3-Dimensional TQFTs Through the Lens of the Cobordism Hypothesis

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Work in progress with Constantin Teleman

The Wess-Zumino-Witten (WZW) model

G	compact Lie group
$\lambda \in H^4(BG; \mathbb{Z})$	*level*

2d conformal field theory $W_{(G,\lambda)}$ with classical fields $\phi\colon Y^2 \to G$ an infinite dimensional group. Hilbert space:

$$W_{(G,\lambda)}(S^1) \cong \bigoplus_{\alpha} \mathcal{H}_\alpha \otimes \overline{\mathcal{H}_\alpha}$$

where the finite sum is over irreducible positive energy reps α of LG. The partition function of a closed oriented 2-manifold factors similarly.

2d conformal theories with this factorization are called *rational.*

Witten (late '80s) realized, influenced by Segal's work on CFT, that this factorization is encoded in a 3d topological quantum field theory $F_{(G,\lambda)}$ with classical fields G-connections and the Chern-Simons functional.

Open problem: Construct $F_{(G,\lambda)}$ directly from (G, λ).

FHT construct the 2d reduction—Verlinde ring—via twisted K-theory.

3d Topological Theories from Modular Tensor Categories

Let A be a *modular tensor category*, i.e., a discrete 1-category with a braided monoidal structure, a ribbon structure, and internal duals. It is semisimple with finitely many simple objects, and it satisfies a nondegeneracy condition.

Example 1: Given (G, λ) with G simple, connected, simply connected, there is an associated quantum group at a root of unity and A is a quotient of a category of its representations.

Example 2: If G is a torus group, then (G, λ) gives rise to (F, q) where F is a finite abelian group and $q\colon F \to \mathbb{Q}/\mathbb{Z}$ a quadratic form. Then $A = A(F, q)$ has as its K-theory ring the representation ring of F and q determines the ribbon structure, hence the braiding.

Example 3: If G is finite, then $A = \mathbf{Vect}_G^{\lambda}[G]$ the category of G-equivariant vector bundles on G with monoidal structure by λ-twisted convolution, i.e., twisted pushforward via multiplication $G \times G \to G$.

Reshetikhin-Turaev constructed a 1-2-3-dimensional topological field theory of bordisms with "signature (σ) structure"

$$\widehat{F}_A \colon \mathrm{Bord}_{\langle 1,2,3\rangle}^{(w_1,\sigma)} \longrightarrow \mathbf{Cat}_{\mathbb{C}}$$

attached to a modular tensor category A. It encodes invariants of links and 3-manifolds (quantum Chern-Simons theory).

Theorem (in progress): Let A be a MTC. There exists a symmetric monoidal 3-category $\mathcal{C}_A$ and a 3-dualizable object $x_A \in \mathcal{C}_A$ which generates a 0-1-2-3-dimensional topological field theory of bordisms with p_1-structure

$$F_A \colon \mathrm{Bord}_3^{(w_1,p_1)} \longrightarrow \mathcal{C}_A.$$

The composition $\mathrm{Bord}_{\langle 1,2,3\rangle}^{(w_1,\sigma)} \to \Omega\,\mathrm{Bord}_3^{(w_1,p_1)} \xrightarrow{\Omega F_A} \Omega\mathcal{C}_A$ is $\widehat{F}_A$.

$\mathcal{C}_A$ contains the 3-category $\mathbf{Cat}_{\mathbb{C}}^{\otimes}$ of tensor categories as a full subcategory and is formally $\mathcal{C}_A = \mathbf{Cat}_{\mathbb{C}}^{\otimes}[\mathbf{x}, \mathbf{x}^\vee]/(\mathbf{x} \otimes \mathbf{x}^\vee \cong \mathbf{A})$.

Remarks:

- Walker has a related picture of Chern-Simons theory using bounding manifolds (as we will do presently).
- Bartels-Douglas-Henriques use conformal nets to study the WZW and related Chern-Simons theories.

In the early '90s Chern-Simons theory gave rise to the notion of extended quantum field theories. In particular, it was understood that in a 3d theory if

$$F_A(S^1) = A$$

then A is a braided tensor category.

Also, if G is a finite group, then it was known that

$$F_G(\mathrm{pt}) = \mathbf{Vect}[G] \in \mathbf{Cat}_{\mathbb{C}}^{\otimes}$$

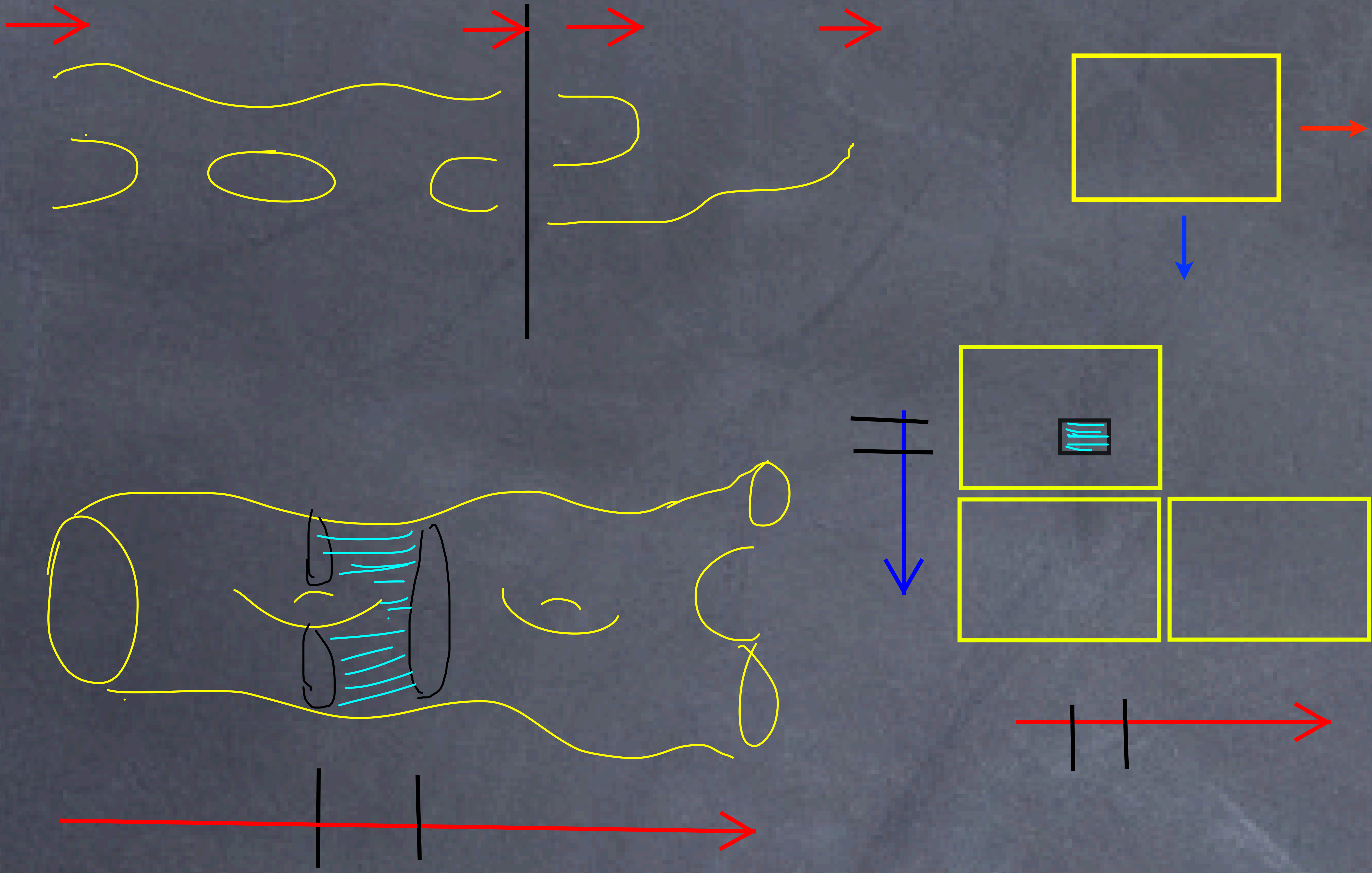
where $\mathbf{Vect}[G]$ is the category of vector bundles on G under convolution: with monoidal structure pushforward by multiplication $G \times G \to G$. There is a twisted version for (G, λ).

