Appendix A

Curve Spaces and Banach Bundles

In this chapter we present a brief introduction of Banach manifolds and associated Banach bundles which form the foundation of the entire analytic approach to Morse homology chosen in this monograph. We discuss explicitly the structure of an infinite-dimensional Banach manifold defined on curve spaces as was stated in Proposition 2.7. The first section of this chapter deals with these manifolds themselves. The second section comprises an analysis of certain Banach bundles on these manifolds. We shall partly use notations and schemes of proofs taken from [Eli]. However, it is quite important to point out that our more specialized situation differs from that in [Eli] with respect to the non-compactness of the curves' domain $\mathbb{R}$. There are several steps in the discussion where we need a more refined treatment.

A.1 The Manifold of Maps $\mathcal{P}^{1,2}_{x,y}(\mathbb{R}, M)$

Definition A.1 *A covariant functor*

$$\mathfrak{S} : \mathrm{Vec}_{C^\infty}(\overline{\mathbb{R}}) \to \mathrm{Ban}$$

is called a section functor if it associates to each smooth vector bundle ξ on $\overline{\mathbb{R}}^1$ a vector space $\mathfrak{S}(\xi)$ of sections in ξ together with a Banach space topology, so that $\mathfrak{S}$ maps each smooth bundle homomorphism $\varphi : \xi \to \eta$ to a linear map $\mathfrak{S}_*\varphi \in \mathcal{L}(\mathfrak{S}(\xi); \mathfrak{S}(\eta))$ defined by $(\mathfrak{S}_*\varphi) \cdot s = \varphi \cdot s$, and

$$\mathfrak{S}_* : C^\infty(\mathrm{Hom}(\xi, \eta)) \to \mathcal{L}(\mathfrak{S}(\xi); \mathfrak{S}(\eta))$$

[1]endowed with the already defined differentiable structure

is continuous.

Auxiliary Proposition A.2 *The mappings* $H^{1,2}_{\overline{\mathbb{R}}}, L^2_{\overline{\mathbb{R}}} : \mathrm{Vec}_{C^\infty}(\overline{\mathbb{R}}) \to \mathrm{Ban}$ *defined in Definition 2.5 are section functors.*

Proof. We have already verified that the Banach space topology on $H^{1,2}_{\overline{\mathbb{R}}}(\xi)$ and $L^2_{\overline{\mathbb{R}}}(\xi)$ is independent of the respectively chosen trivialization ϕ. Each pair of norms $\|\cdot\|^{\phi}_{1,2}$ and $\|\cdot\|^{\phi}_{0,2}$ induced by such trivializations from the canonically normed vector spaces $H^{1,2}(\mathbb{R},\mathbb{R}^n)$ and $L^2(\mathbb{R},\mathbb{R}^n)$, respectively, are equivalent, because ϕ is equipped with asymptotical differentiability. Hence, we may start without loss of generality from the trivial bundle $\xi = \overline{\mathbb{R}} \times \mathbb{R}^n$, i.e.

$$H^{1,2}_{\overline{\mathbb{R}}}(\xi) = H^{1,2}(\mathbb{R},\mathbb{R}^n) \quad \text{and} \quad L^2_{\overline{\mathbb{R}}}(\xi) = L^2(\mathbb{R},\mathbb{R}^n) \ .$$

Now let $A \in C^\infty\big(\mathrm{Hom}(\xi,\eta)\big)$, that is, without loss of generality, $A \in C^\infty\big(\overline{\mathbb{R}}, \mathrm{M}(m \times n, \mathbb{R})\big)$. Then the following estimates hold for $s \in H^{1,2}$ and respectively L^2, where we denote $(As)(t) = A(t)\cdot s(t)$:

$$\text{(A.1)} \qquad \|As\|_{0,2} \leqslant \|A\|_\infty \cdot \|s\|_{0,2}$$

$$\begin{aligned} \|(As)'\|^2_{0,2} &= \|A's + As'\|^2_{0,2} \\ &= \int_{\mathbb{R}} \big(|A's|^2 + |As'|^2 + 2\langle A's, As' \rangle \big)\, dt \\ &\leqslant 2\Big(\|A's\|^2_{0,2} + \|As'\|^2_{0,2} \Big) \\ &\leqslant 2\Big(\|A'\|^2_{0,2}\, \|s\|^2_\infty + \|A\|^2_\infty\, \|s'\|^2_{0,2} \Big) \ . \qquad \text{(A.2)} \end{aligned}$$

According to Lemma 2.2, $\overline{\mathbb{R}}$-differentiability yields the finite norms $\|A'\|_{0,2}$, $\|A\|_\infty < \infty$. Moreover, the estimate

$$\text{(A.3)} \qquad \|s\|_\infty \leqslant \|s\|_{1,2} \quad \text{for all } s \in H^{1,2}(\mathbb{R},\mathbb{R}^n) \ .$$

follows from the simple calculation

$$\begin{aligned} \Big| |s(t_1)|^2 - |s(t_0)|^2 \Big| &= \Big| \int_{t_0}^{t_1} \frac{d}{dt} |s(t)|^2\, d\tau \Big| \\ = \Big| \int_{t_0}^{t_1} 2\big\langle s(\tau), \dot{s}(\tau) \big\rangle\, d\tau \Big| &\leqslant \int_{t_0}^{t_1} \Big(|s(\tau)|^2 + |\dot{s}(\tau)|^2 \Big)\, d\tau \\ &\leqslant \|s\|^2_{1,2} \ . \end{aligned}$$

Thus (A.1) and (A.2) give rise to the estimate

$$\|As\|^2_{1,2} \leqslant \mathrm{const}\Big(\|A'\|^2_{0,2} + \|A\|^2_\infty \Big)\, \|s\|^2_{1,2} \ ,$$

that is

$$\|A\|_{\mathcal{L}(H^{1,2};H^{1,2})} \leqslant \text{const} \sqrt{\|A'\|^2_{0,2} + \|A\|^2_\infty} \ . \tag{A.4}$$

Consequently, the mapping $\mathfrak{S}_* : C^\infty(\mathrm{Hom}(\xi,\eta)) \to \mathcal{L}(\mathfrak{S}(\xi); \mathfrak{S}(\eta))$ is continuous. □

Remark It is worth mentioning that the estimate (A.4) with $\|A'\|_{0,2} < \infty$ relies essentially on the special choice of the differentiable structure on $\overline{\mathbb{R}}$!

