Problem Set # 1

M392C: Index Theory

1. Let $x^1, \dots, x^n$ and $y^1, \dots, y^n$ be two coordinate systems on an open set of affine space $\mathbb{A}^n$.

   (a) Recall the transformation law which expresses $\partial/\partial x^i$ in terms of the partial derivatives $\partial/\partial y^\alpha$. Derive an expression for $\partial^2/\partial x^i \partial x^j$ in terms of partial derivatives in the y-coordinate system.

   (b) Transform the differential operator

   $$L = A^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + B^i \frac{\partial}{\partial x^i} + C$$

   into the y-coordinate system. Note particularly the transformation law for the highest order term.

   (c) Generalize to higher order (more derivatives) differential operators.

2. Now let $x^1, \dots, x^n$ be local coordinates in Euclidean space $\mathbb{E}^n$ (affine space with the standard metric), and $g_{ij} = \langle \partial/\partial x^i, \partial/\partial x^j \rangle$ the expression for the metric in these local coordinates. By "local coordinates" we mean that the x^i are defined on an open set $U \subset \mathbb{E}^n$.

   (a) The operator d maps compactly supported functions f on U to compactly supported 1-forms ω on U. Explicitly,

   $$df = \frac{\partial f}{\partial x^i}\, dx^i.$$

   Compute the expression for the adjoint differential operator d^* from 1-forms to functions which, on compactly supported functions and 1-forms, satisfies the adjointness relation

   $$\int_U \langle df, \omega \rangle d\mu = \int_U f d^* \omega \, d\mu,$$

   where μ is the standard volume form on $\mathbb{E}^n$ (restricted to U).

   (b) Now compute an expression in local coordinates for the laplacian $\Delta = d^* d$ acting on functions.

   (c) Investigate how Δ behaves under change of coordinates, as in the previous problem.

3. Let M be a Riemannian manifold. Write the laplacian Δ on functions in terms of d and the Hodge star $*$. Do the same for Δ acting on $\Omega^q(M)$. Recall that $\Delta = dd^* + d^* d$, so the problem is really to write d^* in terms of d and $*$.

4. Write out a careful proof that any vector bundle admits a covariant derivative.

5. Let $E \to X$ be a vector bundle with a covariant derivative ∇.

(a) A local trivialization of E may be specified by a local basis of sections $e_1, \ldots, e_N$ of E. Write

$$\nabla e_j = A^i_j e_i,$$

where A^i_j are local 1-forms. (Here 'local' means 'defined in some open set of X'.) Write the covariant derivative of an arbitrary section s in terms of A^i_j.

(b) If E has a metric, then we can choose the basis e_j to be unitary (or, in the real case, orthonormal). What can you then say about A^i_j?