These ideas gave rise in the mid '90s to the Baez-Dolan cobordism hypothesis, proved in the late '00s by Lurie (with Hopkins in 2d case). In this work we bring the modern understanding of the cobordism hypothesis to bear on these old ideas. In particular, we derive the Reshetikhin-Turaev theorem from the cobordism hypothesis.

A contemporary motivation for this project, which we will not discuss today, is renewed interest in a 6-dimensional conformal field theory with associated 7-dimensional topological field theory. In physics it goes by the name "the (0,2)-superconformal theory in 6d". Some of us call it Theory $\mathcal{X}$. It looks like an emerging central object in low dimensional geometry with implications for geometric representation theory, knot invariants, ... The structures we find in 2 and 3 dimensions will help unravel the structure of Theory $\mathcal{X}$.

Remark: The technical underpinnings of algebra in higher categories, as well as the proof of the cobordism hypothesis, are under development (by others!).

Bordism Multi-Categories



The Cobordism Hypothesis

Let $\mathrm{Bord}_n^{w_1}$ denote the (∞, n)-category of oriented bordisms. Let $\mathcal{C}$ be an arbitrary symmetric monoidal (∞, n)-category. A (fully extended) n-dimensional topological field theory is a homomorphism

$$F\colon \mathrm{Bord}_n^{w_1} \longrightarrow \mathcal{C}$$

Remark: The definition leaves great flexibility in the choice of $\mathcal{C}$, and we will take advantage several times, as in our main theorem.

The cobordism hypothesis asserts that F is determined by $F(\mathrm{pt}_+)$. Furthermore, any n-dualizable, SO_n-invariant object $X \in \mathcal{C}$ determines a theory F with $F(\mathrm{pt}_+) = X$.

n-dualizability is a condition, that certain (constrained) data attached to Morse handles exists. SO_n-invariance is extra data.

Example: $n = 2$, $\mathcal{C} = \mathrm{Alg}_k$ the Morita 2-category of algebras over a field k. Then $A \in \mathcal{C}$ is 2-dualizable if it is finite dimensional semisimple and SO_2-invariance data is a Frobenius structure (trace).

Invertible Field Theories

$\alpha\colon \mathrm{Bord}_n^{w_1} \to \mathcal{C}$ is *invertible* if $\alpha(M)$ is invertible for every bordism M. The cobordism hypothesis implies α is invertible iff $\alpha(\mathrm{pt}_+)$ is invertible.

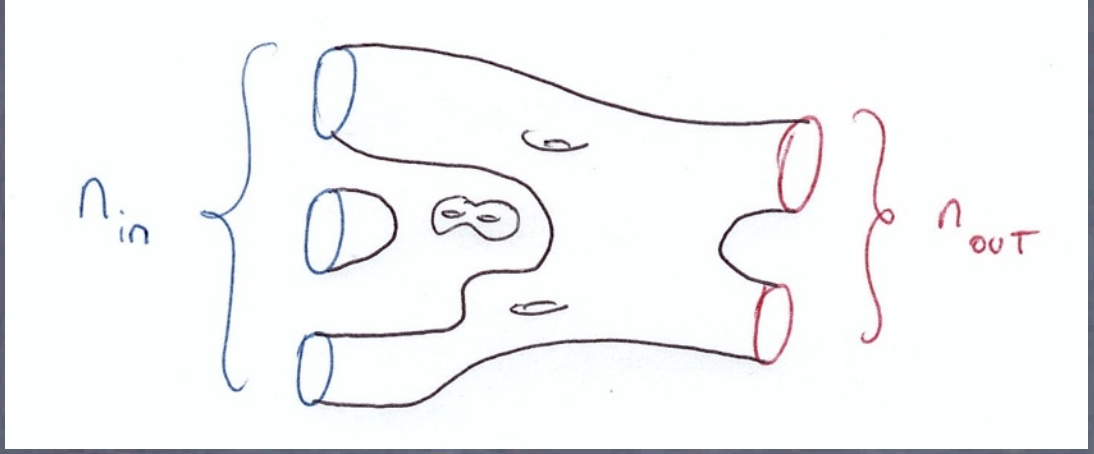
Example: An algebra $A \in \mathrm{Alg}_k$ is invertible iff A is central simple. There is a "super" version for $\mathbb{Z}/2\mathbb{Z}$-graded algebras (Wall). For example, the Clifford algebra $k \oplus ke$ with $e^2 = 1$ is invertible and has an odd Frobenius structure, so defines an invertible oriented 2d theory.

Invertible theories factor through the geometric realization $|\,\mathrm{Bord}_n^{w_1}\,|$, which is an infinite loop space ($\mathrm{Bord}_n^{w_1}$ is symmetric monoidal). The Madsen-Weiss theorem identifies $|\,\mathrm{Bord}_n^{w_1}\,|$ as the 0-space of the Madsen-Tillmann spectrum $\Sigma^n MTSO_n$. So invertible theories can be studied via homotopy theory.

Example (con't): If $\tau(e) = \lambda \in k^\times$,

$$\alpha(S^1) = k, \qquad \alpha(Y) = \lambda^{\chi(Y)+n_{\mathrm{in}}-n_{\mathrm{out}}}$$

Note the exponent is even.



Theorem: Suppose $\alpha\colon \mathrm{Bord}_n^{w_1} \to \mathcal{C}$ and $\alpha(S^k)$ is invertible and $n \geqslant 2k$. Then α is invertible.

This is a kind of localization theorem for $\mathrm{Bord}_n^{w_1}$: if we invert S^k then we invert every bordism.

