Remarks on Chern-Simons Theory

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MSRI: 1982

Mathematical Statistics
August 01, 1982 to July 31, 1983
Organized By: L. LeCam, D. Siegmund (chairman), C. Stone

Nonlinear Partial Differential Equations
August 01, 1982 to July 31, 1983
Organized By: A. Chorin, I. M. Singer (chairman), S.-T. Yau

Ergodic Theory and Dynamical Systems
August 01, 1983 to July 31, 1984
Organized By: J. Feldman (chairman), J. Franks, A. Katok, J. Moser, R. Temam

Infinite-Dimensional Lie Algebras
August 01, 1983 to July 31, 1984
Organized By: H. Garland, I. Kaplansky (chairman), B. Kostant

Classical Chern-Simons

Characteristic forms and geometric invariants

By SHIING-SHEN CHERN AND JAMES SIMONS*

1. Introduction

This work, originally announced in [4], grew out of an attempt to derive a purely combinatorial formula for the first Pontrjagin number of a 4-manifold. The hope was that by integrating the characteristic curvature form (with respect to some Riemannian metric) simplex by simplex, and replacing the integral over each interior by another on the boundary, one could evaluate these boundary integrals, add up over the triangulation, and have the geometry wash out, leaving the sought after combinatorial formula. This process got stuck by the emergence of a boundary term which did not yield to a simple combinatorial analysis. The boundary term seemed interesting in its own right and it and its generalization are the subject of this paper.

The Weil homomorphism is a mapping from the ring of invariant polynomials of the Lie algebra of a Lie group, G, into the real characteristic cohomology ring of the base space of a principal G-bundle, cf. [5], [7]. The map is achieved by evaluating an invariant polynomial P of degree l on the

Quantum Chern-Simons

Witten (1989): Integrate over space of connections—obtain a topological invariant of a closed oriented 3-manifold X

$G = SU(n)$

$P = X \times G$

$A \in \Omega^1(X; \mathfrak{g})$

$\langle \cdot, \cdot \rangle$ basic inner product

$$S(A) = \int_X \langle A \wedge dA \rangle - \frac{1}{6} \langle A \wedge A \wedge A \rangle$$

$$F_k(X) = \text{“} \int_{\mathcal{F}_X} e^{ikS(A)} \, dA, \text{”} \qquad k \in \mathbb{Z}$$

$\mathcal{F}_X$ is the space (stack) of connections on X

Warning: This path integral is “only” a motivating heuristic

$$F_k(X) = \text{“}\int_{\mathcal{F}_X} e^{ikS(A)}\, dA\text{”}, \qquad k \in \mathbb{Z}$$

First sign of trouble: F(X) depends on orientation + *another topological structure* (2-framing, p_1-structure)

Extend to compact manifolds with boundary: path integral with boundary conditions on the fields (connections)

Obtain invariants of knots and links:

- Jones polynomial
- HOMFLYPT = Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, Pzytycki, Traczyk

$$F_k(X) = \text{``}\int_{\mathcal{F}_X} e^{ikS(A)}\, dA\text{''}, \qquad k \in \mathbb{Z}$$

Question: How can we make mathematics out of the path integral heuristic? Focus on topological case.

Plan:

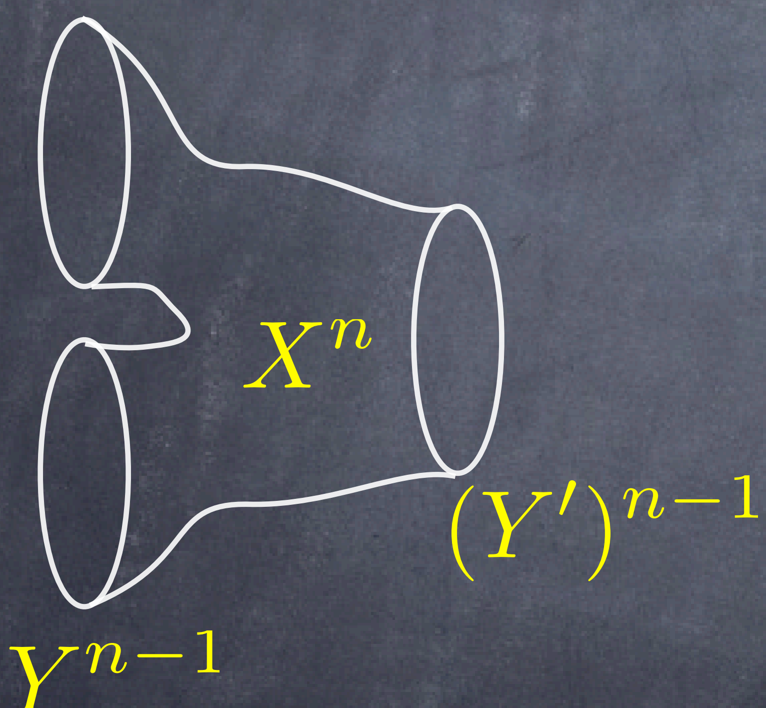
- Axiomatization: define a topological quantum field theory (TQFT)
- Constructions: generators and relations vs. *a priori*
- State a theorem inspired by this physics
- General observations about geometry-physics interaction

