Problem Set # 7

M392C: Morse Theory

1. Let $F\colon Y \times Z \to W$ be a smooth map of smooth (Banach) manifolds. Suppose $w \in W$ is a regular value of F. Define the submanifold $X = F^{-1}(w) \subset Y \times Z$. Let $\pi\colon X \to Y$ be projection. Prove that $y \in Y$ is a regular value of π if and only if w is a regular value of $F\Big|_{\{y\}\times Z}$.

2. Let V be a Banach space.
   (a) Prove that a finite dimensional subspace of V has a complement. (Hint: Use the Hahn-Banach theorem.)
   (b) Prove that a closed subspace of V of finite codimension has a complement.

3. Now suppose V, W are Banach spaces. Let $\mathrm{Hom}(V, W)$ denote the space of continuous linear maps $V \to W$. Review the operator norm on $\mathrm{Hom}(V, W)$ and the fact that it is complete, so $\mathrm{Hom}(V, W)$ is a Banach space. Let $\mathrm{Fred}(V, W) \subset \mathrm{Hom}(V, W)$ denote the subset of Fredholm operators, topologized with the subspace topology.
   (a) Let $F \subset W$ be a finite dimensional subspace. Define $\mathcal{O}(F) \subset \mathrm{Fred}(V, W)$ as the set of Fredholm operators $T\colon V \to W$ transverse to F, i.e., $T(V) + F = W$. Prove that $\mathcal{O}(F)$ is open. Produce an open cover of $\mathrm{Fred}(V, W)$. Prove that $\mathrm{Fred}(V, W) \subset \mathrm{Hom}(V, W)$ is open, hence a Banach manifold. (Hint: Model $T \in \mathcal{O}(F)$ as its restriction $T^{-1}(F) \to F$.)
   (b) Prove that $\mathrm{index}\colon \mathrm{Fred}(V, W) \to \mathbb{Z}$ is locally constant (hence continuous).
   (c) Show that $\dim\ker T$ is *not* a locally constant on $\mathrm{Fred}(V, W)$.
   (d) Construct a line bundle $\mathrm{Det} \to \mathrm{Fred}(V, W)$ whose fiber at T is isomorphic to the line $\mathrm{Det}\,\mathrm{coker}\,T \otimes (\mathrm{Det}\ker T)^{-1}$. You need to glue these lines together smoothly. (Hint: For $T \in \mathcal{O}(F)$ the fiber is isomorphic to $\mathrm{Det}\,F \otimes \big(\mathrm{Det}\,T^{-1}(F)\big)^{-1}$.)