Example: $n = 2$, $k = 1$, $\mathcal{C} = \mathrm{Alg}_k$. If A is a 2-dualizable (finite dimensional, semisimple) Frobenius algebra, then it defines $\alpha\colon \mathrm{Bord}_2^{w_1} \to \mathrm{Alg}_k$ with $\alpha(S^1)$ equal to the center of A. So α is invertible if the center of A is k.

Proof Sketch

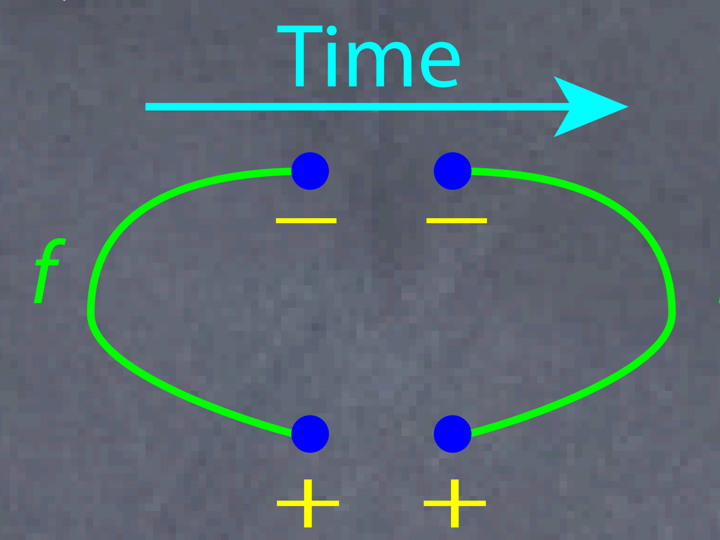
First, by the cobordism hypothesis (easy part) it suffices to prove that $\alpha(\mathrm{pt}_+)$ is invertible; '+' denotes the orientation. We omit 'α' and simply say 'pt_+ is invertible'.

We aim to prove that the 0-manifolds pt_+ and pt_- are inverse:

$$S^0 = \mathrm{pt}_+ \amalg \mathrm{pt}_- = \mathrm{pt}_+ \otimes \mathrm{pt}_- \cong \varnothing^0 = 1$$

with inverse isomorphisms given by

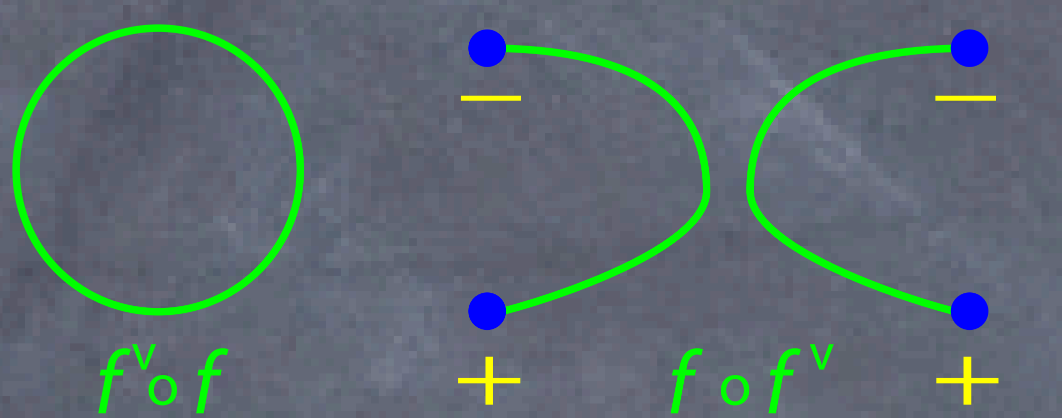
$$f = D^1 : 1 \longrightarrow S^0$$
$$f^\vee = D^1 : S^0 \longrightarrow 1$$



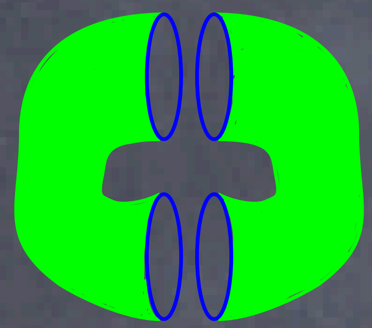
We are reduced to a statement about 1d bordisms: the compositions

$$f^\vee \circ f = S^1 : 1 \longrightarrow 1$$
$$f \circ f^\vee \qquad : S^0 \longrightarrow S^0$$

must be proved to be identity.



Let's now consider $n = 2$ where we assume that S^1 is invertible. We apply an easy algebraic lemma which asserts that invertible objects are dualizable and the dualization data is invertible. For S^1 these data are dual cylinders, and so the composition $S^1 \times S^1$ is also invertible.



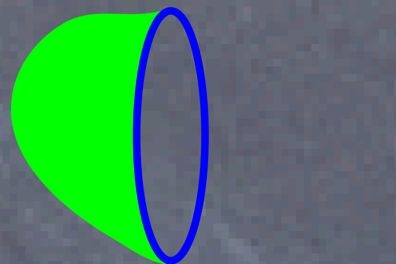
Lemma: Suppose $\mathcal{D}$ is a symmetric monoidal category, $x \in \mathcal{D}$ is invertible, and $g\colon 1 \to x$ and $h\colon x \to 1$ satisfy $h \circ g = \mathrm{id}_1$. Then $g \circ h = \mathrm{id}_x$ and so each of g, h is an isomorphism.

Proof: x^{-1} is a dual of x, $\quad g^\vee = x^{-1}g\colon x^{-1} \to 1$, $h^\vee = x^{-1}h\colon 1 \to x^{-1}$, $\quad$ so the lemma follows from $(h \circ g)^\vee = \mathrm{id}_1$.

Apply the lemma to the 2-morphisms

$$g = D^2\colon 1 \longrightarrow S^1$$

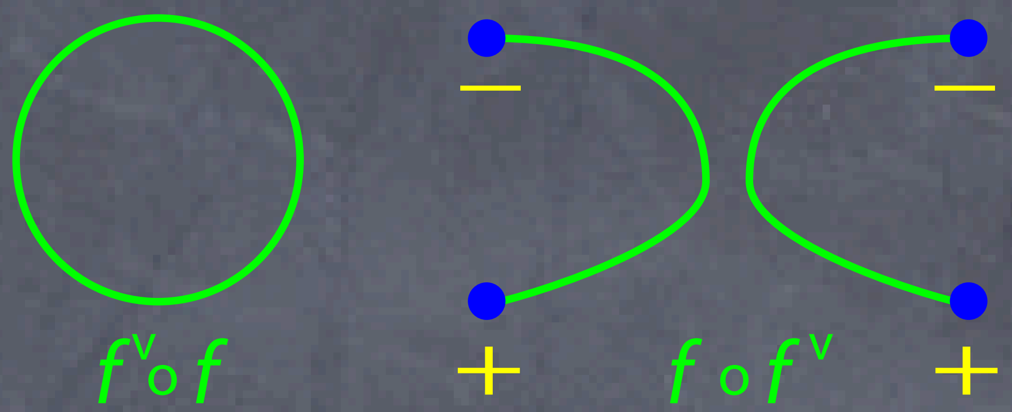
$$h = S^1 \times S^1 \backslash D^2\colon S^1 \longrightarrow 1$$



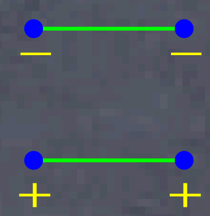
Conclude that $S^1 \cong 1$ and $S^2 = g^\vee \circ g$ is invertible.
Also, $g \circ g^\vee = \mathrm{id}_{S^1} \otimes S^2$, a simple surgery.