Path Integrals: Formal Structure

Fix a dimension $n \geq 1$

$\mathcal{F}_X$ — fields on an n-manifold X

$S_X : \mathcal{F}_X \longrightarrow \mathbb{R}$ — action functional



$$\begin{array}{ccc} & \mathcal{F}_X & \\ {\scriptstyle p_{\text{in}}} \swarrow & & \searrow {\scriptstyle p_{\text{out}}} \\ \mathcal{F}_Y & & \mathcal{F}_{Y'} \end{array}$$

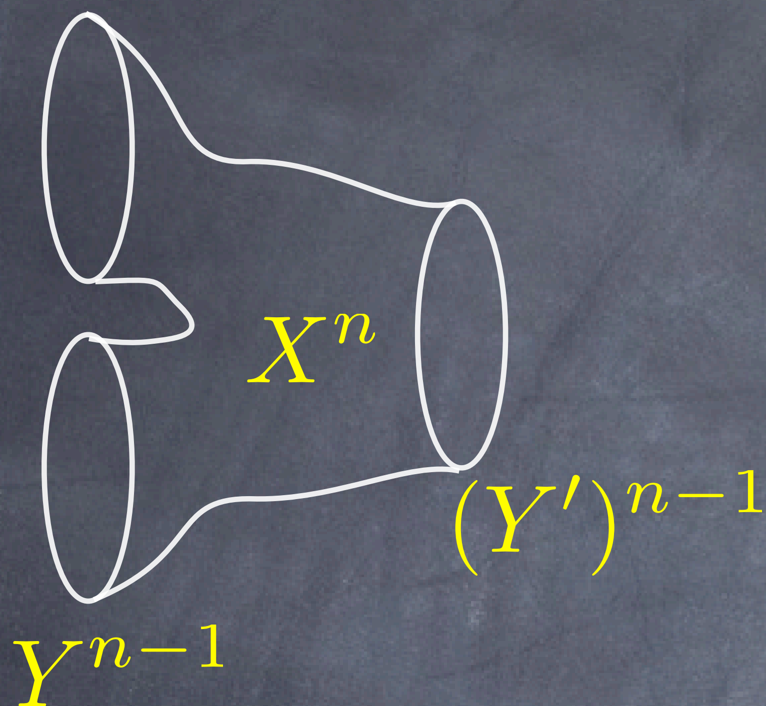
Need measures to define

$\partial X = Y \amalg Y'$

$$L^2(\mathcal{F}_Y) \xrightarrow{(p_{\text{out}})_* \, e^{iS_X} \, (p_{\text{in}})^*} L^2(\mathcal{F}_{Y'})$$

linearization of correspondence diagram

Path Integrals



$$L^2(\mathcal{F}_Y) \xrightarrow{(p_{\text{out}})_* \; e^{iS_X} \; (p_{\text{in}})^*} L^2(\mathcal{F}_{Y'})$$

