Two-dimensional Ising model revisited

Dan Freed
University of Texas at Austin

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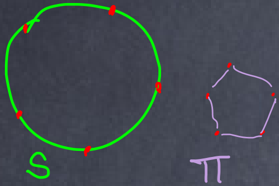
Joint work with Constantin Teleman

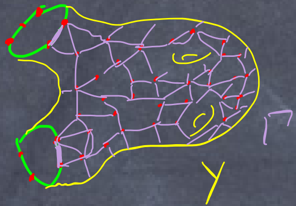
Latticed 1- and 2-manifolds

Definition:

(i) A *latticed 1-manifold* (S, Π) is a closed 1-manifold S equipped with a finite subset; $\Pi \subset S$ is an embedded graph, each component of which is a polygon.

(ii) A *latticed 2-manifold* (Y, Γ) is a compact 2-manifold Y equipped with a smoothly embedded finite graph $\Gamma \subset Y$ such that the closure of each face (component of $Y \setminus \Gamma$) is a smoothly embedded solid n-gon with $n \geq 2$. Furthermore, if e is an edge of Γ, then either (a) $e \cap \partial Y = \emptyset$, (b) $e \cap \partial Y$ is a single boundary vertex of e, or (c) $e \subset \partial Y$.





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- No choice of embedding of n-gons
- Loops are disallowed by the conditions
- Faces may share multiple edges

Ising model

$A = \mu_2 = \{\pm 1\}$ — abelian group of “spins”

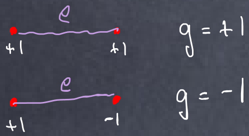
$\beta \in \mathbb{R}^{>0}$ — inverse temperature

$$\theta_\beta \colon A \longrightarrow \mathbb{R}^{\geq 0}$$
$$\pm 1 \longmapsto e^{\pm\beta}$$

weight function

$\mathcal{S}_{(Y,\Gamma)} = \mathrm{Map}\big(\mathrm{Vertices}(\Gamma), A\big)$ — configuration space of spins

$g \colon \mathcal{S}_{(Y,\Gamma)} \times \mathrm{Edges}(\Gamma) \to A$ — ratio of boundary spins



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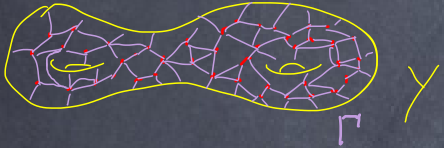

Y closed:

$$I(Y,\Gamma) = \sum_{s \in \mathcal{S}_{(Y,\Gamma)}} \prod_{e \in \mathrm{Edges}(\Gamma)} \theta_\beta\big(g(s;e)\big)$$

This is the Ising *partition function.* Note limits $\beta \to \infty$, $\beta \to 0$.

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The model can be defined for more general data:

G	finite group
$\theta \colon G \longrightarrow \mathbb{R}^{\geq 0}$	admissible function (later!)

Probabalistic interpretation:

$$\delta_s = \frac{\prod_{e \in \text{Edges}(\Gamma)} \theta_\beta\big(g(s;e)\big)}{I(Y,\Gamma)}$$

is a probability measure on $\mathcal{S}_{(Y,\Gamma)}$.

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Expectation value of a function

$$f : \mathcal{S}_{(Y,\Gamma)} \longrightarrow \mathbb{C}$$

such as $f(s) = s(v_1)s(v_2)$ for vertices v_1, v_2 (*order operator*):

$$\langle f \rangle = \sum_{s \in \mathcal{S}_{(Y,\Gamma)}} f(s)\delta_s$$

Quantum mechanical interpretation (Wick-rotated time):

Construct a functor

$$I\colon\ \mathrm{Bord}^{\mathrm{latticed}}_{\langle 1,2\rangle} \longrightarrow \mathrm{Vect}_{\mathbb{C}},$$

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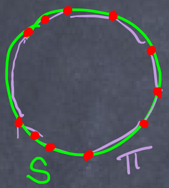
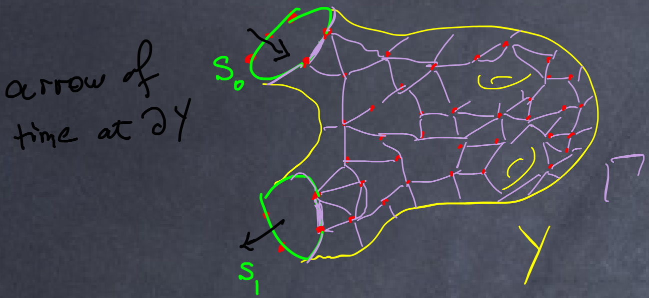
Objects: closed latticed 1-manifold (S,Π) maps to the vector space

$$I(S,\Pi) = \mathrm{Fun}(\mathcal{S}_{(S,\Pi)}) = \mathrm{Map}(\mathcal{S}_{(S,\Pi)},\mathbb{C})$$