Recall that we must prove that the compositions

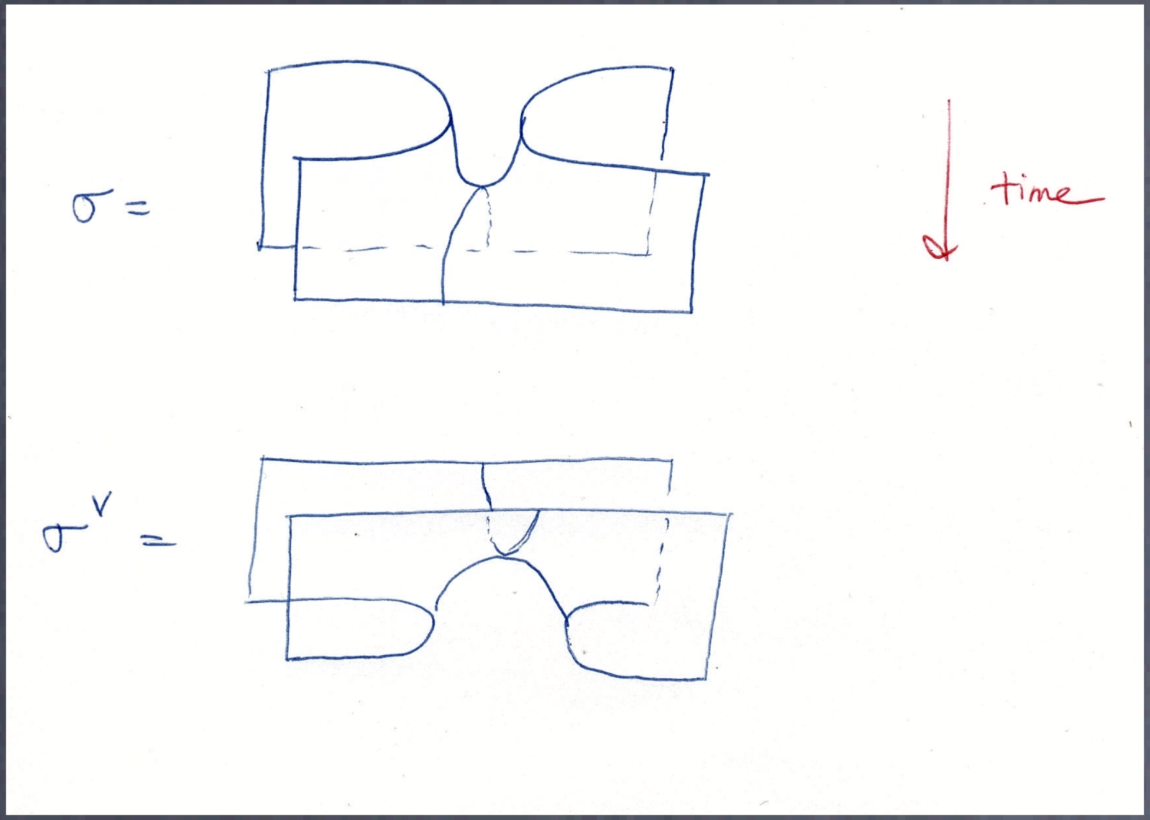
$$f^\vee \circ f = S^1 : 1 \longrightarrow 1$$
$$f \circ f^\vee \qquad : S^0 \longrightarrow S^0$$



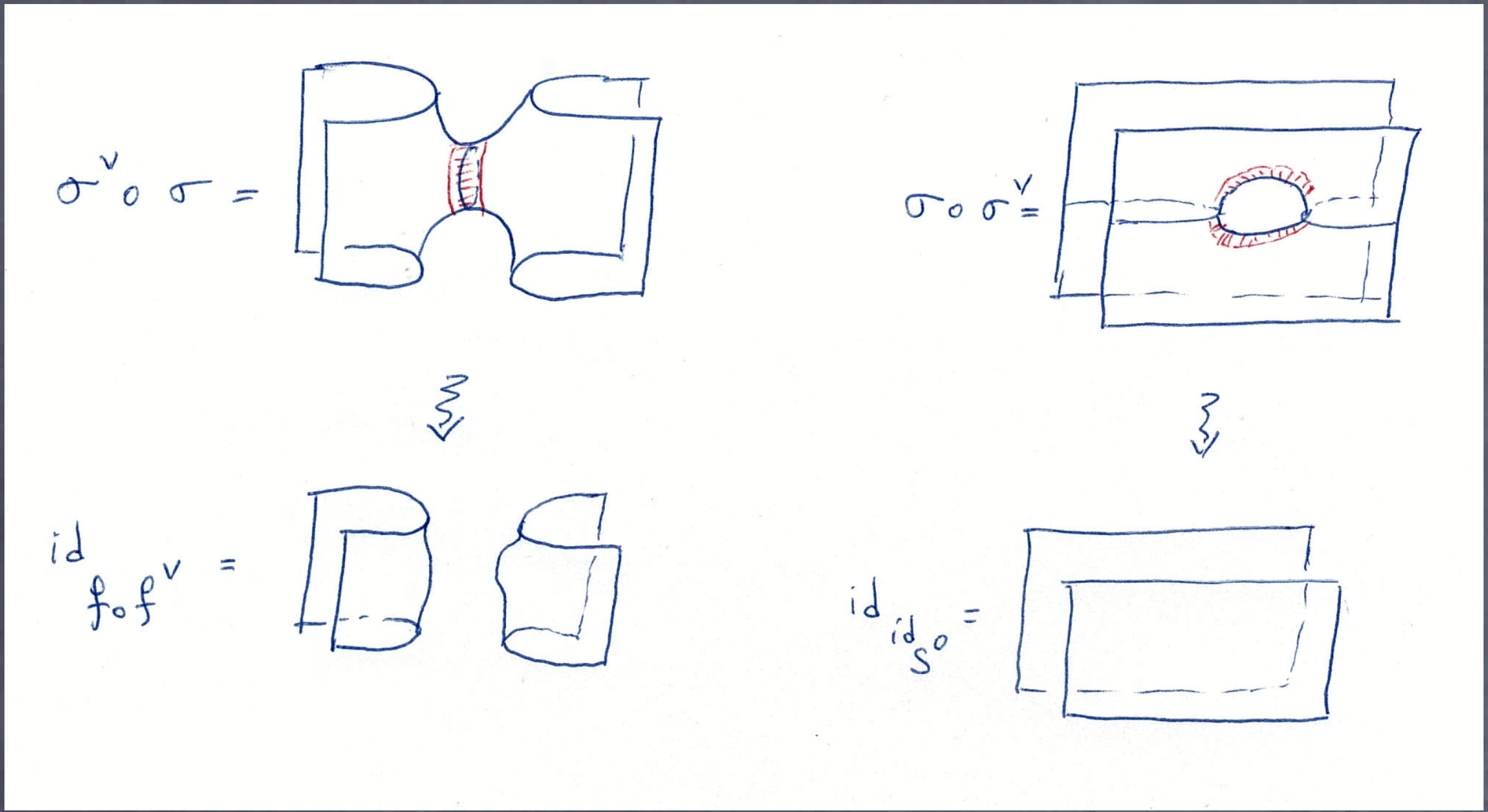
are the identity. We just did the first.