Closed manifold of dimension	F(-)
n	complex #
n-1	Hilbert space

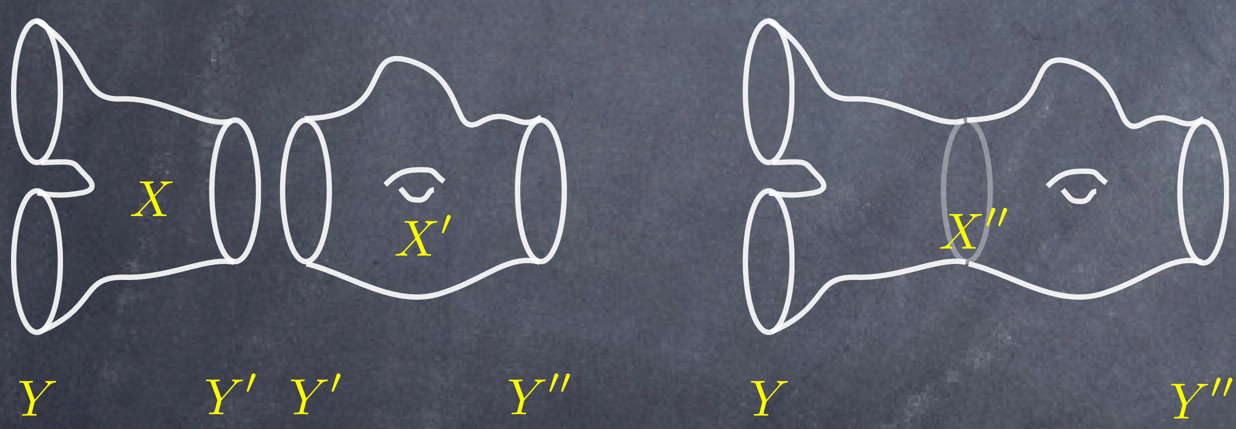
Path Integrals: Multiplicativity

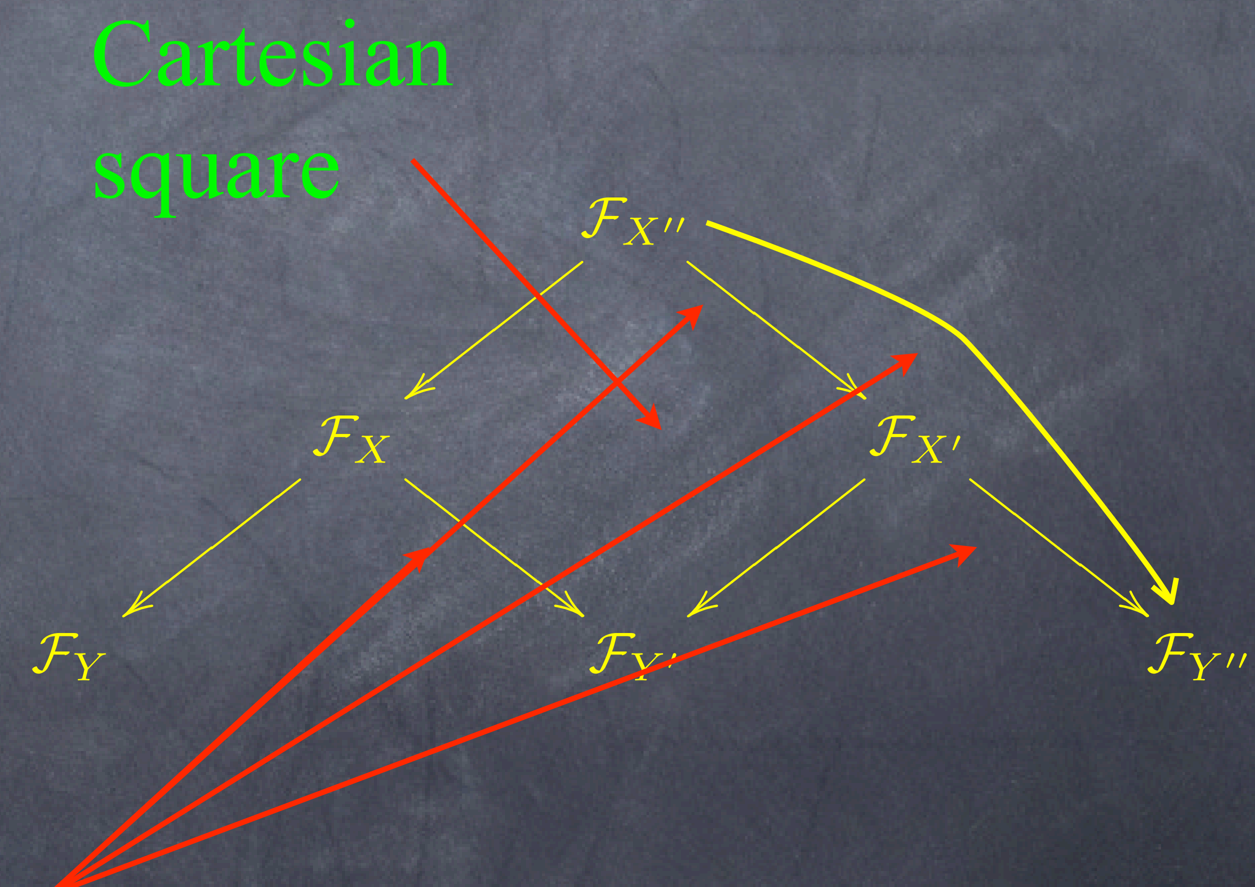
Disjoint union:

