In Example (i) there are of course no reductions ($H^2(S^4)$ is zero) and we see a smooth moduli space whose dimension, five, agrees with our index formula. Similarly in Example (ii) we get a five-dimensional space, and this is in line with the index formula since while $\overline{\mathbb{CP}}^2$ has $b_2 = 1$ the positive part b_+ is zero. But in Example (ii) we get a singular point, the vertex of the cone corresponding to the unique reduction $L \oplus L^{-1}$, where $c_1(L)$ is a generator of $H^2(\mathbb{CP}^2)$. Now our general theory says that a neighbourhood of this singular point is modelled on $H_A^1/\Gamma_A = H_A^1/S^1$. But H_A^1 has six real dimensions, by the index formula, and lies wholly in the L^2 part of $\mathfrak{g}_E$ in the splitting $\mathfrak{g}_E = \mathbb{R} \oplus L^2$. So we can regard it as $\mathbb{C}^6$ with the standard circle action (more precisely, Γ_A acts with weight 2). Thus the theory gives the local model $\mathbb{C}^3/S^1$, which is indeed an open cone over $\mathbb{CP}^2$. (Of course our general theory makes no predictions about the global structure of the moduli space.) We can see explicitly in the formula for the connection matrices J_t that J_0 is reducible, involving only the basis element $\mathbf{i}$ of $SU(2)$. This is indeed the standard connection on the Hopf line bundle over $\mathbb{CP}^2$.

Turning to Example (iii), we have now changed orientation so $b_+ = 1$ and we have a ten-dimensional space predicted by the index formula. There are no reductions since the intersection form is positive definite. Similarly in Example (iv) the spaces have no reductions and their dimensions, zero and eight, agree with those given by the index formula for $SO(3)$ bundles.

Example (v) is the most complicated. The dimension is ten as expected, but we again have a reducible solution, corresponding to the quadric Q in our description of the moduli space. The position is summarized by the diagram of $H^2(S^2 \times S^2)$ (Fig. 9).

The reduction corresponds to the class $(1, -1)$ in the standard basis, and this is in the ASD subspace by symmetry between the two factors. Now our deformation complex breaks up into two pieces, corresponding to the terms $\mathbb{R}$ and L^2 in $\mathfrak{g}_E$. The trivial factor contributes cohomology $\mathbb{R}$ to H_A^2, a copy of $H^+(S^2 \times S^2)$, but nothing to H_A^1, since $H^1(S^2 \times S^2) = 0$. On the other hand, as we will see in Section 6.4.3, there is no contribution to H_A^2 from the L^2 factor. So we get in sum a local model $f^{-1}(0)/S^1$ where $f: \mathbb{C}^6 \to \mathbb{R}$. In Chapter 5 we will show that there is a natural decomposition $H_A^1 = \mathbb{C}^3 \times \mathbb{C}^3$ in which a suitable representative f has the form

$$f(z_1, z_2) = |z_1|^2 - |z_2|^2. \tag{4.2.32}$$

To identify a neighbourhood in the moduli space we reverse the complex structure on the second $\mathbb{C}^3$ factor, so $e^{i\theta} \in \Gamma_A$ acts as $e^{2i\theta}$ on the first factor and as $e^{-2i\theta}$ on the second. Now the map $(z_1, z_2) \mapsto z_1 \otimes z_2$ induces a homeomorphism between $f^{-1}(0)/S^1$ and the space of 3×3 complex matrices with rank ≤ 1. It is an interesting exercise to match up this description of a neighbourhood of the singular point with the description in terms of quadrics in Section 4.1.