For the second, $\mathrm{id}_{S^0} =$ 

and we will show that the saddle $\sigma\colon f \circ f^\vee \to \mathrm{id}_{S^0}$ is an isomorphism with inverse $\sigma^\vee \otimes S^2$.



The saddle σ is diffeomorphic to $D^1 \times D^1$, which is a manifold with corners. Its dual $\sigma^\vee$ is the time-reversed bordism.



Inside each composition $\sigma^\vee \circ \sigma$ and $\sigma \circ \sigma^\vee$ we find a cylinder $\mathrm{id}_{S^1} = D^1 \times S^1$, which is $(S^2)^{-1} \otimes g \circ g^\vee = (S^2)^{-1} \otimes (S^0 \times D^2)$ by a previous argument. Making the replacement we get the desired isomorphisms to identity maps.

This completes the proof of the theorem in $n = 2$ dimensions.

In higher dimensions we see a kind of Poincaré duality phenomenon: we prove invertibility by assuming it in the middle dimension. A new ingredient—a dimensional reduction argument—also appears.

Application to Modular Tensor Categories

Let A be a braided tensor category with braiding $\beta_{x,y}\colon x \otimes y \to y \otimes x$.
Müger and others prove that the nondegeneracy condition on a MTC is equivalent to

$$\begin{aligned}\{x \in A : \beta_{y,x} \circ \beta_{x,y} = \mathrm{id}_{x\otimes y} \text{ for all } y \in A\} &= \{\text{multiples of } 1 \in A\} \\ &= \mathbf{Vect}_{\mathbb{C}}\end{aligned} \qquad (*)$$

Braided tensor categories form the objects of a 4-category:

object	category #
element of $\mathbb{C}$	-1
$\mathbb{C}$-vector space	0
$\mathbf{Vect}_{\mathbb{C}}$	1
$\mathbf{Cat}_{\mathbb{C}}$	2
$\mathbf{Cat}_{\mathbb{C}}^{\otimes} = \mathbf{E_1}(\mathbf{Cat}_{\mathbb{C}})$	3
$\mathbf{Cat}_{\mathbb{C}}^{\beta\otimes} = \mathbf{E_2}(\mathbf{Cat}_{\mathbb{C}})$	4

A MTC A is 4-dualizable and carries SO_4-invariance data, so defines $\alpha_A\colon\ \mathrm{Bord}_4^{w_1} \to \mathbf{Cat}_{\mathbb{C}}^{\beta\otimes}$ with $\alpha_A(\mathrm{pt}_+) = A$, and by $(*)$ we see $\alpha_A(S^2) = \mathbf{Vect}_{\mathbb{C}}$ is invertible. By the theorem α_A is invertible. (Crane-Yetter theory)

Corollary: A modular tensor category $A \in \mathbf{Cat}_{\mathbb{C}}^{\beta\otimes}$ is invertible.

SO_4-invariance

The SO_4-invariance is data, and it amounts to two nonzero complex numbers $\lambda, \mu \in \mathbb{C}^\times$. If W is a closed oriented 4-manifold, then

$$\alpha_A(W) = \lambda^{\mathrm{Sign}(W)}\, \mu^{\chi(W).}$$

We choose $\mu = 1$ so that $\alpha_A(W)$ depends only on the oriented bordism class of W.

We believe, but haven't yet checked carefully, that the SO_3-invariance of the module theory (to be introduced next) forces

$$\lambda = e^{2\pi i c/8},$$

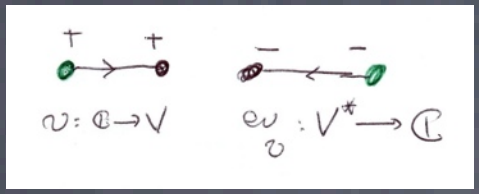
which can be computed from the MTC structure of A.

(c, which is only determined mod 8, is the *central charge*.)

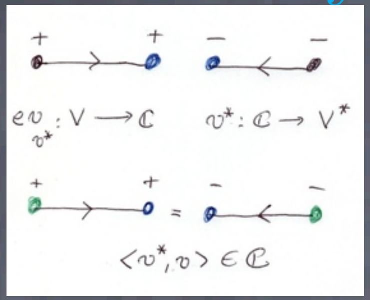
Boundary Conditions

A finite dimensional $V \in \mathbf{Vect}_{\mathbb{C}}$ determines a 1d oriented topological theory Z with $Z(\mathrm{pt}_+) = V$.

A vector $v \in V$ gives a boundary condition, so new pictures:



A vector $v^* \in V^*$ gives another boundary condition, so more pictures:

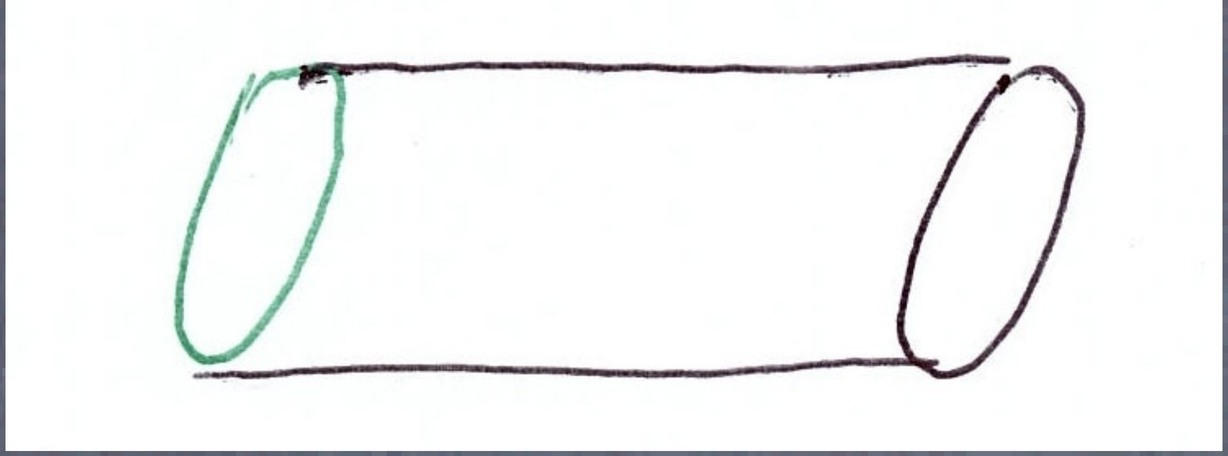


Similarly, left and right modules for an algebra $A \in \mathrm{Alg}_k$ give boundary conditions in a 2d field theory, given enough finiteness.