$$\mathcal{F}_{X \amalg X'} \simeq \mathcal{F}_X \times \mathcal{F}_{X'}$$

$$S_{X \amalg X'}(\phi \amalg \phi') = S_X(\phi) + S_{X'}(\phi')$$

Gluing bordisms:





Need measures which are *consistent* under gluing to define the various pushforward maps (path integrals)

Axiomatization: Definition

Witten, Segal, Atiyah, ...

Bord_n bordism category of compact n-manifolds

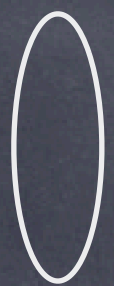
$\mathrm{Vect}_{\mathbb{C}}$ category of finite dim'l complex vector spaces

Definition: A TQFT is a monoidal functor

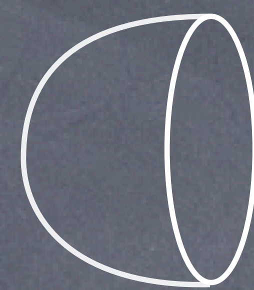
$$F : \mathrm{Bord}_n \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

Closed manifold of dimension	F(-)
n	element of C
n-1	C-module

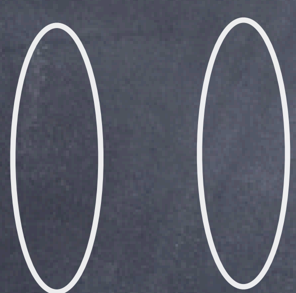
Structure: n=2 on oriented manifolds



A



$\mathbb{C} \longrightarrow A$



$A \otimes A$

$A \longrightarrow \mathbb{C}$

$\emptyset$

$\mathbb{C}$

$A \otimes A \longrightarrow A$

Frobenius algebra: associative (commutative) algebra with identity and *nondegenerate* trace