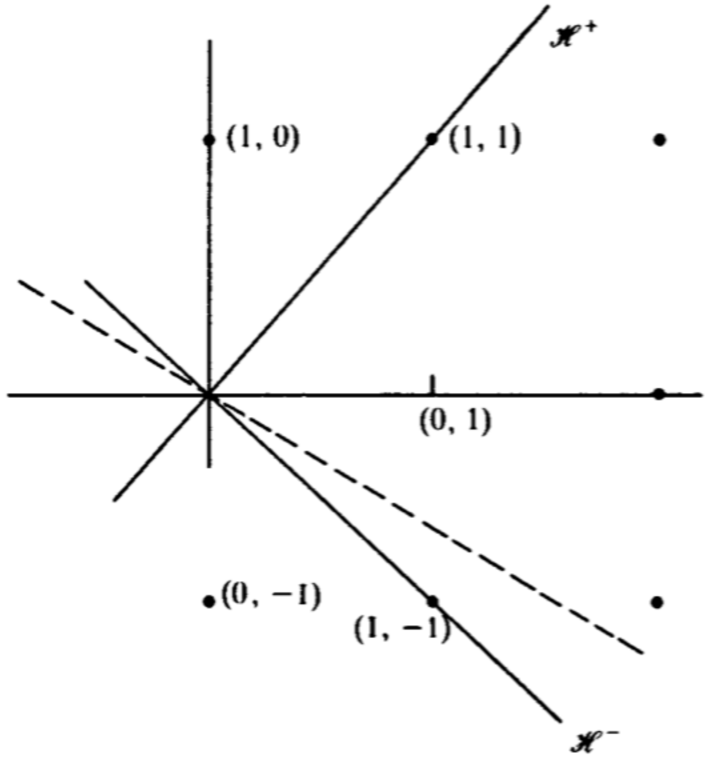

Fig. 9

Notice that this singularity has rather a different nature to that in Example (ii). In the latter case a singular point is present for any metric on the base space, since $b_+ = 0$ and all the classes are represented by purely ASD forms. For $S^2 \times S^2$, by contrast, we can make a small perturbation of the metric under which the reduction in the ASD moduli space disappears. It suffices to take a product metric on round two-spheres with different radii ρ_1, ρ_2, say. Then the ASD subspace is spanned by $(\rho_1^2, -\rho_2^2)$ and avoids the reduction $(1, -1)$. Another interesting exercise is to write down explicit models for the moduli space after such a variation of metric and to see how their topological type changes (cf. Section 4.3.3).

4.3 Transversality

4.3.1 Review of standard theory

We have developed techniques for analysing the local structure of the ASD moduli spaces and tested them against the explicit examples of Section 4.1. In this section we will take the theory further by introducing arguments based on the notion of 'general position'. We have seen that the part of the moduli space M consisting of irreducible connections can be regarded as the zero set of a section Ψ of a bundle $\mathscr{E}$ over the Banach manifold $\mathscr{B}^*$. This depends on a choice of Riemannian metric g on the underlying four-manifold X, so we may write Ψ_g to indicate this dependence. In fact only the conformal class $[g]$ of the metric is relevant, so the abstract picture is that we have a family of equations,

$$\Psi_g([A]) = 0, \qquad (4.3.1)$$

for $[A]$ in $\mathscr{B}$, parametrized by the space $\mathscr{C}$ of all conformal structures on X.

Let us then briefly review some standard properties of 'families of equations', beginning in finite dimensions. The simplest situation to consider is a smooth map $F: P \to Q$ between manifolds of dimension p, q respectively. We can regard this map as a family of equations $F(x) = y$ for $x \in P$, parametrized by $y \in Q$. That is, we are looking at the different fibres of the map F. Recall that a point x in P is called a *regular point* for F if the derivative $(DF)_x$ is surjective, and a point y in Q is a *regular value* for F if all the points in the fibre $F^{-1}(y)$ are regular points. If y is a regular value, the implicit function theorem asserts that the fibre $F^{-1}(y)$ is a smooth submanifold of dimension $p - q$ in P. The well-known theorem of Sard affirms that regular values exist in abundance. Recall that a subset of a topological space is of *second category* if it can be written as a countable intersection of open dense sets. By the Baire category theorem, a second category subset of a manifold is everywhere dense. We state the Sard theorem in two parts:

Proposition (4.3.2). *Let $F: P \to Q$ be a smooth map between finite dimensional manifolds.*

(i) Each point $x \in P$ is contained in a neighbourhood $P' \subset P$ such that the set of regular values of the restriction $F|_{P'}$ is open and dense in Q.

(ii) The regular values of F on P form a second category subset of Q.