Morphisms: 2d latticed bordism $(Y, \Gamma)\colon (S_0, \Pi_0) \to (S_1, \Pi_1)$ gives a correspondence diagram of spin configuration spaces

$$\begin{array}{ccc} & \mathcal{S}_{(Y,\Gamma)} & \\ {\scriptstyle r_0} \swarrow & & \searrow {\scriptstyle r_1} \\ \mathcal{S}_{(S_0,\Pi_0)} & & \mathcal{S}_{(S_1,\Pi_1)} \end{array}$$



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$$\mathcal{S}_{(S_0,\Pi_0)} \xleftarrow{\;r_0\;} \mathcal{S}_{(Y,\Gamma)} \xrightarrow{\;r_1\;} \mathcal{S}_{(S_1,\Pi_1)}$$

Define the linear map by push-pull

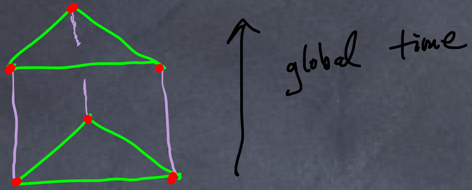
$$I(Y,\Gamma) = (r_1)_* \circ K \circ (r_0)^*\colon I(S_0,\Pi_0) \longrightarrow I(S_1,\Pi_1)$$

where the "kernel" K is the weight function

$$K(s) = \prod_e \theta_\beta\big(g(s;e)\big), \qquad e \text{ incoming or interior}$$

Wick-rotated discrete time evolution via product bordism ("prism")

$$(Y, \Gamma) = [0,1] \times (S, \Pi)$$



The resulting endomorphism of $I(S, \Pi)$ is called the *transfer matrix*. We write it as e^{-H}, where H is the *Hamiltonian*. Eigenvalues of H are energies (possibly infinite).

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- more general classes of models
- whole story in context of extended *topological* field theory
- higher dimensional abelian models (stable homotopy theory)

Fibering over BG

If a group G acts as a symmetry on mathematical object M (condition), we can try to extend (data) to a fibering

$$\begin{array}{ccc} M & \hookrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \hookrightarrow & BG \end{array}$$

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Equivariance $\longrightarrow$ Families

‘Fibering over BG’ in Ising Model

Definition: Z manifold. $\mathrm{Bun}_G(Z)$ groupoid. Objects: $P \to Z$ principal G-bundle. Morphisms: isos of G-bundles covering id_Z.

$$\mathrm{Bun}_G(\mathrm{pt}) \approx *//G$$

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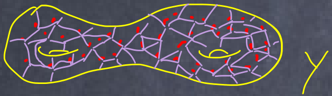
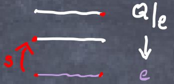
G-Ising model on Y^2: *background* lattice $\Gamma \subset Y$ and G-bundle $Q \to Y$
fluctuating field a “discrete gauged σ-model”

$$\mathcal{S}_{(Y,\Gamma)}[Q] = \text{sections of } Q \to Y \text{ over } \mathrm{Vertices}(\Gamma)$$

The ratio of spins defined via parallel transport

$$g\colon \mathcal{S}_{(Y,\Gamma)}[Q] \times \mathrm{Edges}(\Gamma) \longrightarrow G$$





The partition function of $I = I_{(G,\theta)}$ is now a function of a G-bundle:

$$I(Y,\Gamma)\colon\ \mathrm{Bun}_G(Y) \longrightarrow \mathbb{C}$$

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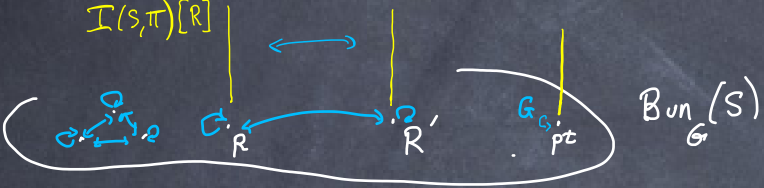
To a latticed 1-manifold (S,Π) we obtain a vector bundle

$$I(S,\Pi) \longrightarrow \operatorname{Bun}_G(S)$$

These are "twisted sectors"; the old state space is the fiber at $\mathrm{pt} \in BG$



Observation: The 3-dimensional finite gauge theory F_G satisfies:

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This is a general picture of symmetry in field theory. The novelty is to apply full force of F_G as an *extended* field theory.