Corollary A.3 *As to the section functor* $\mathfrak{S} = H^{1,2}_{\mathbb{R}}$*, the estimate* $\|A\|_\infty \leqslant \|A\|_{1,2}$ *for* $A \in H^{1,2}(\mathbb{R}, \mathrm{M}(m \times n, \mathbb{R}))$ *implies the continuity of the map*

$$\mathfrak{S}_* : \mathfrak{S}(\mathrm{Hom}(\xi,\eta)) \to \mathcal{L}(\mathfrak{S}(\xi); \mathfrak{S}(\eta)) \ ,$$

which is stronger than the mere section functor property.

The next step provides us with properties of this special section functor, which correspond to the conditions of a so-called manifold model in [Eli].

Auxiliary Proposition A.4 *The section functor* $\mathfrak{S} = H^{1,2}_{\mathbb{R}}$ *on* $\mathrm{Vec}_{C^\infty}(\overline{\mathbb{R}})$ *has the properties*

(a) $\mathfrak{S}(\xi) \hookrightarrow C^0(\xi)$ *is continuous for each* $\xi \in \mathrm{Vec}_{C^\infty}(\overline{\mathbb{R}})$ *and*

(b) $\begin{array}{rcl} \mathfrak{S}(\mathrm{Hom}(\xi,\eta)) & \to & \mathcal{L}(\mathfrak{S}(\xi); \mathfrak{S}(\eta)) \\ A & \mapsto & (A_* : s \mapsto As) \end{array}$ *is continuous*

for all $\xi, \eta \in \mathrm{Vec}_{C^\infty}(\overline{\mathbb{R}})$.

(c) Given $\xi, \eta \in \mathrm{Vec}_{C^\infty}(\overline{\mathbb{R}})$ *and an open subset* $\mathcal{O} \subset \xi$ *such that there is a section* $\gamma \in C^0(\xi)$ *with compact support in* $\mathbb{R}$ *and* $\gamma(\overline{\mathbb{R}}) \subset \mathcal{O}$*, each smooth bundle map* $f \in C^\infty(\mathcal{O}, \eta)$ *satisfying* $f(0_{\pm\infty}) = 0_{\pm\infty}$ *induces a well-defined and continuous map on* $\mathfrak{S}(\mathcal{O}) = \{ s \in \mathfrak{S}(\xi) \mid s(\overline{\mathbb{R}}) \subset \mathcal{O} \}$ [2]

$$\begin{array}{rrcl} f_* : & \mathfrak{S}(\mathcal{O}) & \to & \mathfrak{S}(\eta) \\ & s & \mapsto & f \circ s \ . \end{array}$$

Proof. (a) has been already proved by (A.3) and (b) follows likewise from (A.3) together with (A.4).

As to (c): Since f is a smooth bundle map, its fiber restrictions f_t are in particular Lipschitz continuous, uniformly in $t \in \mathbb{R}$. Thus, with respect to any trivialization of ξ, we obtain the estimates

$$|\, f_t(x_t) - f_t(y_t) \,| \leqslant \text{const} \ | \, x_t - y_t \, | \quad \text{for all } t \in \overline{\mathbb{R}} \tag{A.5}$$

2Note that this subset of sections is open within $\mathfrak{S}(\xi)$ due to item (a).

and

$$\left|f_t(s(t))\right| \leqslant \text{const}\left|s(t)\right| + \left|f_t(0)\right| \ .$$

Since $f_t(0) \in C^\infty(\overline{\mathbb{R}}, \eta)$, this estimate together with Corollary 2.4 implies that the map $f_* : H^{1,2}_{\mathbb{R}}(\mathcal{O}) \to H^{1,2}_{\mathbb{R}}(\eta)$ is well-defined. The continuity of f_* with respect to the section functor $\mathfrak{S} = H^{1,2}_{\mathbb{R}}$ is obtained from (A.5) together with an analogous uniform estimate involving first derivatives of the $\overline{\mathbb{R}}$-smooth bundle map f. □

Now these properties enable us to deduce the crucial lemma concerning the construction of the manifold of maps within our framework.

Fundamental Lemma A.5 *Let $\mathfrak{S}, \mathcal{O} \subset \xi$ and $f \in C^\infty(\mathcal{O}, \eta)$ be as in the auxiliary Proposition A.4. Then the map $f_* : \mathfrak{S}(\mathcal{O}) \to \mathfrak{S}(\eta)$ is smooth and the k-th derivative is given by $D^k f_*(s) = \mathfrak{S}_*(F^k f \circ s)$, which is well-defined. Here, $F^k f : \mathcal{O} \to \mathrm{Hom}(\xi \oplus \ldots \oplus \xi\,; \eta)$ denotes the k-th fibre derivative of f.*

Proof. We prove the lemma by induction on k.

$k = 0$: The continuity of f_* has been already verified as item (c) in the auxiliary Proposition A.4.