For 'most' points y in Q, then, the fibre $F^{-1}(y)$ is a submanifold of the correct dimension $(p - q)$. (For p less than q this is taken to mean that the fibre is empty.) If the map F is proper (e.g. if P is compact) we do not need to introduce the notion of category—the regular values are then open and dense in Q, since if Q' is a compact neighbourhood in Q we can cover $F^{-1}(Q') \subset P$ by a finite number of patches of the form P' as in (4.3.2(i)).

Suppose now that y_0, y_1 are two regular values, so we have two smooth fibres. If the points are sufficiently close together (and the map is, say, proper) these fibres will be diffeomorphic. An extension of the ideas above provides information in the general case when p and q are not close. We assume Q is connected and choose a smooth path $\gamma: [0, 1] \to Q$ between y_0 and y_1. Then we can embed the fibres $F^{-1}(y_0)$, $F^{-1}(y_1)$ in a space:

$$W_\gamma = \{(x, t) \in P \times [0, 1] \mid F(x) = \gamma(t)\}. \qquad (4.3.3)$$

As we shall see in a moment, it is always possible to choose a path γ so that W_γ is a $(p - q + 1)$-dimensional manifold-with-boundary, giving a cobordism between the manifolds $F^{-1}(y_0)$, $F^{-1}(y_1)$. The projection map from W_γ to $[0, 1]$ decomposes the cobordism into a one-parameter family of fibres $F^{-1}(\gamma(t))$. We can think of these as a one-parameter family of spaces

interpolating between $F^{-1}(y_0)$ and $F^{-1}(y_1)$, much as we considered in Section 1.2.3. (We could go on to perturb the projection map slightly to make it a Morse function, so that, as in the proof of the h-cobordism theorem, the fibres change by standard surgeries. But this refinement will not be necessary here.)

We can sum up this discussion, for the family of equations $F(x) = y$ parametrized by $y \in Q$, in the slogan: *for generic parameter values the solutions form a manifold of the correct dimension, and any two such solution sets differ by a cobordism within $P \times [0, 1]$.* The same ideas apply to other 'families of equations', depending on parameters, and in particular, as we shall see, to the ASD equations (4.3.1).

A common framework for the 'general position' arguments that we need is provided by the notion of *transversality*. Let $F: P \to Q$ be a smooth map as above, and R be a third manifold. A smooth map $h: R \to Q$ is said to be *transverse* to F if for all pairs (x, r) in $P \times R$ with $F(x) = h(r)$ the tangent space of Q at $F(x)$ is spanned by the images of $(DF)_x, (Dh)_r$. When this condition holds the set:

$$Z = \{(x, r) \in P \times R \mid F(x) = h(r)\} \qquad (4.3.4)$$

is a smooth submanifold (possibly empty) of $P \times R$, with codimension dim Q.

Transversality is a generic property; any map h can be made transverse to F by a small perturbation. If R is compact we can prove this as follows. We consider a family of maps h_s parametrized by an auxiliary manifold S (which we can take to be a ball in a Euclidean space). Precisely, we have a total map:

$$\underline{h}: R \times S \longrightarrow Q \qquad (4.3.5)$$

and $h_s(r) = \underline{h}(r, s)$. We suppose that there is a base point $s_0 \in S$ such that $h_{s_0} = h$. Suppose we have constructed a family of this form such that $\underline{h}$ is transverse to F. Then the space,

$$\underline{Z} = \{(x, r, s) \in P \times R \times S \mid h_s(r) = F(x)\}, \qquad (4.3.6)$$

is a submanifold of $P \times R \times S$ with a natural projection map $\pi: \underline{Z} \to S$. It is easy to see that the regular values $s \in S$ of π are precisely the parameter values for which h_s is transverse to F. We use Sard's theorem to find a regular value arbitrarily close to s_0, and this gives the desired small, transverse, perturbation of the original map h. The remaining step in the proof of generic transversality is the construction of the transverse family h_s. How best to do this depends on the context. First, suppose that the image space Q is a finite-dimensional vector space U. We can then take S to be a neighbourhood of 0 in U and put

