The Atiyah-Singer Index Theorem

Dan Freed
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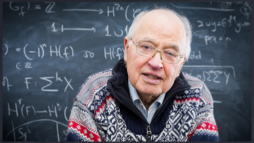
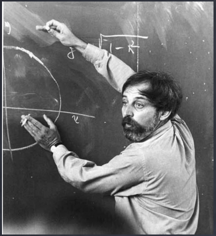
April 20, 2021

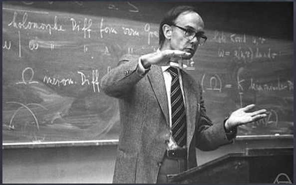
Gang of Four



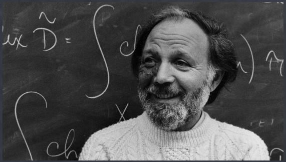
Atiyah



Bott



Hirzebruch



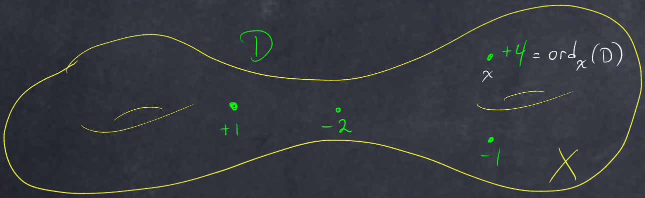
Singer

- 1952–1963: Hirzebruch Riemann-Roch, Bott periodicity, Atiyah-Hirzebruch K-theory, Atiyah-Singer index theorem
- Variations on the theme
- Global topological invariants $\rightsquigarrow$ local geometric invariants (of Dirac operators)
- An application to physics

Riemann-Roch theorem

X	smooth projective curve of genus g
D	divisor on X
$\mathcal{L}(D)$	meromorphic functions on X with pole of order $\leqslant \mathrm{ord}_x(D)$ at each $x \in X$

Problem: Compute $\dim \mathcal{L}(D)$



$\deg D = +1 - 2 + 4 - 1$

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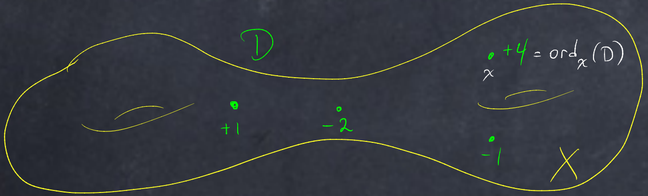
Theorem: If K is a canonical divisor of X, then

$$\dim \mathcal{L}(D) - \dim \mathcal{L}(K - D) = \deg(D) - g + 1$$



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X	smooth projective variety of dimension n
$V \longrightarrow X$	holomorphic vector bundle

Problem: Compute the *Euler characteristic* $\chi(X, V) = \sum_{q=0}^{n} (-1)^q \dim H^q(X, V)$

For $n = 2$ and $V \to X$ trivial of rank one, the *Noether formula* is classical:

$$\chi(X) = \frac{1}{12}\big(c_1^2(X) + c_2(X)\big)[X]$$

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$TX = L_1 \oplus \cdots \oplus L_n$ $\qquad$ $y_i = c_1(L_i) \in H^2(X;\mathbb{Z})$

$V = K_1 \oplus \cdots \oplus K_r$ $\qquad$ $x_i = c_1(K_i)$

splitting principle $\qquad$ first Chern classes

$$\boxed{\mathrm{Todd}(X) = \prod_{i=1}^{n} \frac{y_i}{1 - e^{-y_i}}} \qquad \boxed{\mathrm{ch}(V) = \sum_{i=1}^{r} e^{x_i}}$$

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Theorem: $\chi(X, V) = \mathrm{Todd}(X)\,\mathrm{ch}(V)[X]$

Integrality of the $\hat{A}$ genus

X compact smooth manifold of dimension $4k$

$TX \otimes \mathbb{C} = L_1 \oplus \overline{L_1} \cdots \oplus L_{2k} \oplus \overline{L_{2k}}$ $\quad y_i = c_1(L_i)$

$$\hat{A}(X) = \prod_{i=1}^{2k} \frac{y_i/2}{\sinh y_i/2}$$

X (almost) complex: $c_1(X) \equiv w_2(X) \pmod 2$

$\mathrm{Todd}(X) = e^{c_1(X)/2}\hat{A}(X)$ is a function of $c_1(X)$ and $p_i(X)$

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Question (Hirzebruch 1954): If X is compact smooth and $c \in H^2(X;\mathbb{Z})$ satisfies $c \equiv w_2(X) \pmod 2$, then is $e^{c/2}\hat{A}(X)$ an integer?