$k = 1$: Let $\mathcal{O}$ be fibrewise convex without loss of generality and let $s_0 \in \mathfrak{S}(\mathcal{O})$ be fixed. Then, for $x, y \in \mathcal{O}$ from the same fibre, i.e. $\pi(x) = \pi(y)$, we define

$$\text{(A.6)} \qquad \begin{aligned} &\Theta : \mathcal{O} \oplus \mathcal{O} \to \mathrm{Hom}(\xi, \eta) \\ &\Theta(x, y) \cdot z = \left[\int_0^1 Ff\big(x + t(y - x)\big)dt - Ff(x)\right] \cdot z \ . \end{aligned}$$

Since $Ff : \mathcal{O} \to \mathrm{Hom}(\xi, \eta)$ is smooth and fibre respecting, the same is true for Θ. Moreover, we easily verify the equations

$$\text{(A.7)} \quad \Theta(x, y) \cdot (y - x) = f(y) - f(x) - Ff(x) \cdot (y - x), \;\; \Theta(0, 0) = 0 \ .$$

The section functor $\mathfrak{S}$ with properties (b) and (c) from the auxiliary Proposition A.4 gives rise to the composition of the continuous maps

$$\mathfrak{S}(\Theta) : \mathfrak{S}(\mathcal{O}) \oplus \mathfrak{S}(\mathcal{O}) \to \mathfrak{S}\big(\mathrm{Hom}(\xi, \eta)\big) \to \mathcal{L}\big(\mathfrak{S}(\xi); \mathfrak{S}(\eta)\big) \ ,$$

so that

$$\text{(A.8)} \qquad \mathfrak{S}(\Theta)\,(s_0, s_0) = 0 \quad \text{and} \quad \lim_{s \to s_0} \mathfrak{S}(\Theta)\,(s_0, s) = 0 \ .$$

Since (A.7) implies the identity

$$f_*(s) - f_*(s_0) - \mathfrak{S}_*(Ff \circ s_0) \cdot (s - s_0) = \mathfrak{S}(\Theta)\,(s_0, s) \cdot (s - s_0) \ ,$$

the map f_* is differentiable at s_0 with the representation of the differential

$$Df_*(s_0) = \mathfrak{S}_*(Ff \circ s_0) \in \mathcal{L}\big(\mathfrak{S}(\xi); \mathfrak{S}(\eta)\big) \ . \tag{A.9}$$

It should be mentioned that, due to (A.4), this is also well-defined if it holds that $Ff(\pm\infty, 0) \neq 0$, i.e. $Ff \circ s_0 \notin \mathfrak{S}\big(\mathrm{Hom}(\xi, \eta)\big)$. The continuity of $Df_* : \mathfrak{S}(\mathcal{O}) \to \mathcal{L}\big(\mathfrak{S}(\xi); \mathfrak{S}(\eta)\big)$ follows from the continuity of the mapping

$$\begin{array}{rcl} \mathfrak{S}(\mathcal{O}) & \to & \mathfrak{S}\big(\mathrm{Hom}(\xi, \eta)\big) \subset \mathcal{L}\big(\mathfrak{S}(\xi); \mathfrak{S}(\eta)\big) \\ s & \mapsto & Ff \circ s - Ff \circ s_0 \ , \end{array}$$

because

$$\begin{array}{rcl} \vartheta : \mathcal{O} & \to & \mathrm{Hom}(\xi, \eta) \\ (t, x) & \mapsto & Ff(t, x) - Ff\big(t, s_0(t)\big) \end{array}$$

is a smooth,[3] fibre respecting map, and it satisfies the condition $\vartheta(\pm\infty, 0) = (\pm\infty, 0)$ as with (c) in the auxiliary Proposition A.4.

$k \rightsquigarrow k+1$: Let us now start from the k-times continuously differentiable map f_* together with

$$D^k f_*(s_0) = \mathfrak{S}_*(F^k f \circ s_0) \in \mathcal{L}\big(\mathfrak{S}(\xi), \dots, \mathfrak{S}(\xi); \mathfrak{S}(\eta)\big)$$

and let us denote by $f^{(k)}$ the map

$$\begin{array}{l} f^{(k)} : \mathcal{O} \to \mathrm{Hom}(\xi \oplus \dots \oplus \xi, \eta) \\ f^{(k)}(t, \xi) = F^k f(t, \xi) - F^k f\big(t, s_0(t)\big) \ . \end{array}$$

Then $f^{(k)}$ again satisfies the inital condition on f, namely in particular

$$f^{(k)}(\pm\infty, 0) = 0$$

with $\eta_{\mathrm{new}} = \mathrm{Hom}(\xi \oplus \dots \oplus \xi, \eta)$. Thus, the step '$k = 1$' yields the continuous differentiability of $f^{(k)}{}_* : \mathfrak{S}(\mathcal{O}) \to \mathcal{L}\big(\mathfrak{S}(\xi), \dots, \mathfrak{S}(\xi); \mathfrak{S}(\eta)\big)$ together with

$$Df^{(k)}{}_*(\tilde{s}_0) = \mathfrak{S}_*\big(Ff^{(k)} \circ \tilde{s}_0\big) = \mathfrak{S}_*\big(F^{k+1} f \circ \tilde{s}_0\big) \ .$$

□

We now assume (M, g) to be a paracompact, Riemannian C^∞-manifold together with the associated exponential map. Furthermore, let $\mathcal{D}$ denote an open zero section neighbourhood within the tangent bundle $\tau : TM \to M$ such that

$$\begin{array}{rcl} \mathcal{D} & \xrightarrow{\approx} & V(\Delta) \subset M \times M \\ \xi & \mapsto & \big(\tau(\xi), \exp(\xi)\big) \end{array} \tag{A.10}$$

3 Here, without loss of generality, $s_0 \in C^\infty(\xi)$.

represents a diffeomorphism onto a diagonal neighbourhood within $M \times M$. We observe that smooth, compact curves $h \in C^\infty(\overline{\mathbb{R}}, M)$ give rise to the pull-back bundles

$$h^*TM = \{ (t,\xi) \in \overline{\mathbb{R}} \times TM \,|\, \tau(\xi) = h(t) \} \in \mathrm{Vec}_{C^\infty}(\overline{\mathbb{R}}) \ ,$$

on which the above section functor $H^{1,2}_{\mathbb{R}}$ is well-defined. From now on we shall again use the notation $\mathfrak{S}$ for this section functor. We obviously obtain from $\mathcal{D}$ the open zero section neighbourhoods

$$h^*\mathcal{D} = \{ (t,\xi) \in \overline{\mathbb{R}} \times \mathcal{D} \,|\, \tau(\xi) = h(t) \} \overset{\text{open}}{\subset} h^*TM \ .$$