$$h_s(r) = h(r) + s.$$

This clearly has the desired property, since the image of the derivative of h alone is surjective. It may be possible to be more economical; if $V \subset U$ is a linear subspace which generates the cokernel of $(DF)_x + (Dh)_r$ for all $(x, r) \in Z$ we can use these same variations with S a neighbourhood of 0 in V. In general, cover Q with coordinate balls B_i and find a finite cover of R by open sets $R_n (n = 1, \ldots, N)$ with $h(R_n) \subset \frac{1}{2} B_{i(n)}$. Let R'_n be slightly smaller open sets which still cover R and ψ_n be cut-off functions, supported in R_n and equal to 1 on R'_n. Then take

$$S = \prod_{n=1}^{N} \tfrac{1}{2} B_{i(n)} \tag{4.3.7}$$

and

$$h(r, s_1, \ldots, s_N) = h(t) + \psi(t)s_1 + \ldots + \psi_N(t)s_N.$$

Here the 'addition' of $\psi_n(t)s_n$ is done using the coordinates of $B_{i(n)}$.

(If R is not compact we can still find a transverse perturbation of h, using the argument above on successive compact pieces.)

One application of this theory is the proof of the assertion above on the choice of a path $\gamma:[0, 1] \to Q$ such that W_γ is a submanifold. We take $R = [0, 1]$ and $h = \gamma_0$, for any path γ_0 from y_0 to y_1. Then we find a perturbation γ transverse to F. (Note that we can assume that γ has the same end points, since the map is already transverse there.) Other applications are:

(1) If $K \subset Q$ is a countable, locally-finite union of submanifolds whose codimension exceeds dim R then any map $h:R \to Q$ can be perturbed slightly so that its image does not meet K. In fact the locally finite condition may be dropped, but then one needs a rather longer argument, applying the Baire category theorem in the function space of maps from R to Q.

(2) A section Ψ of a vector bundle $V \to P$ can be perturbed so that it is transverse to the zero section. The zero set is then a smooth submanifold of the base space. To fit this into the framework above we can take F to be the inclusion of the zero section in the total space and h to be the section, regarded as a map from P to V. However, in this situation, if x is a zero of Ψ, we shall usually write $(D\Psi)_x$ for the intrinsic derivative mapping $(TP)_x$ to the fibre V_x. The transversality condition is just that $(D\Psi)_x$ be surjective for all points of the zero set. Since this is a situation we shall want to refer to frequently in this book we introduce the following terminology. A point x in the zero set of a section Φ of a vector bundle V will be called a *regular point* of the zero set if $(D\Phi)_x$ is surjective. We say that the zero set is *regular* if all its points are regular points.

In the context (2) of vector bundles we can formulate the construction above of a section with a regular zero set in the following way. Given any section Φ we consider an auxiliary space S and a bundle,

$$\underline{V} \longrightarrow P \times S, \tag{4.3.8}$$

whose restriction to $P \times \{s_0\}$ is identified with V. In fact we may as well assume that $\underline{V}$ is the pull-back of V to the product. We choose S so that there is a section $\underline{\Phi}$ of $\underline{V}$ which agrees with Φ on $P \times \{s_0\}$, and which has a regular zero set $\underline{Z} \subset P \times S$. Then, as before, we apply Sard's theorem to the projection map from $\underline{Z}$ to S. A regular value s of the projection map yields a perturbation $\Phi_s = \underline{\Phi}|_{P \times \{s\}}$ of Φ, having a regular zero set in P.

4.3.2 The Fredholm case

Transversality theory in finite dimensions does not go over wholesale to the infinite-dimensional setting of smooth maps between Banach manifolds, but to a large extent it does extend to situations where the linear models are Fredholm operators. We begin with the extension, due to Smale, of the Sard theorem. Let $F:\mathscr{P} \to \mathscr{Q}$ be a smooth Fredholm map between paracompact Banach manifolds, and let x be a point of $\mathscr{P}$. We can choose a coordinate patch $\mathscr{P}' \subset \mathscr{P}$ containing x, and a coordinate system so that F is represented, in a neighbourhood of x, by a map:

$$(\xi, \eta) \longmapsto (L(\xi), \alpha(\xi, \eta)),$$

as in (4.2.19), with L a linear isomorphism between Banach spaces and $\alpha: U_0 \times \mathbb{R}^p \to \mathbb{R}^q$. A point (ζ, θ) is a regular value of $F|_{\mathscr{P}'}$ if and only if θ is a regular value for the finite dimensional map,

$$f_\zeta = \alpha|_{L^{-1}(\zeta)}: \mathbb{R}^p \longrightarrow \mathbb{R}^q.$$

