NOTES ON LECTURE 7 (RIEMANNIAN GEOMETRY)

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Equation (1.7) below was misstated in the lecture, so I repeat the entire derivation.

1. Derivation of local formula for the Lie derivative of a vector field

Suppose ξ is a vector field on a smooth manifold X. Fix $p \in X$, and let φ_t be the local flow about p. If η is another vector field on X we define the Lie derivative by

$$\mathcal{L}_\xi \eta = \frac{d}{dt}\Big|_{t=0} (\varphi_{-t})_* \eta. \tag{1.1}$$

Since the flow is only local, this is only well-defined in a neighborhood of p. (So we cover X by such neighborhoods to define the Lie derivative.) In lecture I derived the formula in local coordinates, and I'll repeat that here.

Let $x^1, \dots, x^n$ be local coordinates on X about p. Write

$$\xi = \xi^i \frac{\partial}{\partial x^i}, \qquad \xi = \xi^j \frac{\partial}{\partial x^j} \tag{1.2}$$

and write the local flow as

$$\varphi_t(x) = \varphi(t;x) = \big(\varphi^1(t;x), \dots, \varphi^n(t;x)\big) = \big(\varphi^i(t;x)\big)_i \tag{1.3}$$

in the x-coordinates. Then by the definition and elementary properties of the flow we have

$$\begin{aligned} \varphi^i(0,x) &= x^i \\ \dot\varphi^i(t;x) &= \xi^i_{\varphi(t;x)} \\ \frac{\partial \varphi^i}{\partial x^j}\Big|_{t=0} &= \delta^i_j \\ \frac{\partial^2 \varphi^i}{\partial x^k \partial x^j}\Big|_{t=0} &= 0. \end{aligned} \tag{1.4}$$

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Then

$$(\varphi_{-t})_*\eta = \big(\varphi_t^*\eta^i\big)\Big((\varphi_{-t})_*\frac{\partial}{\partial x^i}\Big). \tag{1.5}$$

We evaluate

$$\varphi_t^*\eta^i(x) = \eta^i\big(\varphi(t;x)\big). \tag{1.6}$$

The next bit is the reason for the note: it is more complicated than stated in lecture. We must flow the curve whose tangent is $\partial/\partial x^i$ at the point $\varphi(t;x)$ by φ_{-t}:

$$\begin{aligned}\Big((\varphi_{-t})_*\frac{\partial}{\partial x^i}\Big)(x) &= \frac{d}{ds}\Big|_{s=0}\varphi\big(-t;\varphi^1(t;x),\dots,\varphi^i(t;x)+s,\dots,\varphi^n(t;x)\big)\\ &= \frac{d}{ds}\Big|_{s=0}\Big(\varphi^j\big(-t;\varphi^1(t;x),\dots,\varphi^i(t;x)+s,\dots,\varphi^n(t;x)\big)\Big)_j\\ &= \frac{\partial\varphi^j}{\partial x^i}\big(-t;\varphi(t;x)\big)\frac{\partial}{\partial x^j}.\end{aligned} \tag{1.7}$$

Combining (1.1), (1.5), (1.6), and (1.7) we compute

$$\begin{aligned}(\mathcal{L}_\xi\eta)(x) &= \frac{d}{dt}\Big|_{t=0}\Big\{\eta^i\big(\varphi(t;x)\big)\frac{\partial\varphi^j}{\partial x^i}\big(-t;\varphi(t;x)\big)\frac{\partial}{\partial x^j}\Big\}\\ &= \frac{\partial\eta^i}{\partial x^k}\dot\varphi^k\frac{\partial\varphi^j}{\partial x^i}\frac{\partial}{\partial x^j} - \eta^i\frac{\partial\dot\varphi^j}{\partial x^i}\frac{\partial}{\partial x^j} + \frac{\partial^2\varphi^j}{\partial x^k\partial x^i}\dot\varphi^k\frac{\partial}{\partial x^j}\\ &= \frac{\partial\eta^j}{\partial x^i}\xi^k\frac{\partial}{\partial x^i} - \eta^i\frac{\partial\xi^j}{\partial x^i}\frac{\partial}{\partial x^j},\end{aligned} \tag{1.8}$$