Special case ($c = 0$): Is the $\hat{A}$ genus of a spin manifold an integer?

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SOME PROBLEMS ON DIFFERENTIABLE AND COMPLEX MANIFOLDS

Friedrich Hirzebruch

(Received March 31, 1954)

A conference with the title *Fiber bundles and differential geometry* was held at Cornell University from May 3 to May 7, 1953.* It was supported by a grant from the National Science Foundation. The purpose of the present paper is to record those problems presented at the conference which concern differentiable,

The Riemann-Roch theorem of algebraic geometry [17][4] makes it rather natural to consider the multiplicative sequence of polynomials in the p_i which belongs to the power series

$$\frac{2\sqrt{z}}{\sinh 2\sqrt{z}}.$$

We denote this sequence of polynomials by $\{A_k\}$ and define the A-genus of an M^{4k} by

$$A(M^{4k}) = A_k(p_1, \cdots, p_k)\,[M^{4k}]. \text{ For example,}$$

$$A(M^4) = -\tfrac{2}{3}p_1[M^4]$$

$$A(M^8) = \tfrac{2}{45}(-4p_2 + 7p_1^2)[M^8]$$

Problem 7. *Determine the greatest integer $b(k)$ such that for all manifolds M^{4k} with vanishing second Stiefel-Whitney class the A-genus $A(M^{4k})$ is divisible by $2^{b(k)}$.* (Examples show that $b(k) \leqq 4k + 1$. Rohlin's theorem states $b(1) = 5$.)

In Section 2.1 of this report we shall point out that Problem 7 is related to certain problems concerning the Todd arithmetic genus.

We have mentioned above that the A-genus of an M^{4k} is always an integer. Actually this is a special case of a more general theorem [2a] which is motivated by the Riemann-Roch theorem $(M_{n,1})$.[4]

Problem 16. *Is the Todd genus $T(M_n)$ an integer for every almost-complex manifold M_n?*

Characteristic Classes and Homogeneous Spaces, II

A. Borel; F. Hirzebruch

American Journal of Mathematics, Vol. 81, No. 2. (Apr., 1959), pp. 315-382.

We do not know how far in 25.5 "integral exc 2" could be replaced by "integral." We can only dare the following *conjectures* which are motivated by the theorem of Riemann-Roch (see [18]). Let X be a compact oriented differentiable manifold and η a principal $\boldsymbol{U}(k)$-bundle over X.

1) *Let* w_2 *denote the second Stiefel-Whitney class of* X, ($w_2 \in H^2(X, \mathbf{Z}_2)$). *If* $d \in H^2(X, \mathbf{Z})$ *reduced* mod 2 *is* w_2, *then* $\hat{A}(X, d/2, \eta)$ *is an integer.*

2) *If* $w_2 = 0$ *and* $\dim X \equiv 4 \pmod 8$, *then* $\hat{A}(X)$ *is an even integer.*

2*) *If* $w_2 = 0$, $\dim X \equiv 4 \pmod 8$ *and if the structural group of* η *can be reduced to* $\boldsymbol{SO}(k)$, *then* $\hat{A}(X, 0, \eta)$ *is an even integer.*

These conjectures would be generalizations of Rohlin's theorem [24] that the

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Proposition. *Let* X *be a compact weakly almost complex manifold. Then for every* $d \in H^2(X, \mathbf{Z})$, *the number* $T(X, d)$ *is an integer.*

3. Integrality theorems for differentiable manifolds. For the definition of $\hat{A}(X, d)$ and $\hat{A}(X, d, \eta)$ we refer to [1, §§ 25.4, 25.5].

