

# NOTES ON LECTURE 6 (RIEMANNIAN GEOMETRY)

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A few remarks left over from the lecture. Please also look at the readings on the web from Warner and Spivak about Lie derivatives and the Frobenius theorem as we talk about those topics in the next lecture.

## 1. Tensors and linearity over functions

I said a bit in lecture, so will say more here. First, let me emphasize that in any theorem stated globally for a manifold  $X$  you can apply the theorem on an open subset  $U \subset X$ , since  $U$  is itself a manifold.

(1.1) *The theorem.* Let  $X$  be a smooth manifold and  $\mathcal{X}(X)$  the (typically infinite dimensional) vector space of smooth vector fields on  $X$ . Let

$$(1.2) \quad T: \mathcal{X}(X) \longrightarrow \Omega_X^0$$

be a linear functional from vector fields to functions. We say  $T$  is *linear over functions* if

$$(1.3) \quad T(f\xi) = fT(\xi), \quad f \in \Omega_X^0, \quad \xi \in \mathcal{X}(X).$$

**Theorem 1.4.** *If  $T$  is linear over functions, then there exists a unique 1-form  $\tau \in \Omega_X^1$  such that*

$$(1.5) \quad T(\xi)(p) = \tau_p(\xi_p), \quad p \in X.$$

Here  $\xi_p \in T_p X$  is the value of the vector field  $\xi$  at  $p$ .

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*Proof.* We use (1.5) to define the functional  $\tau_p$  and must prove that if  $\xi, \xi' \in \mathcal{X}(X)$  satisfy  $\xi_p = \xi'_p$ , then

$$(1.6) \quad T(\xi)(p) = T(\xi')(p).$$

First, if  $\xi = \xi'$  on some open set  $U$  containing  $p$ , then let  $f \in \Omega_X^0$  be a smooth function supported on  $U$  with  $f(p) = 1$ . Then

$$(1.7) \quad fT(\xi) = T(f\xi) = T(f\xi') = fT(\xi'),$$

and evaluating at  $p$  we conclude (1.6). Now suppose that  $\xi_p = \xi'_p$  and we take  $U$  to be the domain of a chart about  $p$  with local coordinates  $x^1, \dots, x^n$ . Write

$$(1.8) \quad \xi = \xi^i \frac{\partial}{\partial x^i}, \quad \xi' = (\xi')^i \frac{\partial}{\partial x^i}$$

and let  $f$  be a cutoff function as above. Then  $f\partial/\partial x^i$  is a global vector field and we have

$$(1.9) \quad \begin{aligned} T(\xi)(p) &= f(p)T(\xi)(p) = T(\xi^i f \frac{\partial}{\partial x^i})(p) = \xi^i(p)T(f \frac{\partial}{\partial x^i})(p) \\ &= (\xi')^i(p)T(f \frac{\partial}{\partial x^i})(p) = T((\xi')^i f \frac{\partial}{\partial x^i})(p) = f(p)T(\xi')(p) = T(\xi')(p). \end{aligned}$$

□

**(1.10)** *d of a 1-form as a function on vector fields.* Here's an application which uses the Lie bracket (which we discuss in the next lecture, so you may want to revisit this). Suppose  $\alpha \in \Omega_X^1$ . Define

$$(1.11) \quad T(\xi, \eta) = \xi\alpha(\eta) - \eta\alpha(\xi) - \alpha([\xi, \eta]), \quad \xi, \eta \in \mathcal{X}(X).$$

Check that  $T$  is linear over functions in both variables and is skew-symmetric. Therefore  $T$  defines an element in  $\Omega_X^2$ . You should check that it is in fact  $d\alpha$ . Equation (1.11) is a *very* useful formula.

## 2. Symmetries of the curvature tensor

Again something we will return to, but you can verify now from the formula in local coordinates. Recall that the Riemann tensor is

$$(2.1) \quad R = R^i_{jkl} \frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^\ell.$$

Lower an index by setting

$$(2.2) \quad R_{ijkl} = g_{im}R_{jkl}^m.$$

Then we have

$$(2.3) \quad \begin{aligned} R_{jikl} &= -R_{ijkl} \\ R_{ijlk} &= -R_{ijkl} \\ R_{klij} &= R_{ijkl}. \end{aligned}$$

If we work at a point  $p$  and let  $V$  be the tangent space at that point, then these symmetries mean that

$$(2.4) \quad \frac{1}{4}R_{ijkl}(dx^i \wedge dx^j) \otimes (dx^k \otimes dx^\ell)$$

is a symmetric bilinear form

$$(2.5) \quad K_p: \wedge^2 V \times \wedge^2 V \longrightarrow \mathbb{R}.$$

A 2-plane  $\Pi \subset V$  determines an element in  $\wedge^2 V$  up to scale by associating  $\xi_1 \wedge \xi_2$  for any ordered basis  $\xi_1, \xi_2$  of  $V$ . Using the metric we can restrict to orthonormal bases  $e_1, e_2$ , in which case  $e_1 \wedge e_2 \in \wedge^2 V$  is independent of the basis up to sign. Therefore,

$$(2.6) \quad K_p(e_1 \wedge e_2, e_1 \wedge e_2)$$

is independent of the basis and just depends on  $\Pi \subset V$ . It is called the *sectional curvature* at  $p$  of the plane  $\Pi$ . On a Riemannian 2-manifold  $\Sigma$  it gives a function  $K: \Sigma \rightarrow \mathbb{R}$ , and the Gauss *Theorema Egregium* states that this function is the Gauss curvature if  $\Sigma \subset E$  is a submanifold of a Euclidean 3-space.