Definition A.6 *Starting from* $s \in \mathfrak{S}(h^*\mathcal{D}) \subset C^0(h^*\mathcal{D})$, *we define the continuous curve*

$$\exp_h s = \exp \circ s \in C^0(\overline{\mathbb{R}}, M), \ (\exp \circ s)(t) = \exp_{h(t)} s(t) \ .$$

Thus, provided any fixed endpoints $x, y \in M$, *the set of curves*

$$\mathcal{P}^{1,2}_{x,y}(\mathbb{R}, M) = \{ \exp \circ s \in C^0(\overline{\mathbb{R}}, M) \,|\, s \in \mathfrak{S}(h^*\mathcal{D}),\ h \in C^\infty_{x,y}(\overline{\mathbb{R}}, M) \}$$

is well-defined.

Theorem 10 $\mathcal{P}^{1,2}_{x,y}(\mathbb{R}, M)$ *is equipped with the differentiable structure of an infinite-dimensional Banach manifold. The family*

$$\{ \mathfrak{S}(h^*\mathcal{D}),\, \mathfrak{S}(\exp_h) \}_{h \in C^\infty_{x,y}(\overline{\mathbb{R}}, M)} \,,$$

$$\text{where} \quad \mathfrak{S}(\exp_h) : \begin{array}{ll} \mathfrak{S}(h^*\mathcal{D}) & \to \mathcal{P}^{1,2}_{x,y} \\ s & \mapsto \exp \circ s \end{array} \ ,$$

represents an associated atlas of charts. Moreover, the Banach manifold $\mathcal{P}^{1,2}_{x,y}$ *together with this differentiable structure is independent of the Riemannian metric* g *on* M.

Proof. We consider the following smooth mappings with respect to $h \in C^\infty_{x,y}(\overline{\mathbb{R}}, M)$:

$$\text{(A.11)} \qquad \begin{array}{rcl} \phi_h : h^*\mathcal{D} & \to & \overline{\mathbb{R}} \times M \\ (t,\xi) & \mapsto & (t, \exp_{h(t)} \xi) \ . \end{array}$$

According to the choice of $\mathcal{D}$, this is an embedding onto an open neighbourhood of the graph of h. We consequently define

$$\begin{array}{rl} \mathcal{U}_h & = \phi_h(h^*\mathcal{D}) \subset \overline{\mathbb{R}} \times M \ , \\ \mathfrak{S}(\mathcal{U}_h) & = \{ g \in C^0(\overline{\mathbb{R}}, M) \,|\, \mathrm{graph}\, g \subset \mathcal{U}_h,\ \phi_h^{-1} \circ (\mathrm{id}, g) \in \mathfrak{S}(h^*\mathcal{D}) \} \\ & \subset C^0_{x,y}(\overline{\mathbb{R}}, M) \end{array}$$

and

$$\mathfrak{S}(\phi_h^{-1}) : \begin{array}{ll} \mathfrak{S}(\mathcal{U}_h) & \to \mathfrak{S}(h^*\mathcal{D}) \\ g & \mapsto \phi_h^{-1} \circ (\mathrm{id}, g) \end{array} . \tag{A.12}$$

It holds that

$$\mathcal{P}^{1,2}_{x,y}(\mathbb{R}, M) = \bigcup_{h \in C^\infty_{x,y}(\overline{\mathbb{R}}, M)} \mathfrak{S}(\mathcal{U}_h) ,$$

and that $\{ \mathfrak{S}(\phi_h^{-1}) \}_{h \in C^\infty_{x,y}(\overline{\mathbb{R}}, M)}$ is a family of one-to-one correspondences. Let us now endow this curve space with a topology such that these bijections become homeomorphisms. The fact that this topology is well-defined is particularly due to the following proof of the diffeomorphism property for changes of the charts $\mathfrak{S}(\phi_f^{-1}) \circ \mathfrak{S}(\phi_h^{-1})^{-1}$.

Let us choose any two supporting curves $f, h \in C^\infty_{x,y}(\overline{\mathbb{R}}, M)$ with $\mathcal{U}_h \cap \mathcal{U}_f \neq \emptyset$. Then $\mathcal{O}_h = \phi_h^{-1}(\mathcal{U}_h \cap \mathcal{U}_f) \subset h^*\mathcal{D}$ and $\mathcal{O}_f = \phi_f^{-1}(\mathcal{U}_h \cap \mathcal{U}_f) \subset f^*\mathcal{D}$ are open subsets satisfying the condition in (c) in the auxiliary Proposition A.4. The non-void intersection $\mathfrak{S}(\mathcal{U}_h) \cap \mathfrak{S}(\mathcal{U}_f)$ is open within $\mathfrak{S}(\mathcal{U}_h)$ and $\mathfrak{S}(\mathcal{U}_f)$, respectively. We now consider the mapping

$$\begin{aligned} \Phi_{fh} &= \phi_f^{-1} \circ \phi_h : h^*TM \supset \mathcal{O}_h \to \mathcal{O}_f \subset f^*TM \\ \Phi_{fh}(t, \xi) &= \left(t, \exp^{-1}_{f(t)}(\exp_{h(t)} \cdot \xi) \right) , \end{aligned} \tag{A.13}$$

which describes the change of the charts. It fulfills the condition $\Phi_{fh}(\pm\infty, 0) = 0$ as we can see from the fixed endpoints $h(\pm\infty) = f(\pm\infty) = x, y$. Thus, the fundamental Lemma A.5 states that

$$\Phi_{fh*} = \mathfrak{S}(\phi_f^{-1}) \circ \mathfrak{S}(\phi_h^{-1})^{-1} : \mathfrak{S}(\mathcal{O}_h) \to \mathfrak{S}(\mathcal{O}_f)$$

is a smooth map with

$$\Phi_{fh_*}^{-1} = \Phi_{hf*}$$

as its inverse, hence the diffeomorphism property.