where we use (1.4) in passing to the last equation. Rearranging the indices we obtain the desired formula:

$$\mathcal{L}_\xi\eta = \Big(\xi^i\frac{\partial\eta^j}{\partial x^i} - \eta^i\frac{\partial\xi^j}{\partial x^i}\Big)\frac{\partial}{\partial x^j}. \tag{1.9}$$

It is instructive for you to go slowly through (1.7) and (1.8). Be sure you understand where every piece of the formula is being evaluated, as that is (deliberately) dropped from the notation.

2. Vector fields as derivations

Let $\xi \in \mathfrak{X}(X)$ be a vector field on a smooth manifold X. Then ξ defines a derivation on functions

$$\begin{aligned}D_\xi\colon \Omega^0_X &\longrightarrow \Omega^0_X\\ f &\longmapsto \xi f\end{aligned} \tag{2.1}$$

using the directional derivative. The derivation property is

$$D_\xi(fg) = (D_\xi f)g + f(D_\xi g), \qquad f,g\in\Omega^0_X. \tag{2.2}$$

Proposition 2.3. *Let $D\colon \Omega^0_X \to \Omega^0_X$ be a derivation. Then there exists a vector field $\xi \in \mathfrak{X}(X)$ such that $D = D_\xi$.*

First we prove the following.

Lemma 2.4. *Suppose h is a real-valued function defined on a convex neighborhood of $0 \in \mathbb{A}^n$, and assume $h(0) = 0$. Then there exist functions g_i, $i = 1, \dots, n$ in this neighborhood such that*

$$h(x) = g_i(x)x^i. \tag{2.5}$$

Furthermore,

$$g_i(0) = \frac{\partial h}{\partial x^i}(0). \tag{2.6}$$

Proof. For any x in this neighborhood write

$$h(x) = \int_0^1 dt\, \frac{d}{dt} h(tx) = \int_0^1 dt\, \frac{\partial h}{\partial x^i}(tx)\, x^i, \tag{2.7}$$

so set

$$g_i(x) = \int_0^1 dt\, \frac{\partial h}{\partial x^i}(tx). \tag{2.8}$$

Equation (2.6) follows immediately from (2.8) upon setting $x = 0$. □

Proof of Proposition 2.3. First, use the derivation property to show that $D(1) = 0$ and then D applied to a constant function vanishes. Next, suppose f_1, f_2 are functions which agree in a neighborhood of a point $p \in X$. Then if g is a cutoff function—$g(p) = 1$ and g vanishes outside that neighborhood—we have $f_1 g = f_2 g$, and applying D we conclude that $D(f_1)(p) = D(f_2)(p)$. Therefore, $Df(p) \in \mathbb{R}$ is well-defined even if f is only defined in a neighborhood of p: extend f arbitrarily to a global function and evaluate. Let $x^1, \dots, x^n$ be local coordinates on a neighborhood U of p such that the image of U is convex and $x^i(p) = 0$. Using the lemma we find functions g_i such that

$$f(x) = f(0) + g_i(x)x^i \tag{2.9}$$

in this neighborhood, and $g_i(0) = \partial f/\partial x^i(0)$. Then

$$Df(0) = D(f - f(0))(0) = D(g_i x^i)(0) = (Dx^i)(0)g_i(0) = Dx^i(0)\frac{\partial f}{\partial x^i}(0). \tag{2.10}$$

Thus define a vector $\xi_p = (Dx^i)(0)\dfrac{\partial}{\partial x^i}$. Evaluate (2.10) at points x near 0 to conclude that ξ is smooth in a neighborhood of p, hence is smooth. □