1. Prove that $\mathbb{R}P^2$ is not the boundary of a compact 3-manifold $M$ in two different ways.

   (a) If so, consider the double of $M$, which is a closed 3-manifold. What is its Euler characteristic?

   (b) For another proof consider the determinant line bundle $\text{Det} TM \to M$ of the tangent bundle.

   Take a generic section on the boundary $\mathbb{R}P^2$, let $Z$ be its zero set, and consider the self-intersection of $Z$ with itself.

   (c) Do these techniques extend to $\mathbb{R}P^n$ for other $n$? What can you say about $\mathbb{C}P^2$? About $\mathbb{C}P^n$ for arbitrary $n$?

2. (a) Explain how to construct $S^1 \times S^1$ from $S^2$ via a single surgery. Describe the bordism which takes one to the other (trace of the surgery).

   (b) Do the same starting from the empty 2-manifold $\emptyset^2$ instead. Describe the handle attachments which construct the bordism.

3. In this problem you will consider one of Milnor’s exotic 7-spheres. You will prove it is homeomorphic to $S^7$ but will not prove it is not diffeomorphic.

   (a) Let $\mathbb{H}$ be the set of quaternions $a + bi + cj + dk$, $a, b, c, d \in \mathbb{R}$. Identify $S^3 \subset \mathbb{H}$ as the set of unit quaternions, i.e., quaternions of unit norm. (See the formula below for the norm.) Use the diffeomorphism

   \[ \varphi: \mathbb{H} \setminus \{0\} \times S^3 \to \mathbb{H} \setminus \{0\} \times S^3 \]

   \[ (q, r) \mapsto (q/|q|^2, q^2rq^{-1}/|q|) \]

   to glue together two copies of $\mathbb{H} \times S^3$. Show that the result is a closed 7-manifold $M$. Recall

   \[ |a + bi + cj + dk| = \sqrt{a^2 + b^2 + c^2 + d^2}. \]

   (b) Verify that the function

   \[ f(q, r) = \frac{\text{Re}(r)}{\sqrt{1 + |q|^2}} \]

   on $\mathbb{H} \times S^3$ extends to a Morse function on $M$ with precisely two nondegenerate critical points.

   (c) Conclude that $M$ is homeomorphic to $S^7$. 

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