3-dimensional finite gauge theory

G	finite group
$\mathrm{Bord}_3 = \mathrm{Bord}_{\langle 0,1,2,3\rangle}$	(unoriented) bordism 3-category
$\mathrm{TensCat}$	Morita 3-category of tensor categories/$\mathbb{C}$
$F_G\colon\ \mathrm{Bord}_3 \longrightarrow \mathrm{TensCat}$	symmetric monoidal functor

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For Y^2 closed sum (= (co)limit) the constant Vect-valued function $\mathbb{C}$:

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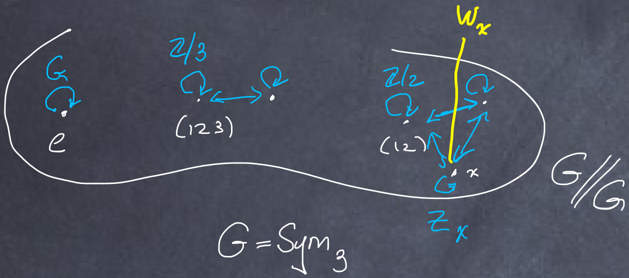
category of conjugation-equivariant G-bundles on G. Modular $\otimes$cat.



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$$F_G(\mathrm{pt}) = \mathrm{Vect}[G] \qquad (*)$$

tensor category of vector bundles on G under convolution—categorified group algebra.

$$(W' * W'')_x = \bigoplus_{x'x''=x} W'_{x'} \otimes W''_{x''}$$

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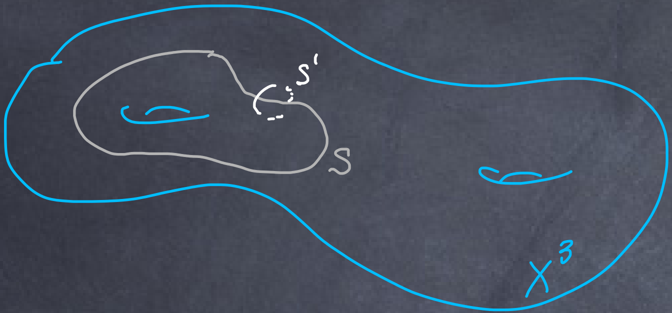
Construction 2—cobordism hypothesis: Simply specify $(*)$

Line operators

$S \subset X$ (oriented and co-oriented) 1d submanifold of X^3 closed

Link S^1 used to label S by objects of $F_G(S^1) = \mathrm{Vect}_G(G)$



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Wilson loops: $\mathrm{Rep}(G) \approx$ full subcategory of $\mathrm{Vect}_G(G)$ with support at $e \in G$. Classical expression using holonomy with character χ:

$$F_G\big(X; (S, \chi)_W\big) = \sum_{[P] \in \pi_0 \, \mathrm{Bun}_G(X)} \frac{h_{S,\chi}(P)}{\# \, \mathrm{Aut}\, P}.$$

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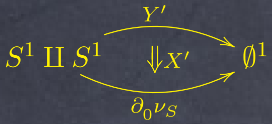
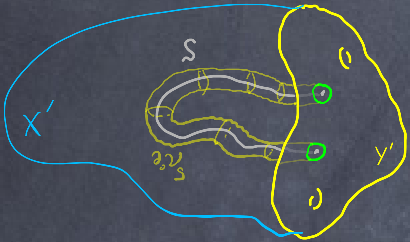
't Hooft loops: Full subcategory of $\mathrm{Vect}_G(G)$ in which centralizers Z_x act trivially on fiber at $x \in G$. Classical model sums bundles on $X \setminus S$ with specified holonomy about S.

