NOTES ON LECTURE 24 (RIEMANNIAN GEOMETRY)

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In class yesterday I did not prove one part of the theorem about homogeneous spaces with a torsionfree linear connection. This note fills in the gap.

As a preliminary I quote a theorem (Theorem 3.20) you'll find in Frank Warner's *Foundations of Differentiable Manifolds and Lie Groups*. The relevant chapter is posted on the website.

Theorem 1. *Let G be a Lie group and $Q \subset G$ a subset which is (i) a subgroup and (ii) admits the structure of an immersed submanifold. Then there is a unique such immersed submanifold struture and Q is a Lie subgroup.*

Now to the theorem at hand. Let G be a Lie group and $H \subset G$ a closed Lie subgroup. Suppose $\mathfrak{m} \subset \mathfrak{g}$ is an H-invariant complement to $\mathfrak{h} \subset \mathfrak{g}$ (under the adjoint action of $H \subset G$ on $\mathfrak{g}$) such that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Then the distribution on G of left translates of $\mathfrak{m}$ is right H-invariant and is a connection on the principal H-bundle $\pi \colon G \to G/H$. (Under an identification of π as a reduction of the bundle of frames of G/H, this connection is torsionfree.) Let $\mathfrak{h}' \subset \mathfrak{h}$ be the linear subspace generated by $[\mathfrak{m}, \mathfrak{m}]$. Then $\mathfrak{h}' \subset \mathfrak{h}$ is a Lie subalgebra, in fact, an ideal. We used the Ambrose-Singer theorem to prove that $\mathfrak{h}'$ is the Lie algebra of the holonomy group (based at the identity $e \in G$) of the connection on π. Now $\mathfrak{k} := \mathfrak{h}' \oplus \mathfrak{m} \subset \mathfrak{g}$ is also a Lie subalgebra, in fact an ideal. Let $K \subset G$ be the connected Lie subgroup with Lie algebra $\mathfrak{k}$. Let $Q \subset G$ be the holonomy bundle based at e.

Theorem 2. $Q = K$.

Proof. From the construction of the holonomy bundle we know that $Q \subset G$ has the structure of an immersed submanifold. Suppose $g, g' \subset Q$ and g_t, g'_t, $0 \leqslant t \leqslant 1$, are horizontal curves from e to g, g': the tangent vectors $g^{-1}\dot{g}_t, (g')^{-1}\dot{g}'_t$ belong to $\mathfrak{m}$ (where defined; the curves are piecewise C^1). Then the left translate gg'_t is also horizontal, and g_t followed by this left translate is a horizontal curve from e to gg'. Hence $Q \subset G$ is a subgroup. It follows from Theorem 1 that $Q \subset G$ is a Lie subgroup. The holonomy bundle Q is connected, so we have only to show that the Lie algebra of Q is $\mathfrak{k}$. But the tangent space to a holonomy bundle at the basepoint is the Lie algebra of the holonomy group direct sum the horizontal space, which is precisely $\mathfrak{k} = \mathfrak{h}' \oplus \mathfrak{m}$. $\square$

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