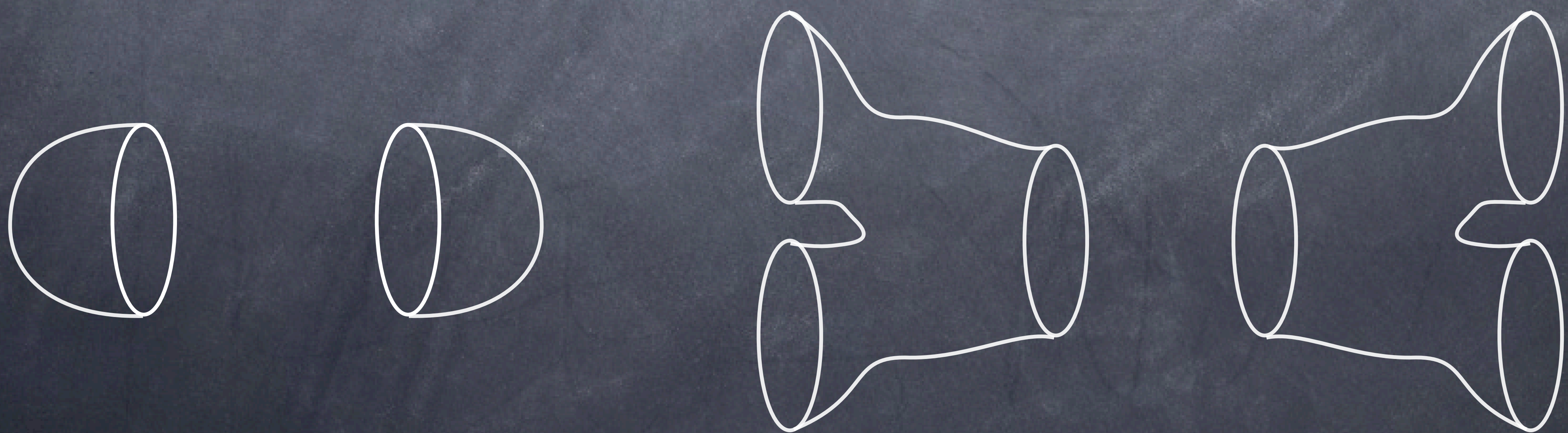
Constructions: Generators & Relations

Theorem: There is an equivalence

$$2d \text{ TQFTs} \longleftrightarrow \text{commutative Frobenius algebras}$$

This is a folk theorem: Dijkgraaf, Abrams, ...

Bordism category of 2-manifolds generated by elementary bordisms:



3-manifolds are more complicated...

Extended TQFT

Extend in two ways:

- families of manifolds
- manifolds of lower dimension

Before: 1-2 theory. Now: 0-1-2 theory, 1-2-3 theory, etc.

Closed manifold of dimension	F(-)
n	element of C
n-1	C-module
n-2	C-linear category

Extended TQFT

Extend in two ways:

- families of manifolds
- manifolds of lower dimension

Before: 1-2 theory. Now: 0-1-2 theory, 1-2-3 theory, etc.

Closed manifold of dimension	F(-)	Category #
n	element of C	-1
n-1	C-module	0
n-2	C-linear category	1

Extended TQFT

Extend in two ways:

- families of manifolds
- manifolds of lower dimension

Before: 1-2 theory. Now: 0-1-2 theory, 1-2-3 theory, etc.

Closed manifold of dimension	F(-)	Category #
n	element of C	-1
n-1	C-module	0
n-2	C-linear category	1
n-3	linear 2-category	2

Definition: The category number of a mathematician is the largest integer n such that he/she can ponder n-categories for a half hour without developing a migraine.

Extended TQFT

Chern-Simons: path integral (heuristic) gives a 2-3 theory
extension to 1-2-3 theory?
extension to 0-1-2-3 theory?

Chern-Simons as a 1-2-3 Theory

$$\text{1-2 TQFTs} \longleftrightarrow \text{commutative Frobenius algebras}$$

Theorem: There is an equivalence

$$\text{1-2-3 TQFTs} \longleftrightarrow \text{modular tensor categories}$$

This is due to Reshetikhin-Turaev (related work: Moore-Seiberg, Kontsevich, Walker, ...)

- The modular tensor category is constructed from a quantum group or loop group and is $F(S^1)$
- Bordism category: oriented manifolds with p_1-structure
- *Unitary* TQFTs $\leftrightarrow$ *unitary* modular tensor categories
- These are semisimple theories, somewhat special
- The Chern-Simons invariant is nowhere in sight

Extended TQFT

Theories which go down to a point (0-1, 0-1-2, 0-1-2-3, etc.) are the most *local* so should have a "simple" structure

Much current work on 0-1-2 theories:

- Elliptic cohomology (Stolz-Teichner)
- Structure (generator and relations) theorems (Moore-Segal; Costello; Lurie-Hopkins)
- Applies to string topology (Chas-Sullivan)

Chern-Simons as a 0-1-2-3 Theory

What is $F(\text{point})$? Should be a 2-category. Try to realize as 2-category of modules over a monoidal 1-category

Unknown for general theories, but will describe for case when G is a finite group. Starting data is

$$\lambda \in H^4(BG;\mathbb{Z})$$

Deriving $F(S^1)$ from $F(\text{point})$:

The *Drinfeld Center* $\mathcal{Z}(\mathcal{C})$ of a monoidal category $\mathcal{C}$ is the braided monoidal category whose objects are pairs (A,θ) of objects A in $\mathcal{C}$ and natural isomorphisms $\theta : A\otimes - \;\to\; - \otimes A.$

Reduction of CS to a 1-2 Theory

Dimensional reduction: $F'(M) = F(S^1 \times M)$

Closed manifold of dimension	F(-)	F′(-)
3	element of C	
2	C-module	element of Z
1	C-linear category	Z-module

Note *integrality* of dimensional reduction: Frobenius *ring*

Some thought about this transition suggests K-theory (refinement of Hochshild homology)

Loop groups and K-Theory

Joint work with Michael Hopkins and Constantin Teleman

Closed manifold of dimension	F(-)	F′(-)
2	C-module	element of Z
1	C-linear category	Z-module