If $\partial X \neq \emptyset$ there are line operators for neat 1d submanifolds $S \subset X$. Evaluate by cutting out tubular neighborhood ν_S.



$$X' = X \setminus \nu_S$$
$$Y' = X' \cap \partial X$$

Can evaluate explicitly on Wilson (parallel transport) and ’t Hooft



Electromagnetic duality

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Theorem: On oriented manifolds there is an isomorphism of theories

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For example, on Y^2 closed oriented, $\mathcal{F}$ is the Fourier transform

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A special case of usual 4d electromagnetism, shifted since A finite

For any finite G use the cobordism hypothesis to define

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Theorem: There is a Morita equivalence $\mathrm{Vect}[G] \approx \mathrm{Rep}(G)$, hence iso

$$\mathcal{F}\colon F_G \xrightarrow{\ \cong\ } \widehat{F_G}$$

of extended topological field theories

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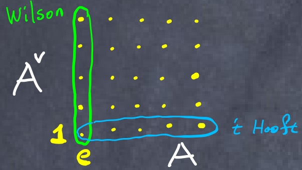
Note $F_A(S^1) = \mathrm{Vect}_A(A) \approx \mathrm{Vect}(A \times A^\vee)$; duality exchanges the factors



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Definition: A *topological boundary theory* for $F_G\colon \mathrm{Bord}_3 \to \mathrm{TensCat}$ is

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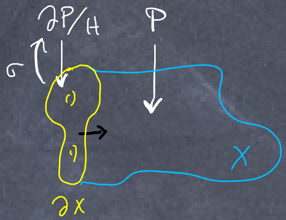
Classical model: boundary field a section of associated G/H-bundle

$$\mathbb{C} \xrightarrow{\beta(\partial X)} F_G(\partial X) \xrightarrow{F_G(X)} \mathbb{C}$$

$$\sum_{[P\to X]} \frac{1}{\#\operatorname{Aut} P} \sum_{\sigma\colon \partial X \to \partial P/H} \lambda_{\widetilde{H}}(\sigma^*\partial P \to \partial X)$$



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Neumann: subgroup $e \subset G$, so no new boundary field
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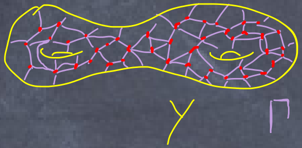
For (Y, Γ) closed obtain a function on $\operatorname{Bun}_G(Y)$:

$$I(Y,\Gamma)[Q] = \sum_{s \in \mathcal{S}_{(Y,\Gamma)}[Q]} \; \prod_{e \in \operatorname{Edges}(\Gamma)} \theta\big(g(s;e)\big)$$



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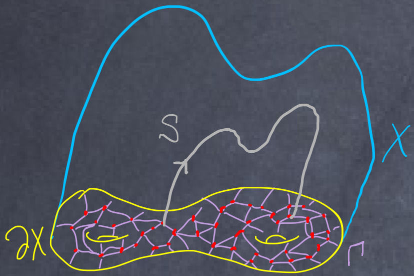

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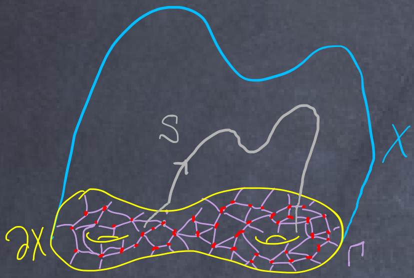

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- prediction for low energy behavior (discuss next)
- more general classes of models
- whole story in context of extended *topological* field theory
- higher dimensional abelian models (stable homotopy theory)

Low energy behavior; phase diagram

$\mathcal{M}$	moduli space of quantum theories
$\Delta \subset \mathcal{M}$	locus of phase transitions
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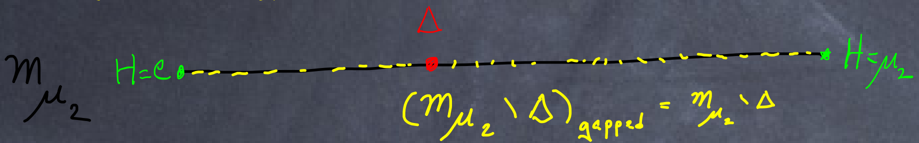

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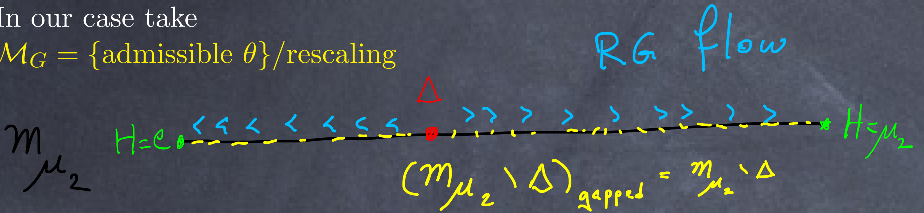