The compatibility of two atlases belonging to different Riemannian metrics follows from the fact that $C^\infty_{x,y}(\overline{\mathbb{R}}, M)$ lies dense within $C^0_{x,y}(\overline{\mathbb{R}}, M)$ and that, for any arbitrary metric g, the exponential $\exp_g$ represents a local diffeomorphism in the sense of (A.10). □

Supplement Moreover, we are able to find a countable subset of $C^\infty_{x,y}(\overline{\mathbb{R}}, M)$ which lies likewise dense within $C^0_{x,y}(\overline{\mathbb{R}}, M)$. This enables us to reduce the atlas given in Theorem 10 to a countable subatlas.

Corollary A.7 *Given Riemannian C^∞-manifolds M and N together with a map $f \in C^\infty(M, N)$,*

$$\begin{array}{rcl} \mathfrak{S}(f) : \mathcal{P}^{1,2}_{x,y}(\mathbb{R}, M) & \to & \mathcal{P}^{1,2}_{f(x),f(y)}(\mathbb{R}, N) \\ \gamma & \mapsto & f \circ \gamma \end{array}$$

is a well-defined, smooth map between Banach manifolds.

Proof. First, any fixed $\gamma \in \mathcal{P}^{1,2}_{x,y}$ is mapped to $f \circ \gamma \in C^0_{f(x),f(y)}(\overline{\mathbb{R}}, N)$. Since $C^\infty_{f(x),f(y)}(\overline{\mathbb{R}}, N)$ lies dense within $C^0_{f(x),f(y)}(\overline{\mathbb{R}}, N)$, we find a representation of the shape

$$f \circ \gamma = \exp_k t,\ k \in C^\infty_{f(x),f(y)}(\overline{\mathbb{R}}, N),\ t \in C^0(k^*\mathcal{D}_N)\ .$$

Secondly, we may express γ as

$$\gamma = \exp_h s,\ h \in C^\infty_{x,y}(\overline{\mathbb{R}}, M),\ s \in \mathfrak{S}(h^*\mathcal{D}_M)\ .$$

Furthermore, we find an open neighbourhood $\widetilde{\mathcal{O}}_h \subset h^*\mathcal{D}_M$, such that the conditions in (c) in the auxiliary Proposition A.4 are satisfied for the smooth fibre respecting map

$$\psi^f_{kh} = \exp_k^{-1} \circ f \circ \exp_h : \widetilde{\mathcal{O}}_h \to k^*\mathcal{D}_N\ ,$$
$$\psi^f_{kh}(\pm\infty, 0) = (\pm\infty, 0)\ .$$

Hence it is due to the fundamental Lemma A.5, that $\psi^f_{kh*} : \mathfrak{S}(\widetilde{\mathcal{O}}_h) \to \mathfrak{S}(k^*\mathcal{D}_N)$ is a smooth map. This proves the corollary. □

A.2 Banach Bundles on $\mathcal{P}^{1,2}_{x,y}(\mathbb{R}, M)$

We henceforth use the notation $\mathfrak{S}$ for the section functor $H^{1,2}_{\mathbb{R}}$. Let us fix a Riemannian metric g on the manifold M, as was required in the last section in order to define the atlas on $\mathcal{P}^{1,2}_{x,y}$. Given such a g we are provided canonically with the following features known from Riemannian geometry:

Let

$$K : T(TM) \to TM,\ \tau : TM \to M$$

denote the unique Levi-Civita connection and the canonical projection in the tangent bundle of M. As a consequence, K and $D\tau : T(TM) \to TM$ yield a decomposition of $T(TM)$ into a horizontal bundle and a vertical bundle:[4]

$$\begin{aligned} T_{\xi,v}(TM) &= \ker\left(D\tau(\xi)\right) \\ T_{\xi,h}(TM) &= \ker\left(K(\xi)\right) \end{aligned}\ ,\ \xi \in TM\ . \tag{A.14}$$

4Note that the vertical bundle comes canonically and independent of any specification of a Riemannian metric.

Referring to the exponential map associated to g and K, $\exp : \mathcal{D} \to M$, we obtain isomorphisms at any $\xi \in \mathcal{D}$, where $\mathcal{D}$ once again denotes the injectivity neighbourhood associated to exp:

$$\text{(A.15)}\quad \begin{aligned} \nabla_1 \exp(\xi) &= D\exp(\xi) \circ \left(D\tau_{|T_{\xi,h}(TM)}\right)^{-1} : T_{\tau(\xi)}M \xrightarrow{\cong} T_{\exp(\xi)}M \;, \\ \nabla_2 \exp(\xi) &= D\exp(\xi) \circ \left(K_{|T_{\xi,v}(TM)}\right)^{-1} : \; T_{\tau(\xi)}M \xrightarrow{\cong} T_{\exp(\xi)}M \;. \end{aligned}$$

It holds that $\nabla_1 \exp(0) = \nabla_2 \exp(0) = \mathrm{id}_{T_{\tau(0)}M}$. Furthermore,

$$\text{(A.16)}\quad \begin{aligned} &\Theta : \mathcal{D} \to \mathrm{Hom}(TM, TM) \\ &\Theta(\xi) = \left(\nabla_2 \exp(\xi)\right)^{-1} \circ \nabla_1 \exp(\xi) : T_{\tau(\xi)}M \xrightarrow{\cong} T_{\tau(\xi)}M \end{aligned}$$

is a smooth fibre respecting map which satisfies the identities

- $\Theta(0) = \mathrm{id}$,
- $F\Theta(0) = 0$ and
- $\Theta(\xi)\cdot \xi = \xi$.