G a finite group, $A = \mathbb{C}[G]$

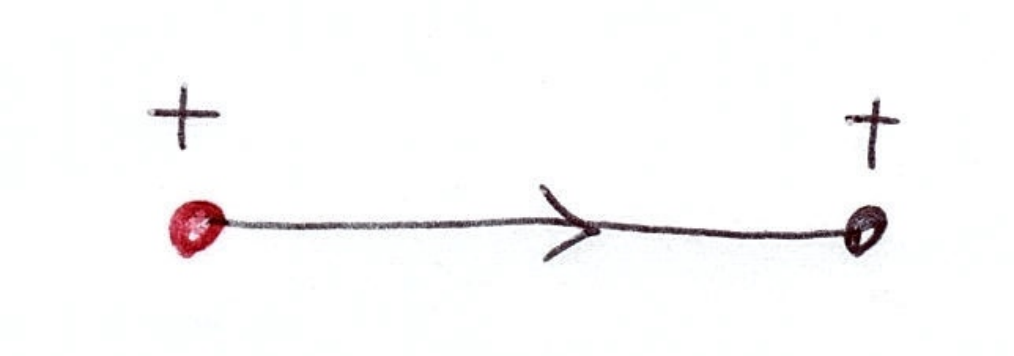
M a representation

This cylinder is the character of M in the center of A



Remark: Boundary conditions, and later domain walls and defects, are all special cases of Lurie's cobordism hypothesis with "singularities".

If $A \in \mathrm{Alg}_k$ is a 2-dualizable Frobenius algebra, then there is a special boundary condition for the associated 2-dimensional field theory, namely A as a left A-module, represented by the oriented interval with one colored endpoint:



A must satisfy a finiteness condition for the theory to exist. Also, whereas the uncolored (bulk) theory depends only on the Morita class of A, this is not true for the theory with boundary condition.

Anomalous Field Theories

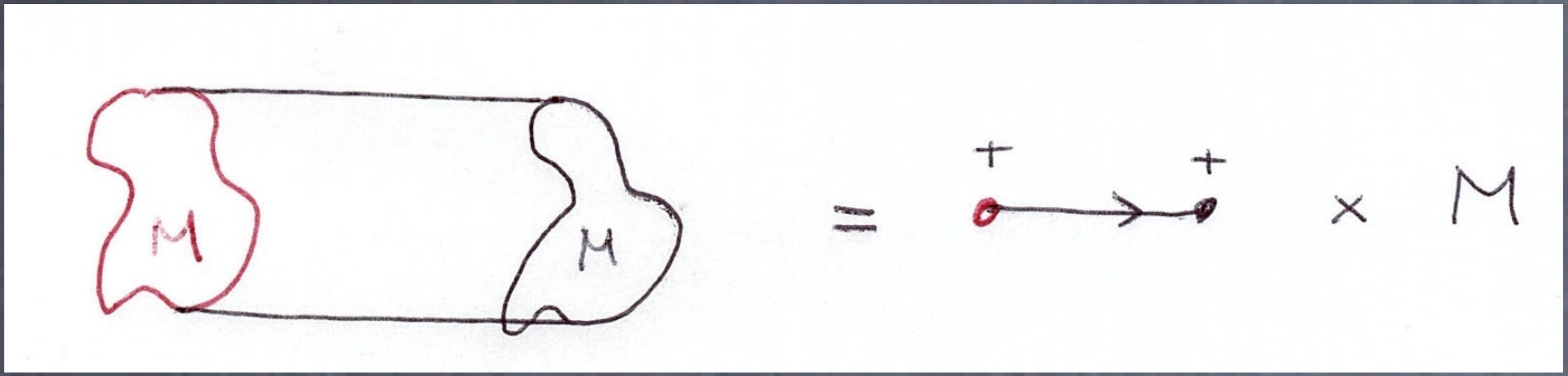
In general, if $\alpha\colon \mathrm{Bord}_n^{w_1} \to \mathcal{C}$ is a theory with $\alpha(\mathrm{pt}_+) = x \in \mathcal{C}$, then a boundary condition is a 1-morphism $1 \to x$ in $\mathcal{C}$.

It gives rise to an $(n-1)$-dimensional theory f *with values in* α. If α is invertible, we say f is *anomalous* with anomaly α.

In terms of the cobordism hypothesis with singularities, if M is any morphism in $\mathrm{Bord}_{n-1}^{w_1}$, then we associate to it the Cartesian product with the half-colored interval. Assume $\Omega^n\mathcal{C} = \mathbb{C}$ and $\Omega^{n-1}\mathcal{C} = \mathbf{Vect}_{\mathbb{C}}$. For example, if X is a closed oriented $(n-1)$-manifold, then $\alpha(X)$ is a complex line and

$$f(X)\colon \mathbb{C} \longrightarrow \alpha(X)$$

is an element of the line $\alpha(X)$.



The 3d Anomalous Oriented Theory

Let A be a modular tensor category, an invertible object in the 4-category $\mathbf{Cat}_{\mathbb{C}}^{\beta\otimes}$. There is an associated invertible 4-dimensional oriented field theory

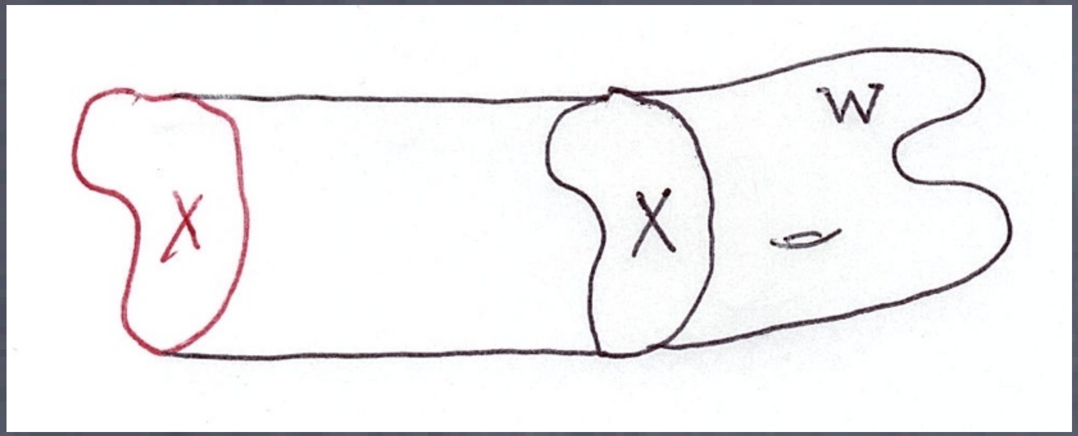
$$\alpha_A \colon \mathrm{Bord}_4^{w_1} \longrightarrow \mathbf{Cat}_{\mathbb{C}}^{\beta\otimes}$$

Now view A as a left A-module, so a 1-morphism $A\colon 1 \to A$ in $\mathbf{Cat}_{\mathbb{C}}^{\beta\otimes}$. ($A$ is a tensor category with the standard half-braiding of A. This distinguished boundary condition determines an anomalous oriented 3-dimensional theory f_A with values in α_A.