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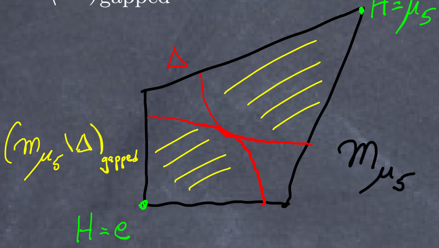

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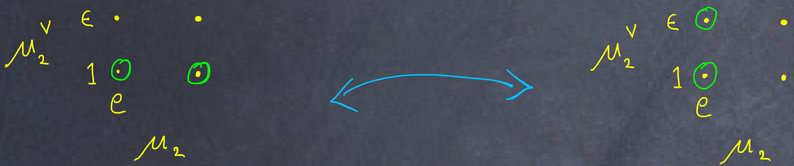

paramagnetic ($\beta \to 0$) ferromagnetic ($\beta \to \infty$)

EM duality

$(G = \mu_2)$

Topological construction; general theories

$\mathcal{T} = \mathrm{Vect}[G]$	categorified group algebra (white)
$\mathcal{B}_1 = \mathrm{Vect}[G]$	Neumann boundary theory (blue)
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Replace lattice Γ by a coloring via Morse function with critical points:

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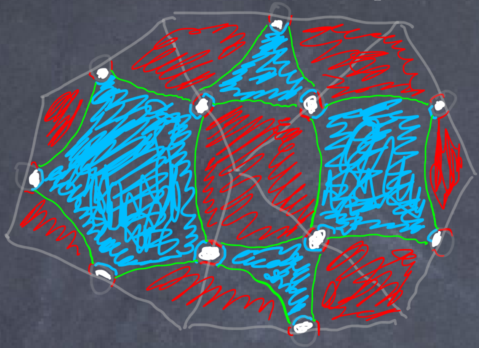
index 1	edges
index 2	faces



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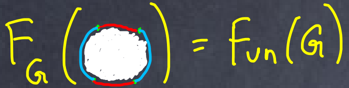

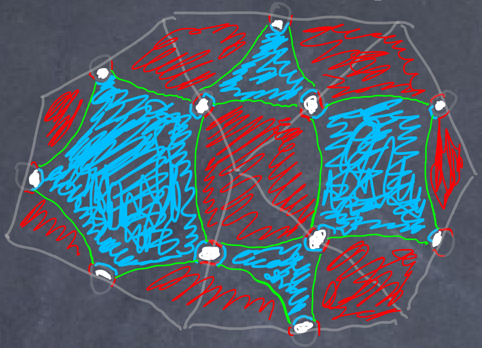

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Generalization: With an additional assumption on $(\mathcal{T}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{D})$, that $\mathrm{End}_{\mathcal{T}}(\mathcal{B}_i) \approx \mathcal{T}$, we can reduce to $\mathcal{B}_1 = \mathcal{T}$, $\mathcal{B}_2 = \mathrm{Vect}$. In that case $\mathcal{T}$ is the representation category of a *Frobenius Hopf algebra* H, exchanged by duality with H^*.

Abelian duality in higher dimensions

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$\mathcal{F}_X$	$\mathrm{Map}(X_+, S)$

n-dimensional theory F_S (finite path integral) with partition function

$$F_S(X) = \sum_{[\varphi]\in\pi_0\mathcal{F}_X} \frac{1}{\#\pi_1(\mathcal{F}_X,\varphi)} \frac{\#\pi_2(\mathcal{F}_X,\varphi)}{\#\pi_3(\mathcal{F}_X,\varphi)} \cdots$$

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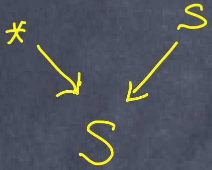

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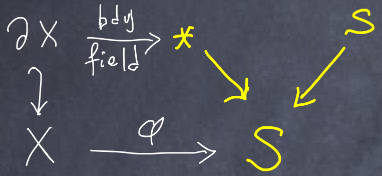

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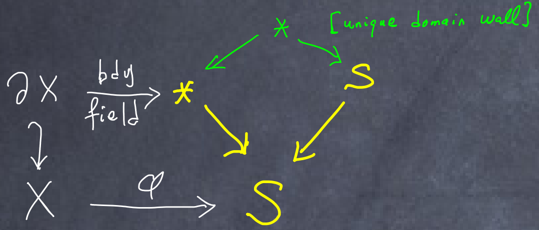

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