Problem Set # 8

M392C: Morse Theory

1. Let M be a smooth manifold with a covariant derivative ∇ on its tangent bundle (linear connection).

   (a) For vector fields ξ_1, ξ_2 on M prove that the expression

   $$T(\xi_1, \xi_2) = \nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1 - [\xi_1, \xi_2]$$

   is linear over functions, i.e., defines a tensor. Verify that it is a skew-symmetric function of ξ_1, ξ_2, so an element $T \in \Omega^2_M(TM)$. It is called the *torsion tensor* of ∇.

   (b) Let $\iota \in \Omega^1_M(TM)$ be the identity endomorphism of $TM \to M$. Combine the covariant derivative and the Cartan d to define a first order differential operator

   $$d_\nabla \colon \Omega^q_M(TM) \longrightarrow \Omega^{q+1}_M(TM)$$

   for all q such that $d_\nabla = d$ for $q = 0$. Prove $T = d_\nabla \iota$. (Hint: Recall the formula which evaluates the differential of a 1-form on two vector fields.)

   (c) Let $f \colon N \to M$ be a smooth map of a smooth manifold N into M. Prove that $d_\nabla df = f^*T_\nabla$. (This is an equality in $\Omega^2_N(f^*TM)$. Make sense of $df \in \Omega^1_N(f^*TM)$. Here 'd_∇' uses the pullback covariant derivative.)

   (d) Suppose ξ, η are vector fields on N. Use them to define sections $\hat{\xi}, \hat{\eta}$ of $f^*TM \to N$. Suppose $[\xi, \eta] = 0$ and $T_\nabla = 0$. Prove $\nabla_\xi \hat{\eta} = \nabla_\eta \hat{\xi}$. This is the identity that was used in lecture in case N is an open subset of an affine plane with affine coordinates s, t and ξ, η are the vector fields $\partial/\partial s, \partial/\partial t$.

2. Let V, W be finite dimensional real vector spaces, $\varphi \colon V \to W$ a C^∞ map all of whose derivatives are bounded. Prove that

   $$L^2_1(\mathbb{R}, V) \longrightarrow L^2_1(\mathbb{R}, W)$$
   $$\gamma \longmapsto \varphi \circ \gamma$$

   is a C^∞ map (between the indicated Sobolev spaces).