NOTES ON LECTURE 3 (RIEMANNIAN GEOMETRY)

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Contents

1. Lie groups and right torsors 1
(**1.1**) Definition and Lie algebra 1
(**1.2**) Maurer-Cartan 1-form 1
(**1.4**) Right G-torsors 2
2. Prescribing curvature of plane curves 2
(**2.2**) Interpretation as equation for an integral curve 2
(**2.3**) Embedding the affine group in a linear group 3
(**2.8**) Differential form version 3

I'll give a few more details about the setup for proving the existence of plane curves with prescribed curvature.

1. Lie groups and right torsors

(1.1) *Definition and Lie algebra.* A *Lie group* G is a smooth manifold which is simultaneously a group such that the group operations of multiplication $G \times G \to G$ and inversion $G \to G$ are smooth. Given an element $g \in G$ there is a diffeomorphism $L_g\colon G \to G$ defined by $L_g(x) = gx$: it is left multiplication by g. Its differential maps vector fields on G to vector fields on G. A vector field ξ is *left-invariant* if $(L_g)_*\xi = \xi$ for all $g \in G$. The left invariant vector fields form a finite dimensional subspace $\mathfrak{g}$ of all vector fields. (It is closed under Lie bracket and is a finite dimensional *Lie algebra*, but we will not use that structure today.) We can evaluate a vector field at any point, for example the identity $e \in G$, and so obtain a linear isomorphism $\mathfrak{g} \to T_eG$.

(1.2) *Maurer-Cartan 1-form.* A Lie group has a tautological 1-form $\theta \in \Omega^1_G(\mathfrak{g})$ defined as follows. At every point $x \in G$ it is a linear map $T_xG \to \mathfrak{g}$, and we define it to be the inverse of the evaluation isomorphism $\mathfrak{g} \to T_xG$. In other words, it takes a tangent vector at x and extends it to a left-invariant vector field.

We will come back to the important *Maurer-Cartan equation*, which encodes much about the structure of the Lie group:

$$d\theta + \frac{1}{2}[\theta \wedge \theta] = 0. \tag{1.3}$$

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The Lie bracket appears in this equation, which is an equation in $\Omega^2_G(\mathfrak{g})$. But we will not need it today. (We may come back and talk about the analogous theorem for surfaces, in which case we'll use (1.3) as an integrability condition for a system of partial differential equations. For ordinary differential equations there is no integrability condition.)

(1.4) *Right G-torsors.* A *right G-torsor* P is a smooth manifold with a simply transitive right action $P \times G \to P$. If we pick a point $p_0 \in P$ then we obtain an isomorphism

$$\begin{aligned} \phi_{p_0} \colon G &\longrightarrow P \\ x &\longmapsto p_0 \cdot x \end{aligned} \tag{1.5}$$

Think of this as a "coordinate system" on P. If $p_1 = p_0 \cdot g$ is another point of P, then you can easily compute that the change of coordinates

$$\phi_{p_0}^{-1} \circ \phi_{p_1} = L_g \tag{1.6}$$

is left translation by g. This means that any left-invariant concept on G transports to P. Thus there is a linear map of $\mathfrak{g}$ into vector fields on P, and there is a tautological 1-form $\theta \in \Omega^1_P(\mathfrak{g})$. You should think these through directly in terms of the right G-action on P.

2. Prescribing curvature of plane curves

Let E be a Euclidean plane, $I \subset \mathbb{R}$ an open interval, and $k \colon I \to \mathbb{R}$ a smooth function. We seek to construct an immersion $\iota \colon I \to E$ and a co-orientation of the image so that the signed curvature is k. Recall that the strategy is to lift ι to a map $\tilde{\iota} \colon I \to \mathcal{B}_O(E)$ which we write as $\tilde{\iota}(t) = \bigl(p(t); e_1(t), e_2(t)\bigr)$. We ask that e_1 be normal to the image and e_2 tangent; we use e_1 to define the co-orientation. Since e_2 is the tangent, we have $\dot{p}(t) = \pm e_2(t)$, and for convenience we choose the $+$ sign. The time derivatives of e_1 and e_2 are given by the curvature, as derived in the lecture, so altogether we have

$$\frac{d}{dt}\begin{pmatrix} p & e_1 & e_2 \end{pmatrix} = \begin{pmatrix} p & e_1 & e_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & k \\ 1 & -k & 0 \end{pmatrix} \tag{2.1}$$

(2.2) *Interpretation as equation for an integral curve.* The 3×3 matrix $A(t)$ depends on $t \in I$ (as do the row vectors). Recall that $\mathcal{B}_O(E)$ is a right G-torsor for $G = \mathrm{Euc}_2$ the Euclidean group of symmetries of the standard Euclidean plane $\mathbb{E}^2$. I claim that we can identify that 3×3 matrix as lying in its Lie algebra $\mathfrak{g}$, and so $A \colon I \to \mathfrak{g}$. Then according to **(1.4)** we get a curve of vector fields on $\mathcal{B}_O(E)$, that is a time-varying vector field. Equation (2.1) is the equation for an integral curve of that vector field, and now the basic theorem on ODE applies.

(2.3) *Embedding the affine group in a linear group.* The expression in terms of matrices applies to affine space of any dimension, but for convenience we write it for $\mathbb{A}^2$ with coordinates x, y. Namely, use the affine linear map

$$\begin{aligned} \mathbb{A}^2 &\longrightarrow \mathbb{R}^3 \\ (x,y) &\longmapsto \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \end{aligned} \tag{2.4}$$

Recall that there is a split group extension

$$1 \longrightarrow \mathbb{R}^2 \longrightarrow \mathrm{Aff}_2 \longrightarrow GL_2\mathbb{R} \longrightarrow 1 \tag{2.5}$$

split by the origin $(0,0) \in \mathbb{A}^2$. Then we have an injection

$$\mathrm{Aff}_2 \longrightarrow GL_3\mathbb{R} \tag{2.6}$$

which fixes the image of (2.4): it takes $(h,k) \in \mathbb{R}^2$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \subset GL_2\mathbb{R}$ to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ h & a & b \\ k & c & d \end{pmatrix} \tag{2.7}$$

The Lie algebra is obtained by differentiating a curve of matrices (2.7) through the identity. For the Euclidean group we restrict to the subgroup $O_2 \subset GL_2\mathbb{R}$; the Lie algebra of O_2 consists of 2×2 skew-symmetric matrices. Putting this together, we see that the matrix in (2.1) lies in the Lie algebra of Euc_2, as claimed.

(2.8) *Differential form version.* I encourage you to think through the translation to the following formulation. Define the $\mathfrak{g}$-valued 1-form

$$\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & k(t)dt \\ dt & -k(t)dt & 0 \end{pmatrix} \in \Omega^1_I(\mathfrak{g}) \tag{2.9}$$

on I. The integral curve equation (2.1) is equivalent to the equation

$$\tilde{\iota}^*\theta = \alpha \tag{2.10}$$

in $\Omega^1_I(\mathfrak{g})$. Here $\theta \in \Omega^1_{\mathcal{B}_O(E)}(\mathfrak{g})$ is the Maurer-Cartan form, as in **(1.4)**.