Problem Set # 2

M392C: Morse Theory

1. (a) In lecture it was suggested that the (modified) negative gradient flow of the function $f(r) = (\log r)^2$ can be used to construct a deformation retraction of $\mathbb{R}^{>0}$ onto $\{1\} \subset \mathbb{R}^{>0}$. Carry this out in detail.

   (b) Do the same for the function $f(A) = \mathrm{Tr}(\log(A^*A))$ on $GL_n(\mathbb{R})$ to construct a deformation retraction onto O_n.

2. Let $C \subset \mathbb{E}^2$ be a co-oriented plane curve, which you may as well assume is connected. Its *evolute* C' is the union of its centers of curvature. Use a parametrization of C by arclength to parametrize C'. When is your parametrization of C' an immersion? Can it be an immersion if C is closed, i.e., an embedded circle in the Euclidean plane. Compute the evolute of an ellipse.

3. The equations

$$x = \cos t$$
$$y = \sin t$$
$$z = t$$

   describe an embedding $\mathbb{R} \to \mathbb{E}^3$ whose image is a helix. Find all focal points of the helix.

4. Recall that a covariant derivative on a Riemannian manifold M is orthogonal and torsionfree if for all vector fields X, Y, Z we have

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$
$$[X, Y] = \nabla_X Y - \nabla_Y X$$

   Prove that there is a unique torsionfree orthogonal covariant derivative on M. (Hint: Give a formula for $\langle \nabla_X Y, Z\rangle$.)

5. Let $M \subset N$ be a submanifold of a Riemannian manifold N, and let ∇ be the Levi-Civita covariant derivative on N. Let X, Y be vector fields on M.

   (a) Prove that the tangential orthogonal projection of $\nabla_X Y$ is equal to $\nabla'_X Y$, where ∇' is the Levi-Civita covariant derivative of the induced metric on M.

   (b) Prove that the bilinear form

$$II(X, Y) = \text{normal orthogonal projection of } \nabla_X Y$$

   is a symmetric tensor. The tensor part means that for functions f, g we have

$$II(fX, gY) = fgII(X, Y).$$