Positive energy representations of loop groups

Twisted equivariant K-theory of G

Theorem (FHT): There is an isomorphism of rings

$$\Phi : \mathcal{R}^{\tau-\sigma}(LG) \longrightarrow K^{\tau}_{G}(G)$$

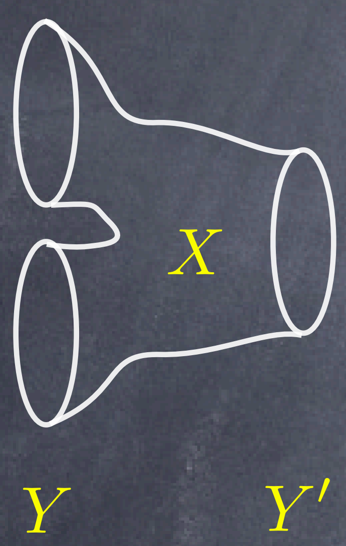
Dirac family

A Priori Construction of F′

Path integral:

$$\mathcal{F}_Y \xleftarrow{p_{\text{in}}} \mathcal{F}_X \xrightarrow{p_{\text{out}}} \mathcal{F}_{Y'}$$

$$L^2(\mathcal{F}_Y) \xrightarrow{(p_{\text{out}})_* \, e^{iS_X} \, (p_{\text{in}})^*} L^2(\mathcal{F}_{Y'})$$



$\mathcal{F}$: infinite dimensional stack of G-connections

Need *consistent measures* to define path integral

Topology:

$$\mathcal{M}_Y \xleftarrow{p_{\text{in}}} \mathcal{M}_X \xrightarrow{p_{\text{out}}} \mathcal{M}_{Y'}$$

$$K(\mathcal{M}_Y) \xrightarrow{(p_{\text{out}})_* \, (p_{\text{in}})^*} K(\mathcal{M}_{Y'})$$

$\mathcal{M}$: finite dimensional stack of *flat* G-connections

Need *consistent orientations* to define pushforward

A Priori Construction of F′

$$\mathcal{M}_{S^1} \cong G/\!\!/G$$
$$K(G/\!\!/G) \cong K_G(G)$$

$$\mathcal{M}_Y \xleftarrow{\;p_{\text{in}}\;} \mathcal{M}_X \xrightarrow{\;p_{\text{out}}\;} \mathcal{M}_{Y'}$$

$$K(\mathcal{M}_Y) \xrightarrow{\;(p_{\text{out}})_*\,(p_{\text{in}})^*\;} K(\mathcal{M}_{Y'})$$

Theorem (FHT): There exist universal, hence consistent, orientations.

Summary:

- Generators and relations construction of 1-2-3 Chern-Simons theory (quantum groups)
- *A priori* construction of the 1-2 dimensional reduction of Chern-Simons (twisted K-theory)
- Classical Chern-Simons invariant has disappeared from the discussion

Chern-Simons in Quantum Physics

Among the many possibilities we mention:

- Large N duality in string theory relates quantum Chern-Simons invariants to Gromov-Witten invariants (Gopakumar-Vafa)
- Quantum Chern-Simons theory and other 1-2-3 theories are at the heart of proposed topological quantum computers (Freedman et. al.)

Geometry/QFT-Strings Interaction

- More spectacular impact on geometry/topology elsewhere: 4d gauge theory (4-manifolds, geometric Langlands) and 2d conformal field theory (mirror symmetry, etc.)
- New links in math (here reps of loop groups and K-theory)
- Deep mathematics, bidirectional influence, big success!
- But...little beyond formal aspects of QFT (and string theory) has been absorbed into geometry and topology
- Over past 25 years good understanding of scale-invariant part of the physics: topological and conformal
- Perhaps more focus needed now on geometric aspects of scale-dependence in the physics

How Little We Know

Our axiomatization does not capture a basic feature of the path integral: stationary phase approximation

$$F_k(X) = \text{“}\int_{\mathcal{F}_X} e^{ikS(A)}\, dA\text{”}, \qquad k \in \mathbb{Z}$$

The discrete parameter k is $1/\hbar$, where $\hbar$ is Planck’s “constant”

The semi-classical limit $\hbar \to 0$ can be computed in terms of classical topological invariants. So we obtain a prediction for the asymptotic behavior of $F_k(X)$ as $k \to \infty$