Given any $v \in T_pM$ und $X \in C^\infty(TM)$, we obtain the covariant derivative of X in the direction of v from the formula

$$\nabla_v X(p) = K \circ D_pX \cdot v \;.$$

Thus, straightforward computation for $h \in C^\infty_{x,y}$ and $\xi \in C^\infty(h^*TM)$ yields the identity

$$\text{(A.17)}\quad \frac{\partial}{\partial t} \exp_h \xi = \nabla_1 \exp(\xi) \cdot \dot{h} + \nabla_2 \exp(\xi) \cdot \nabla_t \xi \;.$$

Let us consider once again the map from (A.13) representing the change of charts Φ_{fh} for smooth $\overline{\mathbb{R}}$-curves $f, h \in C^\infty_{x,y}(\overline{\mathbb{R}}, M)$. Then the fibre derivative $F\Phi_{fh}(t,\xi) \in \mathrm{Hom}\,(T_{h(t)}M, T_{f(t)}M)$ has the representation

$$\text{(A.18)}\quad \begin{aligned} &F\Phi_{fh} : \mathcal{O}_h \to \mathrm{Hom}\,(h^*TM, f^*TM) \\ &F\Phi_{fh}(t,\xi) = \nabla_2 \exp\left(\exp^{-1}_{f(t)}(\exp_{h(t)} \xi)\right)^{-1} \circ \nabla_2 \exp(\xi) \;. \end{aligned}$$

Thus, according to the fundamental Lemma A.5,

$$\text{(A.19)}\quad \begin{aligned} \mathfrak{S}(\mathcal{O}_h) \;&\to\; \mathcal{L}\big(\mathfrak{S}(h^*TM); \mathfrak{S}(f^*TM)\big) \\ s \;&\mapsto\; \mathfrak{S}_*(F\Phi_{fh} \circ s) = D\Phi_{fh*}(s) \end{aligned}$$

is a smooth map.

Remark Similar to the proof of this fundamental lemma, we may verify also that $\Theta : \mathcal{D} \to \text{Hom}\,(TM, TM)$ gives rise to a smooth map

$$\text{(A.20)} \qquad \begin{aligned} \mathfrak{S}(\Theta) : \mathfrak{S}(\mathcal{O}_h) &\to \mathcal{L}(\mathfrak{S}(h^*TM); \mathfrak{S}(h^*TM)) \\ s &\mapsto \mathfrak{S}_*(\Theta \circ s) \ . \end{aligned}$$

Now let $\mathfrak{T} : \text{Vec}_{C^\infty}(\overline{\mathbb{R}}) \to \text{Ban}$ be any section functor satisfying the condition that

$$\begin{aligned} \mathfrak{T}_* : \mathfrak{S}(\text{Hom}\,(\xi, \eta)) &\to \mathcal{L}(\mathfrak{T}(\xi); \mathfrak{T}(\eta)) \\ A &\mapsto (s \mapsto A \cdot s) \end{aligned}$$

yields a continuous map for each two $\xi, \eta \in \text{Vec}(\overline{\mathbb{R}})$, that is, in short terms,

$$\text{(A.21)} \qquad \mathfrak{S}(\text{Hom}) \subset \mathcal{L}(\mathfrak{T}; \mathfrak{T}) \ .$$

Referring to the estimate (A.1) in the auxiliary proposition A.2, we notice that $\mathfrak{T} = L^2_{\overline{\mathbb{R}}}$ fulfills this condition with respect to $\mathfrak{S} = C^0_{\overline{\mathbb{R}}}$ and thus in particular with respect to $\mathfrak{S} = H^{1,2}_{\overline{\mathbb{R}}}$ due to (A.3).

Given a fixed smooth section $s_0 \in C^\infty_{\overline{\mathbb{R}}}(\mathcal{O}_h)$, we are provided with a smooth map as in the proof of the fundamental lemma,

$$\text{(A.22)} \qquad \begin{aligned} \vartheta_* : \mathfrak{S}(\mathcal{O}_h) &\to \mathfrak{S}(\text{Hom}\,(h^*TM, f^*TM)) \\ s &\mapsto F\Phi_{fh} \circ s - F\Phi_{fh} \circ s_0 \ . \end{aligned}$$

It holds that

$$\begin{aligned} &\mathfrak{S}_* \circ \vartheta_*(s) = \mathfrak{S}_*(F\Phi_{fh} \circ s - F\Phi_{fh} \circ s_0) \\ = \ &\mathfrak{S}_*(F\Phi_{fh} \circ s) - \mathfrak{S}_*(F\Phi_{fh} \circ s_0) \in \mathcal{L}(\mathfrak{S}(h^*TM); \mathfrak{S}(f^*TM)) \ . \end{aligned}$$

Thus, property (A.21) of the section functor $\mathfrak{T}$ yields the smooth mapping

$$\text{(A.23)} \qquad \begin{aligned} \mathfrak{S}(\mathcal{O}_h) &\to \mathcal{L}(\mathfrak{T}(h^*Tm); \mathfrak{T}(f^*TM)) \\ s &\mapsto \mathfrak{T}_*(F\Phi_{fh} \circ s - F\Phi_{fh} \circ s_0) \ . \end{aligned}$$

Since $\mathfrak{T}_*(F\Phi_{fh} \circ s_0) \in \mathcal{L}(\mathfrak{T}(h^*TM); \mathfrak{T}(f^*TM))$ is already well-defined according to the section functor property, we obtain the smooth map

$$\text{(A.24)} \qquad \begin{aligned} \mathfrak{T}_* : \mathfrak{S}(\mathcal{O}_h) &\to \mathcal{L}(\mathfrak{T}(h^*TM); \mathfrak{T}(f^*TM)) \\ s &\mapsto \mathfrak{T}_*(F\Phi_{fh} \circ s) \end{aligned}$$

in analogy with

$$\mathfrak{S}_* : \mathfrak{S}(\mathcal{O}_h) \to \mathcal{L}(\mathfrak{S}(h^*TM); \mathfrak{S}(f^*TM)) \ .$$

