space of smooth maps $X \to Y$ may be given the structure of an infinite dimensional Fréchet manifold, but we can alternatively work with it as a sheaf $\mathcal{F}$. Namely, for a test manifold M let $\mathcal{F}(M)$ be the set of smooth maps $M \times X \to Y$.

The text in this section is a small extract from the paper *Chern-Weil forms and abstract homotopy theory*, joint with Mike Hopkins. You can look there for more material along these lines and an application to the geometry of principal bundles.

2. Gauge transformations and the infinitesimal version

Let $\pi \colon P \to X$ be a principal G-bundle for a Lie group G.

Definition 2.1. A *gauge transformation* is a diffeomorphism $\varphi \colon P \to P$ which commutes with the right G-action and covers id_X.

Thus φ maps each fiber P_x, $x \in X$, to itself and is an automorphism of right G-torsors. The group of such automorphisms is noncanonically isomorphic to G: see (1.4) in the Notes on Lecture 3. The set of gauge transformations is denoted[1] $\mathrm{Aut}(\pi)$; it is a group under composition. As mentioned above, unless X is 0-dimensional then as a Lie group $\mathrm{Aut}(\pi)$ is infinite-dimensional. Rather than tackle infinite-dimensional calculus, which is a nice story, we instead use the ideas of §1 to encode smoothness.

Definition 2.2. Let $\mathcal{G}_P \colon \mathbf{Man}^{op} \to \mathbf{Group}$ be the group-valued presheaf defined by

$$\mathcal{G}_P(M) = \mathrm{Aut}(\mathrm{id}_M \times \pi). \tag{2.3}$$

The right hand side of (2.3) is the group of gauge transformations of the principal G-bundle $M \times P \to M \times X$, which is the pullback of π under the projection $M \times X \to X$. Note the similarity of Definition 2.2 to Example 1.13. As explained above, although this is a presheaf of discrete groups (no topology), by varying the test manifold M we see that $\mathcal{G}_P$ does encode smoothness, and so the idea of a Lie group structure. As in most problems we only work with a finite parameter family of gauge transformations—M is the parameter space—then $\mathcal{G}_P$ encodes everything we need.

We might ask for a definition of the Lie algebra of a group-valued presheaf on **Man**; it should be a Lie algebra-valued presheaf on **Man**. I do not know a general definition.[2] A curve of group

[1] Well, I don't know anywhere it is denoted by that symbol, but it makes sense since $\pi \circ \varphi = \pi$. A more standard notation is $\mathcal{G}_P$, which we'll use for the presheaf in Definition 2.2.

[2] If we were to extend **Man** to allow also the test space $\operatorname{Spec} \mathbb{R}[\epsilon]/(\epsilon^2)$, then we can make a definition: do so!

elements through the identity is, for some $\epsilon > 0$, an element in the fiber $\mathrm{ev}_0^{-1}(e_{\mathcal{G}_P(M)})$ of the map

$$\mathcal{G}_P\bigl((-\epsilon,\epsilon)\times M\bigr) \xrightarrow{\;\mathrm{ev}_0\;} \mathcal{G}_P(M) \tag{2.4}$$

which evaluates at $t = 0 \in (-\epsilon, \epsilon)$. An element is then a local flow φ_t on $M \times P$ which is vertical: there is a function $g\colon (-\epsilon,\epsilon)\times M \times P$ such that $\varphi_t(m,p) = (m, p\cdot g_t(m,p))$. Now differentiate at $t = 0$ to obtain a vertical vector field, identified with a function $\zeta\colon M\times P \to \mathfrak{g}$ to the Lie algebra of G. It satisfies the transformation law $R_g^*\zeta = \mathrm{Ad}_{g^{-1}}\,\zeta$. The collection of such vertical vector fields, of equivalently functions ζ, comprise the value of the Lie algebra presheaf $\mathfrak{g}_P$ on the test manifold M.