How Little We Know

$$F_k(X) \sim \frac{1}{2} e^{-3\pi i/4} \sum_{\substack{A \in \mathcal{M}^o_X \\ \text{flat connections}}} e^{i(k+2)S_X(A)}\, e^{-(2\pi i/4) I_X(A)}\, \sqrt{\tau_X(A)}$$

M^o_X	*irreducible* flat connections, assumed isolated
$S_X(A)$	classical Chern-Simons invariant
$I_X(A)$	spectral flow (Atiyah-Patodi-Singer)
$\tau_X(A)$	Reidemeister torsion

LHS: loop groups or quantum groups

RHS: topological invariants of flat connections

How Little We Know

Table 9. Asymptotic values of the Witten invariant for $\Sigma(2,3,17)$

k	Exact value	Asymptotic value	Ratio
141	$0.607899+0.102594i$	$0.596099+0.151172i$	$0.999182-0.081285i$
142	$-0.104966-0.151106i$	$-0.094614-0.157913i$	$0.997181-0.067244i$
143	$0.123614-0.139016i$	$0.132261-0.128045i$	$1.007707-0.075491i$
144	$-0.612014+0.038199i$	$-0.614913-0.008261i$	$0.994271-0.075479i$
145	$-0.291162-0.132171i$	$-0.281928-0.153204i$	$0.993986-0.071336i$
146	$-0.413944+0.674785i$	$-0.465909+0.642185i$	$0.994797-0.077144i$
147	$0.400490-0.286350i$	$0.419276-0.254325i$	$1.001116-0.075706i$
148	$-0.091879+0.669230i$	$-0.143660+0.661309i$	$0.995194-0.077257i$
149	$0.946786-0.263649i$	$0.962119-0.191329i$	$0.999048-0.075356i$
150	$-0.024553-0.058313i$	$-0.021860-0.059113i$	$1.002906-0.044484i$

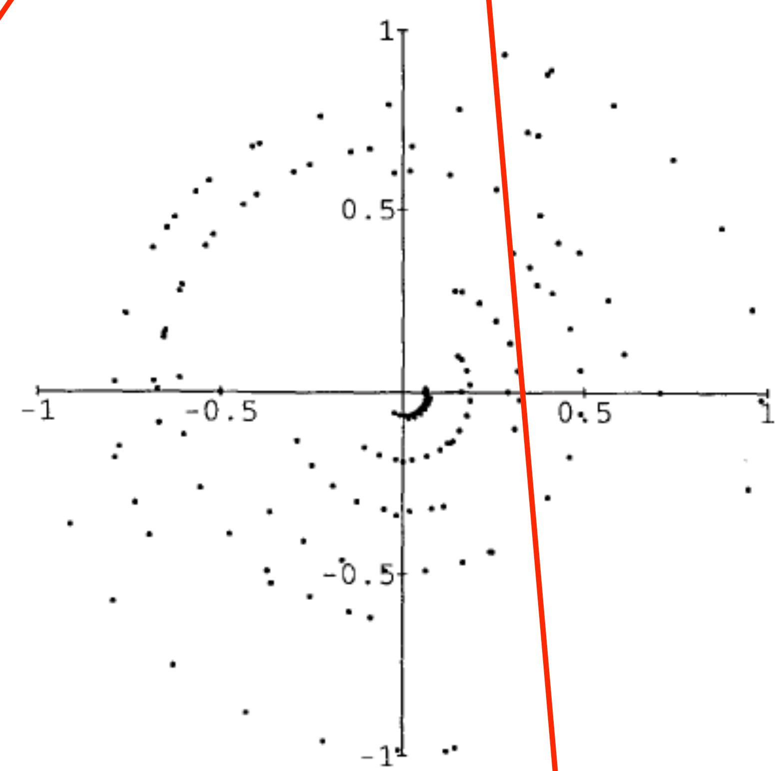


Fig. 11. Witten invariants for $\Sigma(2,3,17)$: $1 \leqq k \leqq 150$

$$F_k(X) \sim \frac{1}{2} e^{-3\pi i/4} \sum_{A \in \mathcal{M}^o_X} e^{i(k+2)S_X(A)} \, e^{-(2\pi i/4) I_X(A)} \sqrt{\tau_X(A)}$$