After these preparations we are able to prove the following

Theorem 11 *The section functor $\mathfrak{T}$ satisfying condition (A.21) admits a unique extension to the continuous vector bundles of the shape*

$$g^*TM,\ \ g \in \mathcal{P}^{1,2}_{x,y}(\mathbb{R}, M) \subset C^0_{x,y}(\overline{\mathbb{R}}, M);\ x, y \in M\ .$$

Moreover,

$$\mathfrak{T}\left(\mathcal{P}^{1,2*}_{x,y}TM\right) = \bigcup_{g\in\mathcal{P}^{1,2}_{x,y}(\mathbb{R},M)} \mathfrak{T}(g^*TM)$$

is a Banach vector bundle on $\mathcal{P}^{1,2}_{x,y}(\mathbb{R}, M)$.

Proof. Let $\alpha, \beta \in C^0_{x,y}(\overline{\mathbb{R}}, M)$ be given, satisfying

$$\beta = \exp\circ\xi,\ \ \xi \in C^0(\alpha^*\mathcal{D}) \subset C^0(\alpha^*TM)\ .$$

We consequently define $J_{\beta\alpha} \in C^0\big(\mathrm{Hom}\,(\alpha^*TM, \beta^*TM)\big)$ by

$$J_{\beta\alpha}(t)\cdot v(t) = \nabla_2 \exp\big(\xi(t)\big)\cdot v(t)\ . \tag{A.25}$$

Thus, $J_{\beta\alpha}(t)$ is a toplinear isomorphism for each $t \in \overline{\mathbb{R}}$. Now let us consider charts based on the supporting curves $h, f \in C^\infty_{x,y}(\overline{\mathbb{R}}, M)$ as in the proof of Theorem 10. Formula (A.18) applied to $s \in \mathfrak{S}(\mathcal{O}_h)$ and $v \in \mathfrak{S}(h^*TM)$ leads to the identity

$$\nabla_2 \exp(\Phi_{fh}\circ s)\circ(F\Phi_{fh}\circ s)\cdot v = \nabla_2\exp(s)\cdot v\ ,$$

that is

$$\mathfrak{S}_*(F\Phi_{fh}\circ s) = \mathfrak{S}_*\big((J_{gf})^{-1}\circ J_{gh}\big),\quad g = \exp_h s \in \mathcal{P}^{1,2}_{x,y}(\mathbb{R}, M)\ . \tag{A.26}$$

We therefore define the fibre at $g = \exp_h s$ as

$$\mathfrak{T}(g^*TM) = \big\{\, J_{gh}\cdot v \,\big|\, v \in \mathfrak{T}(h^*TM)\,\big\}\ .$$

This is independent of the choice of h, because (A.26) implies the equation

$$J_{gf}\circ\mathfrak{S}_*(F\Phi_{fh}\circ s)\cdot v = J_{gh}\cdot v\ . \tag{A.27}$$

Thus, referring to (A.24), the Banach space topology on $\mathfrak{T}(g^*TM)$ is defined in a unique way independent of the underlying curve h. The smoothness of the map

$$\begin{aligned}\mathfrak{T}_* : \mathfrak{S}(\mathcal{O}_h) &\to \mathcal{L}\big(\mathfrak{T}(h^*TM); \mathfrak{T}(f^*TM)\big)\\ s &\mapsto \mathfrak{T}_*(F\Phi_{fh}\circ s)\end{aligned}$$

due to (A.24) implies that the maps

$$J_{gh} : \mathfrak{T}(h^*TM) \to \mathfrak{T}(g^*TM)$$

provide us with a well-defined local trivialization of the bundle $\mathfrak{T}\left(\mathcal{P}^{1,2*}_{x,y}TM\right)$ together with smooth transition maps. □

Since the transition maps due to (A.27) comply with the identity

$$(J_{gf})^{-1} \circ J_{gh} = \mathfrak{S}_*(F\Phi_{fh} \circ s) = D\Phi_{fh*}(s) \tag{A.28}$$

with respect to the special section functor $\mathfrak{T} = \mathfrak{S}$, the Banach bundle

$$\mathfrak{S}\left(\mathcal{P}^{1,2*}_{x,y}TM\right) = \bigcup_{h \in \mathcal{P}^{1,2}_{x,y}(\mathbb{R},M)} \mathfrak{S}(h^*TM)$$

represents exactly the tangent bundle of the Banach manifold $\mathcal{P}^{1,2}_{x,y}(\mathbb{R}, M)$. As a consequence we obtain

Corollary A.8 $T\mathcal{P}^{1,2}_{x,y}(\mathbb{R}, M) = H^{1,2}_{\mathbb{R}}\left(\mathcal{P}^{1,2*}_{x,y}TM\right)$.

Remark A more detailed analysis of the exponential map, its derivatives $\nabla_1 \exp$ and $\nabla_2 \exp$ and of the Levi-Civita connection would yield the possibility of discussing the associated exponential map $\exp_T$ on the Riemannian manifold TM. As a consequence, the functorial behaviour of $\mathcal{P}^{1,2}_{\cdot,\cdot}(\mathbb{R}, \cdot)$ as stated in Corollary A.3 applied to the projection map $\tau : TM \to M$ would enable us to find a representation of the tangent bundle $T\mathcal{P}^{1,2}_{x,y}$ in the following form:

$$T\mathcal{P}^{1,2}_{x,y}(\mathbb{R}, M) = \mathcal{P}^{1,2}_{0_x,0_y}(\mathbb{R}, TM),$$
$$T_\gamma\mathcal{P}^{1,2}_{x,y}(\mathbb{R}, M) = \left\{ \zeta \in \mathcal{P}^{1,2}_{0_x,0_y}(\mathbb{R}, TM) \,\middle|\, \tau \circ \zeta = \gamma \right\}, \quad \gamma \in \mathcal{P}^{1,2}_{x,y} \ .$$

According to this representation we may grasp the tangent space $T_\gamma\mathcal{P}^{1,2}_{x,y}$ as the space of the '$H^{1,2}_{\mathbb{R}}$-vector fields along the $H^{1,2}_{\mathbb{R}}$-curve γ' equipped with a Banach space topology.

