Problem Set # 2

M392C: Index Theory

I do recommend working through some problems if you are to get the most out of the lectures. Here are some problems about connections on princpal bundles, some of which were covered in the lecture.

1. Let X be a manifold and $\Omega^\bullet(X)$ the $\mathbb{Z}$-graded commutative algebra of differential forms. Let ξ, η be vector fields on X. In differential calculus we have three basic operations: d, L_ξ, ι_ξ, where d is the exterior derivative of degree $+1$, L_ξ is the Lie derivative of degree 0, and ι_ξ is the contraction of degree -1. Note that d and L_ξ are first-order differential operators, whereas ι_ξ is algebraic. In this problem you will compute the commutators of these operations. These are the basic equations of differential calculus.

   (a) Verify $[d, \iota_\xi] = L_\xi$. This is the Cartan formula for the Lie derivative on differential forms.

   (b) Show $[d, L_\xi] = 0$. What does this formula express about the operator d?

   (c) Note $[d, d] = 0$ is one of the defining properties of d. What is $[\iota_\xi, \iota_\eta]$?

   (d) Check $[L_\xi, L_\eta] = L_{[\xi,\eta]}$.

   (e) The remaining commutator is $[L_\xi, \iota_\eta] = \iota_{[\xi,\eta]}$.

   (f) Let α be a 1-form. Compute $d\alpha(\xi, \eta) = \iota_\xi \iota_\eta d\alpha$ using the above formulas. Try the next case: $d\beta$ for a 2-form β.

2. Let $E \to X$ be a vector bundle with covariant derivative ∇. Let $e_1, \dots, e_r$ be a local framing of E, and write

$$\nabla e_j = A^i_j e_i$$

   for 1-forms A^i_j.

   (a) Recall that we extend ∇ to an operator d_∇ and define the curvature $F = F_\nabla$ to be d^2_∇. Compute F in terms of A^i_j. (I believe I did that in lecture.)

   (b) Use that formula to show that for vector fields ξ, η

$$F(\xi, \eta) = \nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi - \nabla_{[\xi,\eta]}$$

   as an operator on sections of E. Use the calculus developed in the previous problem—start by writing the left hand side as $\iota_\eta \iota_\xi F$.

3. Let $\pi\colon P \to X$ be a principal bundle with structure group a Lie group G. Let $\mathfrak{g}$ be the Lie algebra of G.

   (a) Recall that if $\xi \in \mathfrak{g}$ then there is a corresponding curve $\exp(t\xi) \in G$ which passes through the identity element at $t = 0$ and whose velocity at $t = 0$ is ξ. Via the right action on P this gives a 1-parameter family of diffeomorphisms whose generator is a vector field $\hat{\xi}$. Show that at each $p \in P$ we obtain an isomorphism of $\mathfrak{g}$ with the vertical tangent space to P at p. (The vertical tangent space is the kernel of the differential $d\pi$.)

   (b) Compute $[\hat{\xi}_1, \hat{\xi}_2]$ for $\xi_1, \xi_2 \in \mathfrak{g}$.

   (c) Suppose $\rho\colon G \to \mathrm{Aut}(\mathbb{E})$ is a representation of G on a vector space $\mathbb{E}$, and $\hat{\rho}\colon \mathfrak{g} \to \mathrm{End}(\mathbb{E})$ the corresponding representation of Lie algebras. Then there is an associated vector bundle $E \to X$, and a section of E is represented by an equivariant function $s\colon P \to \mathbb{E}$. What, precisely, is the equivariance condition? Now, given $\xi \in \mathfrak{g}$, compute the directional derivative $\hat{\xi} \cdot s$. Be careful with signs.

4. A *connection* on a principal G-bundle $\pi\colon P \to X$ is a G-invariant distribution of horizontal subspaces. That is, for each $p \in P$ we have a subspace $H_p \subset T_pP$ such that $d\pi_p$ maps H_p isomorphically onto $T_{\pi(p)}X$ and for any $g \in G$ we have $H_{p\cdot g} = (R_g)^*(H_p)$, where $R_g\colon P \to P$ is the action of g.

   (a) The horizontal subspace H_p determines the projection Θ_p of the tangent space at p onto the vertical subspace with kernel H_p. Identify the vertical subspace with $\mathfrak{g}$, as in the previous problem, and so view Θ as a $\mathfrak{g}$-valued 1-form on P. Compute $R_g^*\Theta$.

   (b) Recall the Maurer-Cartan form θ on a Lie group G. It is the G-valued *left*-invariant 1-form whose value on a vector is the left-invariant vector field (so element of $\mathfrak{g}$) which extends the given vector. Show that θ induces a $\mathfrak{g}$-valued 1-form on each fiber $\pi^{-1}(x) = P_x$ of the principal bundle $\pi\colon P \to X$.

   (c) What is the restriction of the connection form Θ to a fiber of P?

5. Define the *curvature* of Θ to be the $\mathfrak{g}$-valued 2-form

$$\Omega = d\Theta + \frac{1}{2}[\Theta, \Theta].$$

   (a) Verify that Ω restricts to zero on each fiber of P. For this you'll need to use the naturality of d and the Maurer-Cartan equation

$$d\theta + \frac{1}{2}[\theta, \theta] = 0.$$

   Verify the latter using the formula of exercise 1(f) above.

(b) Compute $R_g^*\Omega$. Use this formula to interpret Ω as an element of $\Omega^2_X(\mathfrak{g}_P)$, where the *adjoint* bundle $\mathfrak{g}_P$ is associated to P using the adjoint representation

$$\mathrm{ad}\colon G \longrightarrow \mathrm{Aut}(\mathfrak{g}).$$

(c) Interpret the curvature in terms of the Frobenius theorem. What does the vanishing of curvature say about the distribution of horizontal planes?

(d) Compute $d\Omega$. This gives the Bianchi identity.

6. With all of the notation above, use the connection on P to induce a covariant derivative on an associated vector bundle E. Check the compatibility of the definitions of the curvature.