It follows easily from the ordinary Sard theorem that the regular values for the restriction of F to a small coordinate patch $\mathscr{P}'$ are open and dense in $\mathscr{Q}$. The Baire category theorem applies equally well to Banach manifolds so, taking a countable cover of $\mathscr{P}$, we obtain the Smale–Sard theorem:

Proposition (4.3.8). *If $F:\mathscr{P} \to \mathscr{Q}$ is a smooth Fredholm map between paracompact Banach manifolds, the regular values of F are of second category, hence everywhere dense in $\mathscr{Q}$.*

If $\mathscr{P}$ is connected then for any such regular value $y \in \mathscr{Q}$ the fibre $F^{-1}(y) \subset \mathscr{P}$ is a smooth submanifold of dimension

$$\dim F^{-1}(y) = \text{ind } F. \tag{4.3.9}$$

Similarly we have a Fredholm transversality theorem:

Proposition (4.3.10). *If $F:\mathscr{P} \to \mathscr{Q}$ is a Fredholm map, as in (4.3.8), and $h:R \to \mathscr{Q}$ is a smooth map from a finite-dimensional manifold R, there is a map $h':R \to \mathscr{Q}$, arbitrarily close to h (in the topology of C^∞ convergence on compact sets) and transverse to F. If h is already transverse to F on a closed subset $G \subset T$ we can take $h' = h$ on G.*

Proof. The proof is much as before. There are two possible approaches. For the first we suppose initially that R is compact. Then the construction we gave for the transverse family h is valid, except that now we must use the more economical, finite-rank, perturbations which suffice to generate the cokernels. Then we can take S again to be finite-dimensional. We obtain a finite-dimensional manifold $\underline{\mathscr{Z}} \subset \mathscr{P} \times R \times S$ and apply the ordinary Sard theorem as before. Then we handle the general case by writing S as a union of compact sets. For the second approach, we work with an infinite set of balls $B_{i(n)}$ and use an infinite-dimensional space S, replacing the product in (4.3.7) by the space of bounded sequences. Then S is a Banach manifold and the projection $\pi:\underline{\mathscr{Z}} \to S$ is Fredholm, with index $\operatorname{ind}(\pi) = \operatorname{ind}(g) + \dim R$. We use (4.3.8) to find a regular value s of π and hence a transverse perturbation h_s.

Our main application of the Fredholm theory will be to sections of vector bundles. Suppose that $\mathscr{V} \to \mathscr{P}$ is a bundle of Banach spaces over a Banach manifold, and Φ is a Fredholm section of $\mathscr{V}$, i.e. represented by Fredholm maps in local trivializations of $\mathscr{V}$. We would like to perturb Φ to find a section with a regular zero set. We cannot now proceed directly to apply (4.3.10) since the hypotheses will not be satisfied if $\mathscr{P}$ is infinite-dimensional. We can however apply the same scheme to analyse perturbations. Following the notation at the end of Section 4.3.2 we consider a bundle $\underline{\mathscr{V}} = \pi_1^*(\mathscr{V}) \to \mathscr{P} \times S$, where S is an auxiliary Banach manifold with base point s_0. Let $\underline{\Phi}$ be a section of $\mathscr{V}$, extending Φ, which is Fredholm in the $\mathscr{P}$ variable. That is, in local trivializations $\underline{\Phi}$ is represented by smooth maps to the fibre whose partial derivatives in the $\mathscr{P}$ factor are Fredholm. For s in S we regard the restriction of Φ to $\mathscr{P} \times \{s\}$ as another section Φ_s of $\mathscr{V}$.

