Problem Set # 3

M392C: Index Theory

I do recommend working through some problems if you are to get the most out of the lectures.

1. Let A be an associative algebra with identity.

   (a) A *filtration* of A is a sequence of subalgebras

   $$0 = F^{-1}A \subset F^0A \subset F^1A \subset \cdots \subset F^pA \subset \cdots$$

   such that $\cup F^pA = A$. The *associated graded algebra* $\mathrm{gr}^\bullet A$ is the direct sum of the homogeneous subspaces

   $$\mathrm{gr}^p A = F^pA/F^{p-1}A.$$

   Show that $\mathrm{gr}^\bullet A$ is indeed a graded algebra.

   (b) Let V be a real vector space and consider the exterior algebra $\bigwedge^\bullet V^*$. Define a filtration on it. What is the associated graded algebra?

   (c) Now do the same for the Clifford algebra $C(V^*, B^*)$, where B is a symmetric bilinear form on V and B^* the induced form on the dual space V^*.

2. (a) Construct a *vector space* isomorphism $\bigwedge^\bullet V^* \cong C(V^*, B^*)$. You may do this by choosing a basis of V and checking that the isomorphism you write is independent of a basis. Another idea is to embed the exterior algebra in the tensor algebra (which works in characteristic zero).

   (b) Use this isomorphism to express left and right Clifford multiplication in terms of operations on the exterior algebra.

3. Let C_n be the Clifford algebra, as usual, and fix

   $$\omega_n = i^{n(n+1)/2}\gamma^1 \cdots \gamma^n.$$

   Check that $\gamma^2 = 1$. Let ϵ denote the grading operator on C_n: $\epsilon(a) = a$ if a is even and $\epsilon(a) = -a$ if a is odd. Then show that the *graded* commutator computes the grading $\epsilon(a) = (-1)^{n|a|}\omega_n a \omega_n^{-1}$.

4. Show that the set of elements

$$\pm\gamma^{i_1}\cdots\gamma^{i_p}, \qquad 1 \le i_1 < \cdots < i_p \le n,$$

forms a finite group G_n sitting in the Clifford algebra. (The empty product is 1.) This group is called the *extra-special 2-group*.

(a) Construct a surjective map $G_n \to (\mathbb{Z}/2\mathbb{Z})^n$. What is the kernel?

(b) Work out the dimensions of the irreducible complex representations of G_n. (Recall that the sum of the squares of the dimensions equals the order of G_n. This problem requires that you know some representation theory of finite groups.)

5. Consider the set of elements $\mathfrak{g} = \{\omega_{ij}\gamma^i\gamma^j\}$ in C_n, where ω_{ij} is a collection of real numbers, skew in i, j.

(a) Show that $\mathfrak{g}$ is closed under (graded) commutator: if $a, b \in \mathfrak{g}$, then $[a, b] \in \mathfrak{g}$.

(b) Identify $\mathfrak{g}$ with the Lie algebra $\mathfrak{o}_n$. In other words, write an linear isomorphism from the vector space of skew-symmetric $n \times n$ matrices which preserves the bracket.

(c) Show that if $a \in \mathfrak{g}$, then $\exp(a) \in \mathrm{Spin}_n$. Compute this exponential explicitly in the case $n = 2$.

6. Compute the Weitzenböck formula on 0-forms and 1-forms. That is, compute K in the formula

$$dd^* + d^*d = \nabla^*\nabla + K$$

acting on 0-forms and 1-forms on a Riemannian manifold.

7. Let X be Riemannian and $M \to X$ a $\mathrm{Cliff}(X)$-module (with compatible metric and connection). Let D be the associated Dirac operator and

$$D^2 = \nabla^*\nabla + K$$

the Weitzenböck formula.

(a) Prove that if X is compact, then $D\psi = 0$ if and only if $D^2\psi = 0$, where $\psi \in \Gamma_X(M)$.

(b) Prove that if K is a nonnegative operator on $\Gamma_X(M)$ in the L^2 norm, then $D\psi = 0$ if and only if ψ is parallel, i.e., $\nabla\psi = 0$. The positivity means $\int_X \langle K\psi, \psi\rangle \ge 0$ for all ψ.