Theorem 12 *Let $X \in C^\infty(TM)$ be a smooth vector field on M satisfying $X(x) = X(y) = 0$. Then the mapping*

$$C^\infty_{x,y}(\overline{\mathbb{R}}, M) \ni \gamma \mapsto \dot{\gamma} + X \circ \gamma \in C^\infty(\gamma^*TM)$$

can be extended to a smooth section in the Banach bundle $L^2_{\mathbb{R}}(\mathcal{P}^{1,2*}_{x,y}TM)$:

$$\begin{aligned} \Gamma : \mathcal{P}^{1,2}_{x,y}(\mathbb{R}, M) &\to L^2_{\mathbb{R}}(\mathcal{P}^{1,2*}_{x,y}TM) \\ \gamma &\mapsto \dot{\gamma} + X \circ \gamma \in L^2_{\mathbb{R}}(\gamma^*TM) \ . \end{aligned}$$

Proof. It is sufficient to verify the smoothness of this map with respect to the local representation by a chart together with the overlying local trivialization. We thus consider

$$\begin{aligned} H^{1,2}_{\mathbb{R}}(\mathcal{U}_h) \times L^2_{\mathbb{R}}(h^*TM) &\to L^2_{\mathbb{R}}(\mathcal{P}^{1,2*}_{x,y}TM)_{|\, H^{1,2}_{\mathbb{R}}(\mathcal{U}_h)} \\ (\exp_h \xi, \eta) &\mapsto \nabla_2 \exp(\xi) \cdot \eta \ . \end{aligned}$$

The local representation of the section Γ consequently gets the shape

$$\text{(A.29)} \qquad \begin{array}{rcl} \Gamma_{\text{triv},h} : H^{1,2}_{\mathbb{R}}(h^*\mathcal{D}) & \to & L^2_{\mathbb{R}}(h^*\mathcal{D}) \\ \xi & \mapsto & \nabla_2 \exp(\xi)^{-1}\big(\nabla_1 \exp(\xi)\cdot \dot{h} \\ & & +\nabla_2 \exp(\xi)\cdot \nabla_t \xi + X(\exp_h \xi)\big) \ . \end{array}$$

We may also express this as

$$\text{(A.30)} \qquad \Gamma_{\text{triv},h}(\xi) = \Theta(\xi)\cdot \dot{h} + \nabla_t\xi + \nabla_2 \exp(\xi)^{-1}\cdot (X \circ \exp_h)(\xi) \ .$$

As we already noticed in (A.20), the mapping

$$H^{1,2}_{\mathbb{R}} \ni \xi \mapsto \Theta(\xi) \in \mathcal{L}\big(H^{1,2}_{\mathbb{R}}(h^*TM); H^{1,2}_{\mathbb{R}}(h^*TM)\big)$$

is smooth, therefore in particular the mapping

$$\text{(A.31)} \qquad \xi \mapsto \Theta(\xi)\cdot \dot{h} \in H^{1,2}_{\mathbb{R}}(h^*TM) \quad \text{for} \quad \dot{h} \in C^{\infty}_{\overline{\mathbb{R}}}(h^*TM)$$

is also. We furthermore observe that

$$\text{(A.32)} \qquad H^{1,2}_{\mathbb{R}} \ni \xi \mapsto \nabla_t \xi \in L^2_{\mathbb{R}}$$

is continuously linear and thus also smooth. Considering now the open zero section neighbourhood $\mathcal{O} = h^*\mathcal{D} \subset h^*TM$, we are led to the smooth bundle map

$$\text{(A.33)} \qquad \begin{array}{l} f : \mathcal{O} \to h^*TM \\ f(v) = \nabla_2 \exp(v)^{-1}\cdot X(\exp_h v) \end{array}$$

satisfying the condition $f(0_{\pm\infty}) = 0_{\pm\infty}$ due to the assumption $X(x) = X(y) = 0$. Hence, the fundamental Lemma A.5 guarantees the smoothness of the map

$$f_* : H^{1,2}_{\mathbb{R}}(h^*\mathcal{D}) \to H^{1,2}_{\mathbb{R}}(h^*TM) \ .$$

Piecing all this together proves the assertion. □

Corollary A.9 *Given a smooth Morse function f on M, the mapping*

$$\begin{array}{rcl} F : \mathcal{P}^{1,2}_{x,y} & \to & L^2_{\mathbb{R}}(\mathcal{P}^{1,2\,*}_{x,y}TM), \quad x, y \in \operatorname{Crit} f \\ s & \mapsto & \dot{s} + \nabla f \circ s \end{array}$$

describes a smooth section in the specified L^2-Banach bundle. The local representation at $\gamma \in C^{\infty}_{x,y}$ is of the type

$$\begin{array}{l} F_{\text{loc},\gamma} : H^{1,2}_{\mathbb{R}}(\gamma^*\mathcal{D}) \to L^2_{\mathbb{R}}(\gamma^*TM) \\ F_{\text{loc},\gamma}(\xi)(t) = \nabla_t \xi(t) + g\big(t, \xi(t)\big) \ , \end{array}$$

where $g : \overline{\mathbb{R}} \times \gamma^*\mathcal{D} \to \gamma^*TM$ is smooth and fibre respecting, satisfies the identity $g(\pm\infty, 0) = 0$ and is endowed with the asymptotic fibre derivatives $D_2 g(\pm\infty, 0)$ which are conjugated linear operators of the Hessians $H^2 f(x)$ and $H^2 f(y)$, respectively.