Remark: A is *3-adjointable*, which amounts to data and conditions for Morse handles with boundary. This is the necessary finiteness necessary to define f_A.

Trivializing the Anomaly

Suppose X is a closed oriented 3-manifold, and we write $X = \partial W$ for a compact oriented 4-manifold W with boundary.



The composition

$$\widehat{F}_A(X)\colon 1(X) = 1 \xrightarrow{f_A(X)} \alpha_A(X) \xrightarrow{\alpha_A(W)} \alpha_A(\varnothing^3) = 1$$

is multiplication by a number in $\mathbb{C}$.

$W \rightsquigarrow W'$ multiplies this by $\lambda^{2\pi icn/8}$, where $n = \mathrm{Sign}(W' \cup_X W)$.

Signature structure (σ) makes sense on 1-, 2-, 3-, and 4-manifolds, and every (w_1, σ)-manifold of these dimensions bounds a (w_1, σ)-manifold. Therefore, we recover the Reshetikhin-Turaev 1-2-3-theory $\widehat{F}_A$, defined on bordisms with a signature structure.

To get a proper bordism category we use a tangential structure based on p_1 (Blanchet-Habegger-Masbaum-Vogel). It is, in fact, a stable tangential structure. If M is an oriented bordism, a p_1-structure is a lift of a classifying map of TM:

$$\begin{array}{ccccc} & & BO\langle w_1, p_1\rangle & & \\ & \nearrow & \downarrow & & \\ M & \xrightarrow{TM} & BO\langle w_1\rangle & \xrightarrow{p_1} & K(\mathbb{Z}, 4) \end{array}$$

(w_1, p_1)-bordism groups:

$$\Omega^{(w_1,p_1)}_{\{0,1,2,3,4\}} \cong \{\mathbb{Z}\ ,\ 0\ ,\ 0\ ,\ \mathbb{Z}/3\mathbb{Z}\ ,\ 0\}$$

To define a non-anomalous theory on (w_1, p_1)-bordisms we:

(i) choose a cube root of $e^{2\pi ic/8}$;

(ii) formally extend the theory to pt_+ and pt_-.

The Formal Extension to pt_+ and pt_-

$$\mathcal{D} = \bigoplus_{n\in\mathbb{Z}} \mathcal{D}^n = \cdots \;\oplus\; (A^{\mathrm{op}})^{\otimes 2}\text{-Mod} \;\oplus\; A^{\mathrm{op}}\text{-Mod} \;\oplus\; \mathbf{Cat}_{\mathbb{C}}^{\otimes} \;\oplus\; A\text{-Mod} \;\oplus\; A^{\otimes 2}\text{-Mod} \;\oplus\; \cdots$$

$$\mathrm{End}(\mathcal{D}) = \bigoplus_{n\in\mathbb{Z}} \mathrm{End}^n(\mathcal{D})$$

$$\mathrm{End}^0(\mathcal{D}) = \mathbf{Cat}_{\mathbb{C}}^{\otimes}$$

$$x = A \otimes - \qquad (\deg x = +1)$$

$$y = A^{\mathrm{op}} \otimes - \qquad (\deg y = -1)$$

Since A invertible, as algebras $A \otimes A^{\mathrm{op}} \cong \mathrm{End}(A) \overset{\text{Morita}}{\approx} 1$, so as modules

$$A \otimes A^{\mathrm{op}} \overset{\text{Morita}}{\approx} A \otimes_{A\otimes A^{\mathrm{op}}} (A \otimes A^{\mathrm{op}}) \cong \underline{A}$$

$\mathcal{C}_A$ is the sub-3-category of $\mathrm{End}(\mathcal{D})$ generated by $\mathrm{End}^0(\mathcal{D}), x, y$.

Remark: Toy example (2d theory) with A an invertible superalgebra, $\mathbf{Cat}_{\mathbb{C}}^{\otimes} \rightsquigarrow \mathbf{Vect}_{\mathbb{C}}$, and $p_1 \rightsquigarrow \chi$. Now only 1-categories!

