Problem Set # 3

M392C: Morse Theory

1. Prove that $\mathbb{RP}^2$ is not the boundary of a compact 3-manifold M in two different ways.
   (a) If so, consider the double of M, which is a closed 3-manifold. What is its Euler characteristic?
   (b) For another proof consider the determinant line bundle $\operatorname{Det} TM \to M$ of the tangent bundle. Take a generic section on the boundary $\mathbb{RP}^2$, let Z be its zero set, and consider the self-intersection of Z with itself.
   (c) Do these techniques extend to $\mathbb{RP}^n$ for other n? What can you say about $\mathbb{CP}^2$? About $\mathbb{CP}^n$ for arbitrary n?

2. (a) Explain how to construct $S^1 \times S^1$ from S^2 via a single surgery. Describe the bordism which takes one to the other (*trace* of the surgery).
   (b) Do the same starting from the empty 2-manifold $\emptyset^2$ instead. Describe the handle attachments which construct the bordism.

3. In this problem you will consider one of Milnor's exotic 7-spheres. You will prove it is homeomorphic to S^7 but will not prove it is not diffeomorphic.
   (a) Let $\mathbb{H}$ be the set of quaternions $a + bi + cj + dk,\ a, b, c, d \in \mathbb{R}$. Identify $S^3 \subset \mathbb{H}$ as the set of unit quaternions, i.e., quaternions of unit norm. (See the formula below for the norm.) Use the diffeomorphism
   $$\begin{aligned} \varphi\colon \mathbb{H} \setminus \{0\} \times S^3 &\longrightarrow \mathbb{H} \setminus \{0\} \times S^3 \\ (q, r) &\longmapsto (q/|q|^2, q^2 r q^{-1}/|q|) \end{aligned}$$
   to glue together two copies of $\mathbb{H} \times S^3$. Show that the result is a closed 7-manifold M. Recall
   $$|a + bi + cj + dk| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$
   (b) Verify that the function
   $$f(q, r) = \frac{\operatorname{Re}(r)}{\sqrt{1 + |q|^2}}$$
   on $\mathbb{H} \times S^3$ extends to a Morse function on M with precisely two nondegenerate critical points.
   (c) Conclude that M is homeomorphic to S^7.