Proposition (4.3.11). *If the zero set $\underline{\mathscr{Z}} \subset \mathscr{P} \times S$ is regular then there is a dense (second category) set of parameters $s \in S$ for which the zero sets of the perturbations Φ_s are regular.*

This follows immediately from (4.3.8), applied to the projection map from $\underline{\mathscr{Z}}$ to S, as before. Notice that, as in our first proof of (4.3.10), if it is possible to choose S to be finite-dimensional then we only need the 'ordinary' Sard theorem. We will return to discuss the construction of such families $\underline{\Phi}$, in the abstract setting, in Section 4.3.6.

4.3.3 Applications to moduli spaces

We will now apply this theory to the ASD equations and state our main results. We will defer the proofs of the main assertions, which involve more detailed differential-geometric considerations, to Sections 4.3.4 and 4.3.5. The main results were first proved by Freed and Uhlenbeck and our treatment is not fundamentally different from theirs. Throughout this section we let X be a compact, simply connected, oriented four-manifold. We will use the terminology introduced at the end of Section 4.3.1, so (with a given metric) an irreducible ASD connection A is called *regular* if $H_A^2 = 0$ and we call a moduli space regular if all its irreducible points are regular points. Of course, a regular moduli space of irreducible connections is a smooth manifold of dimension given by the index $s = s(E)$. But the converse is not true; it may happen that the moduli space is homeomorphic to a smooth manifold of the correct dimension, but is not regular. (We will see an example of this in Chapter 10). The regularity condition is equivalent to the condition that as a ringed space the moduli space should be a manifold.

We begin by discussing the natural parameter space in the set-up, the space $\mathscr{C}$ of conformal structures on X. At one point we will want to apply Banach manifold results to this space, so we agree henceforth to work with C^r metrics on X for some fixed large r ($r = 3$ will do). The space $\mathscr{C}$ is the quotient of these metrics by the C^r conformal changes. It is easy to see that $\mathscr{C}$ is naturally a Banach manifold. We can use the construction of Section 1.1.5 to obtain a set of handy charts on $\mathscr{C}$. Given one conformal structure $[g_0] \in \mathscr{C}$ with $\pm$ self-dual subspaces Λ^+, Λ^-, the space $\mathscr{C}$ is naturally identified with the space of C^r maps,

$$m: \Lambda^- \longrightarrow \Lambda^+$$

with $|m_x| < 1$ for all $x \in X$. In particular the tangent space of $\mathscr{C}$ at the given point is naturally identified as:

$$(T\mathscr{C})_{g_0} \cong \operatorname{Hom}(\Lambda^-, \Lambda^+). \tag{4.3.12}$$

We will now consider the abelian reductions in our moduli spaces. We have seen in (2.2.6) that a line bundle $L \to X$ admits an ASD connection if and only if $c_1(L)$ can be represented by an anti-self-dual harmonic form. If we identify $H^2(X; \mathbf{R})$ with the space of harmonic two-forms we have a decomposition $H^2(X; \mathbf{R}) = \mathscr{H}^+ \oplus \mathscr{H}^-$; the condition for an ASD connection is that $c_1(L)$ lies in $\mathscr{H}^-$. If the intersection form of X is negative definite, so $\mathscr{H}^+ = 0$, this is no restriction—any line bundle carries an ASD connection, for any metric on X. If $b^+(X)$ is non-zero on the other hand we see a marked difference—the space $\mathscr{H}^-$ is then a proper subspace of $H^2(X)$ and we would expect that generically it meets the integer lattice $H^2(X; \mathbf{Z}) \subset H^2(X; \mathbf{R})$ only at zero. We introduce some notation. Let Gr be the Grassmann manifold of b^--dimensional subspaces of $H^2(X; \mathbf{R})$ and $U \subset \mathrm{Gr}$ be the open subset of maximal negative subspaces, with respect to the intersection form. So the assignment of the space $\mathscr{H}^-(g)$ of ASD harmonic forms to a conformal class gives a map:

$$P: \mathscr{C} \longrightarrow U. \tag{4.3.13}$$

Now suppose that c is a class in $H^2(X; \mathbf{Z})$ with $c.c < 0$ and define

$$N_c \subset U = \{\mathscr{H}^- \mid c \in \mathscr{H}^-\}. \tag{4.3.14}$$

It is easy to see that N_c is a submanifold of codimension $b^+(X)$ in U. Our main result here is: