NOTES ON LECTURE 2 (RIEMANNIAN GEOMETRY)

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Contents

I will post occasional notes to give supplemental material from a lecture, or to clarify some points and provide more details. These are meant only to supplement your own notes, not to replace them. I also post readings.

1. Smoothness of a Riemannian metric

I said a *Riemannian metric* on a smooth manifold X is a "smoothly varying" assignment of inner products on the tangent spaces T_xX. I gave a definition of 'smoothly varying' in terms of local coordinates; here I explain how to define smoothness of the metric without coordinates. I recommend you also read the handout on fiber bundles. This construction is an instance of the general idea that it is powerful to use our known basic concepts (calculus on manifolds) on more complicated spaces, rather than introduce new concepts.

Recall that an inner product on a real vector space V is a map

$$g_V : V \times V \longrightarrow \mathbb{R} \tag{1.1}$$

which satisfies some properties: it is bilinear, symmetric, and positive. So the *data* is the map (1.1). On a manifold X the data is a collection of maps

$$g_x : T_xX \times T_xX \longrightarrow \mathbb{R}, \qquad x \in X. \tag{1.2}$$

The issue is to formulate the condition that g_x is smooth in x.

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(1.3) *Fiber products.* Here is a very useful general construction and exercise for you. Let

$$\begin{array}{ccc} & & Z \\ & & \big\downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X \end{array} \tag{1.4}$$

be a diagram of smooth maps of smooth manifolds. Then the *fiber product* is a smooth manifold W and dotted maps which make the diagram

$$\begin{array}{ccc} W & \dashrightarrow & Z \\ \downarrow & & \big\downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X \end{array} \tag{1.5}$$

commute. As a set we can define

$$W = \{(y,z) \in Y \times Z : f(y) = g(z)\}. \tag{1.6}$$

Proposition 1.7. *If either f or g is a submersion, then W is a smooth manifold.*

More generally, it suffices to assume that f is transverse to g.

(1.8) *The construction.* Now we construct a manifold W which is the domain of a function $g\colon W \to \mathbb{R}$ that encodes all of the g_x simultaneously. Then we simply ask that g be a smooth map of manifolds. Now we have the tangent bundle $\pi\colon TX \to X$ whose fiber over x is T_xX; we want a new fiber bundle $p\colon W \to X$ whose fiber $p^{-1}(x)$ is $T_xX \times T_xX$. Observe first that $V \times V$ is the fiber product of

$$\begin{array}{ccc} & & V \\ & & \big\downarrow \\ V & \longrightarrow & \mathrm{pt} \end{array} \tag{1.9}$$

and then define W to be the fiber product

$$\begin{array}{ccc} & & TX \\ & & \big\downarrow{\scriptstyle \pi} \\ TX & \xrightarrow{\pi} & X \end{array} \tag{1.10}$$

2. Orientation

Here is some material to remind you about orientations.

(2.1) *Orientation of a real vector space.* Let V be a real vector space of dimension $n > 0$. A *basis* of V is a linear isomorphism $b\colon \mathbb{R}^n \to V$. Let $\mathcal{B}(V)$ denote the set of all bases of V. The group $GL_n(\mathbb{R})$ of linear isomorphisms of $\mathbb{R}^n$ acts *simply transitively* on the right of $\mathcal{B}(V)$ by composition: if $b\colon \mathbb{R}^n \to V$ and $g\colon \mathbb{R}^n \to \mathbb{R}^n$ are isomorphisms, then so too is $b \circ g\colon \mathbb{R}^n \to V$. We say that $\mathcal{B}(V)$ is a *right* $GL_n(\mathbb{R})$*-torsor.* For any $b \in \mathcal{B}(V)$ the map $g \mapsto b \circ g$ is a bijection from $GL_n(\mathbb{R})$ to $\mathcal{B}(V)$, and we use it to topologize $\mathcal{B}(V)$. Since $GL_n(\mathbb{R})$ has two components, so does $\mathcal{B}(V)$.

Definition 2.2. An orientation of V is a choice of component of $\mathcal{B}(V)$.

(2.3) *Determinants and orientation.* Recall that the components of $GL_n(\mathbb{R})$ are distinguished by the determinant homomorphism

$$\det\colon GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^{\neq 0}; \tag{2.4}$$

the identity component consists of $g \in GL_n(\mathbb{R})$ with $\det(g) > 0$, and the other component consists of g with $\det(g) < 0$. On the other hand, an isomorphism $b\colon \mathbb{R}^n \to V$ does not have a numerical determinant. Rather, its determinant lives in the *determinant line* $\operatorname{Det} V$ of V. Namely, define

$$\operatorname{Det} V = \{\epsilon\colon \mathcal{B}(V) \to \mathbb{R} : \epsilon(b \circ g) = \det(g)^{-1}\epsilon(b) \ \text{ for all } b \in \mathcal{B}(V),\ g \in GL_n(\mathbb{R})\}. \tag{2.5}$$

Exercise 2.6. Prove the following elementary facts about determinants and orientations.

(i) Construct a canonical isomorphism $\operatorname{Det} V \xrightarrow{\cong} \bigwedge^n V$ of the determinant line with the highest exterior power. The latter is often taken as the definition.

(ii) Prove that an orientation is a choice of component of $\operatorname{Det} V \backslash \{0\}$. More precisely, construct a map $\mathcal{B}(V) \to \operatorname{Det} V \backslash \{0\}$ which induces a bijection on components.

(iii) Construct the "determinant" of an arbitrary linear map $b\colon \mathbb{R}^n \to V$ as an element of $\operatorname{Det} V$. Show it is nonzero iff b is invertible.

(iv) More generally, construct the determinant of a linear map $T\colon V \to W$ as a linear map $\det T\colon \operatorname{Det} V \to \operatorname{Det} W$, assuming $\dim V = \dim W$.

(v) Part (ii) gives two descriptions of a canonical $\{\pm 1\}$-torsor[1] (=set of two points) associated to a finite dimensional real vector space. Show that it can also be defined as

$$\mathfrak{o}(V) = \{\epsilon\colon \mathcal{B}(V) \to \{\pm 1\} : \epsilon(b \circ g) = \operatorname{sign}\det(g)^{-1}\epsilon(b) \ \text{ for all } b \in \mathcal{B}(V),\ g \in GL_n(\mathbb{R})\}. \tag{2.7}$$

Summary: An orientation of V is a point of $\mathfrak{o}(V)$.

(2.8) *Orienting the zero vector space.* There is a unique zero-dimensional vector space 0 consisting of a single element, the zero vector. There is a unique basis—the empty set—and so by (2.5) the determinant line $\operatorname{Det} 0$ is canonically isomorphic to $\mathbb{R}$ and $\mathfrak{o}(V)$ is canonically isomorphic to $\{\pm 1\}$. Note that $\bigwedge^0(0) = \mathbb{R}$ as $\bigwedge^0 V = \mathbb{R}$ for *any* real vector space V. The real line $\mathbb{R}$ has a canonical orientation: the component $\mathbb{R}^{>0} \subset \mathbb{R}^{\neq 0}$. We denote this orientation as '+'. The opposite orientation is denoted '−'.

[1] $\{\pm 1\}$ is the multiplicative group of square roots of unity, sometimes denoted μ_2.

Exercise 2.9 (2-out-of-3)**.** Suppose

$$0 \longrightarrow V' \xrightarrow{i} V \xrightarrow{j} V'' \longrightarrow 0 \tag{2.10}$$

is a *short exact sequence* of finite dimensional real vector spaces. Construct a canonical isomorphism

$$\operatorname{Det} V'' \otimes \operatorname{Det} V' \longrightarrow \operatorname{Det} V. \tag{2.11}$$

Notice the order: *quotient before sub*. If two out of three of V, V', V'' are oriented, then there is a unique orientation of the third compatible with (2.11). This lemma is quite important in oriented intersection theory.

(2.12) *Real vector bundles and orientation.* Now let X be a smooth manifold and $V \to X$ a finite rank real vector bundle. For each $x \in X$ there is associated to the fiber V_x over x a canonical $\{\pm 1\}$-torsor $\mathfrak{o}(V)_x$—a two-element set—which has the two descriptions given in Exercise 2.6(ii).

Exercise 2.13. Use local trivializations of $V \to X$ to construct local trivializations of $\mathfrak{o}(V) \to X$, where $\mathfrak{o}(V) = \coprod_{x \in X} \mathfrak{o}(V)_x$.

The 2:1 map $\mathfrak{o}(V) \to X$ is called the *orientation double cover associated to* $V \to X$. In case $V = TX$ is the tangent bundle, it is called the *orientation double cover of* X.

Definition 2.14.

(i) An *orientation* of a real vector bundle $V \to X$ is a section of $\mathfrak{o}(V) \to X$.
(ii) If $o\colon X \to \mathfrak{o}(V)$ is an orientation, then the *opposite orientation* is the section $-o\colon X \to \mathfrak{o}(V)$.
(iii) An *orientation* of a manifold X is an orientation of its tangent bundle $TX \to X$.

Orientations may or may not exist, which is to say that a vector bundle $V \to X$ may be orientable or non-orientable. The notation '$-o$' in (ii) uses the fact that $\mathfrak{o}(V) \to X$ is a principal $\{\pm 1\}$-bundle: $-o$ is the result of acting $-1 \in \{\pm 1\}$ on the section o.

Exercise 2.15. Construct the *determinant line bundle* $\operatorname{Det} V \to X$ by carrying out the determinant construction (2.5) (cf. Exercise 2.6) pointwise and proving local trivializations exist. Show that a *nonzero* section of $\operatorname{Det} V \to X$ determines an orientation.

(2.16) *Submanifolds.* Let $Y \subset X$ be a submanifold. Then over Y we have a short exact sequence of vector bundles

$$0 \longrightarrow TY \longrightarrow TX\big|_Y \longrightarrow \nu_{Y \subset X} \longrightarrow 0 \tag{2.17}$$

where the *normal bundle* $\nu_{Y \subset X}$ is defined as the quotient vector bundle $(TX\big|_Y)/TY \to Y$.

Definition 2.18. A *co-orientation* of $Y \subset X$ is an orientation of $\nu_{Y \subset X} \to Y$.