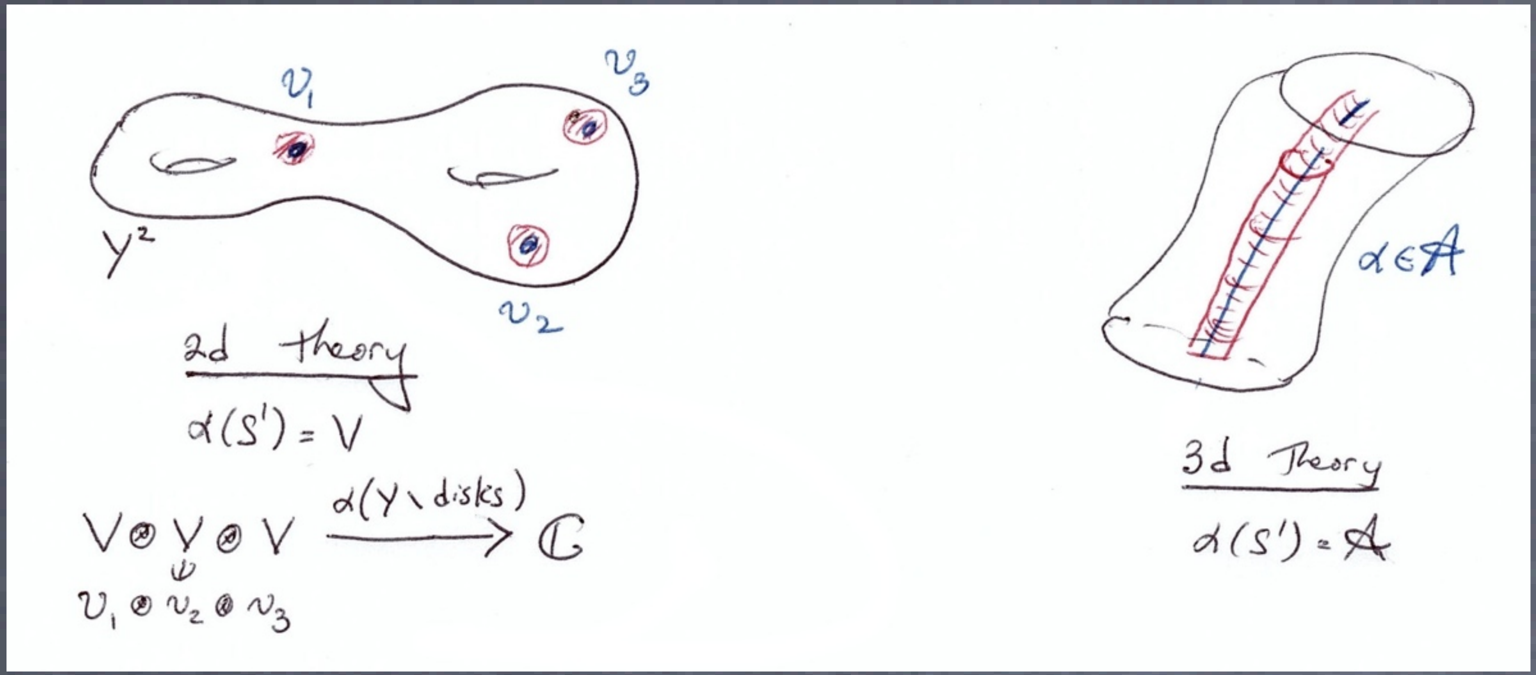
Defects

We end with two speculations. Both involve defects, which we review.

A defect on a bordism W is supported on a sub-bordism $Z \subset W$. Assume that Z is *normally* framed, so the link at each point is an ordinary sphere S^{k-1}, where Z has codimension k.

An ordinary defects in a theory α is labeled by an element in $\alpha(S^k)$, which we "integrate" over Z.

Examples: Let $k = 2$. 2-dimensional theory: label is a vector in the vector space $\alpha(S^1)$. 3-dimensional theory: label is an object in the category $\alpha(S^1)$.



Spin 3d Topological Field Theories

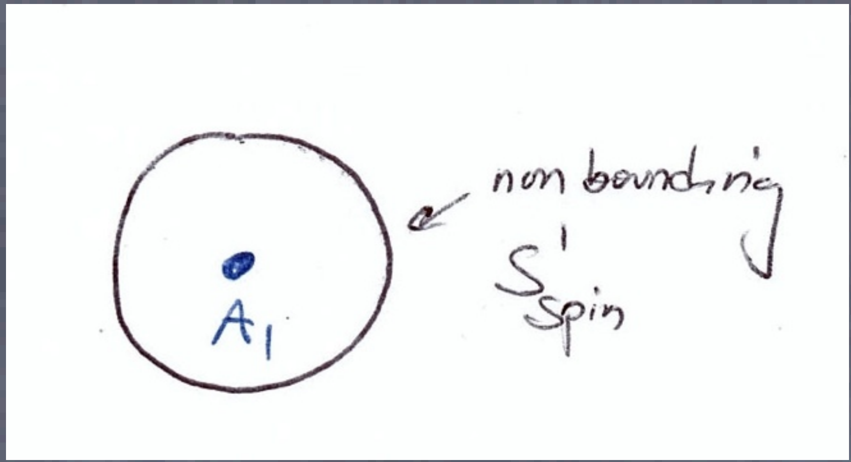
Suppose A_0 is a MTC with associated invertible theory $\alpha_A : \text{Bord}_4^{w_1} \to \mathbf{Cat}_{\mathbb{C}}^{\beta\otimes}$. Now let's try to make a theory of $(w_1, w_2, p_1/2)$-bordisms, or string bordisms.

$$\Omega^{\text{string}}_{\{0,1,2,3,4\}} \cong \{\mathbb{Z}\ ,\ \mathbb{Z}/2\mathbb{Z}\ ,\ \mathbb{Z}/2\mathbb{Z}\ ,\ \mathbb{Z}/24\mathbb{Z}\ ,\ 0\}$$

If X is a 4-framed bordism, we ignore w_2 and write $X = \partial W$ for a (w_1, p_1)-bordism W. Choose $Z \subset W$ of codimension 2 which represents $w_2(W)$. (No normal framing.) Observe $\alpha_{A_0}(S^1) \cong \mathbf{Cat}_{\mathbb{C}}$ (in $\mathbf{Cat}_{\mathbb{C}}^{\otimes}$).

This defect is labeled by a category A_1 which

- is a framed E_2-module over A_0
- has a map $A_1 \otimes A_1 \to A_0$ since $2w_2 = 0$



So $A_0 \oplus A_1$ is a "$\mathbb{Z}/2\mathbb{Z}$-graded MTC" and should define a spin 3d TQFT.

Factorizing 2d Conformal Field Theories

We now return to conformal field theories, in particular the Wess-Zumino-Witten model. The collection $\mathcal{C}$ of 2d conformal theories is a 2-category; a 1-morphism between theories is a codimension 1 defect called a domain wall.

Conjecture: A conformal field theory $T \in \mathcal{C}$ is rational if it can be endowed with a 3-dualizable algebra structure.

The algebra structure is data; the dualizability is a finiteness condition.

The algebra $\widehat{T} \in \mathrm{Alg}(\mathcal{C})$ determines a 3d *topological* theory $F_{\widehat{T}}$ which encodes the factorization of the conformal field theory T.

For T the WZW model, recall

$$T(S^1) \cong \bigoplus_{\alpha} \mathcal{H}_\alpha \otimes \overline{\mathcal{H}_\alpha} \cong \bigoplus_{\alpha} \mathrm{End}(\mathcal{H}_\alpha)$$

is an algebra. This is part of an algebra structure on T.

Remarks:

- We regard the conformal field theory T as a left $\widehat{T}$-module, so as a non-topological boundary condition for the 3d topological theory $F_{\widehat{T}}$. This theory is the chiral, or holomorphic, conformal field theory.
- For T the WZW model, the conformal field theory $F_{\widehat{T}}(S^1)$ should be the "G/G coset model", originally studied by Spiegelglas. It is a topological field theory, the 2-dimensional reduction of the Chern-Simons theory $F_{\widehat{T}}$.
- For G a finite group the conformal theory T is topological, so we can regard $\widehat{T} \in \mathrm{Alg}(\mathrm{TFT}_2)$. The cobordism hypothesis says evaluation on pt_+ is an injective map $\mathrm{TFT}_2 \to \mathbf{Cat}_{\mathbb{C}}$, so $\widehat{T} \in \mathrm{Alg}(\mathbf{Cat}_{\mathbb{C}}) \simeq \mathbf{Cat}_{\mathbb{C}}^{\otimes}$. Working this out recovers the previous description for finite group theories.
- For a general conformal field theory we might take evaluation on pt_+ to map to von Neumann (vN) algebras, so $\widehat{T}$ maps into $\mathrm{Alg}(\mathrm{vN})$. The latter is close to a conformal net...