3.1. Theorem. *Let* X *be a compact oriented differentiable manifold and* d *an element of* $H^2(X, \mathbf{Z})$ *whose restriction* mod 2 *is equal to* $w_2(X)$. *Then* $\hat{A}(X, \frac{1}{2}d)$ *is an integer.*

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What is the integer $\hat{A}(X)[X]$? (X spin)

Grothendieck's Riemann-Roch theorem

- Introduction of K-theory
- Geometry over a base

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X	smooth projective variety
$K(X)$	free abelian group on sheaves $\mathcal{F}$ modulo $\mathcal{F} \sim \mathcal{F}' + \mathcal{F}''$ if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$

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$f : X \longrightarrow S$ proper morphism of nonsingular varieties

$f_! : \mathfrak{F} \longmapsto \sum_q (-1)^q R^q f_*(\mathfrak{F}) \in K(S)$ $R^q f_*(\mathfrak{F})$ sheafification of $U \mapsto H^q(f^{-1}(U), \mathfrak{F})$

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Theorem (Grothendieck 1957): For $\eta \in K(X)$ we have

$$\mathrm{Todd}(Y)\,\mathrm{ch}\big(f_!(\eta)\big) = f_*\big(\mathrm{Todd}(X)\,\mathrm{ch}(\eta)\big)$$

Stable homotopy of the orthogonal group

Theorem (Bott 1957): The homotopy groups of the *stable* orthogonal group O are:

$$\pi_{n-1}\mathrm{O} \cong \begin{cases} \mathbb{Z} & n \equiv 0 \pmod 8 \\ \mathbb{Z}/2\mathbb{Z} & n \equiv 1 \pmod 8 \\ \mathbb{Z}/2\mathbb{Z} & n \equiv 2 \pmod 8 \\ 0 & n \equiv 3 \pmod 8 \\ \mathbb{Z} & n \equiv 4 \pmod 8 \\ 0 & n \equiv 5 \pmod 8 \\ 0 & n \equiv 6 \pmod 8 \\ 0 & n \equiv 7 \pmod 8 \end{cases}$$

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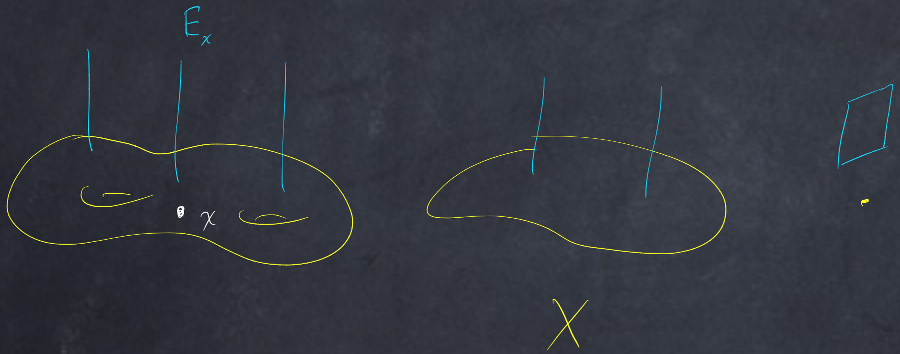
Atiyah-Hirzebruch used this as the cornerstone of topological K-theory, which is modeled on Grothendieck's Riemann-Roch theorem and K-theory in algebraic geometry

Topological K-theory

Let X be a nice compact topological space

$\mathrm{Vect}(X) = \{\text{isomorphism classes of real vector bundles } E \longrightarrow X\}$, $\underbrace{X \longrightarrow X}_{\text{0 vector bundle}}$, $\oplus$



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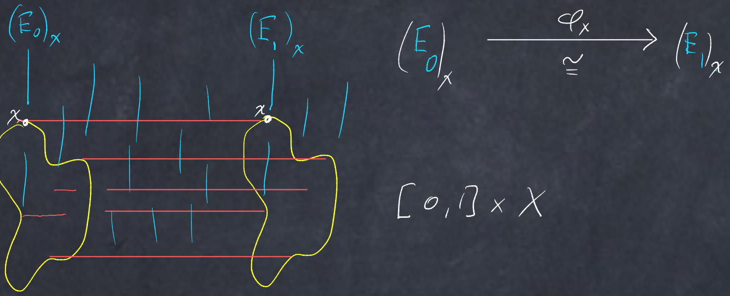

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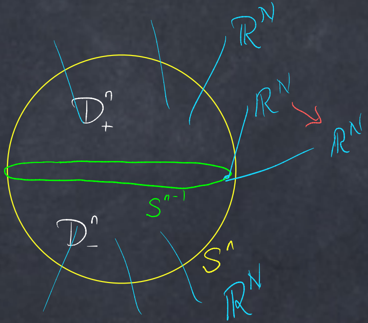

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Bott periodicity: $\boxed{KO^{-n}(\mathrm{pt}) = \widetilde{KO}(S^n) \cong \pi_{n-1}\mathrm{O}}$ and $\boxed{KO^{n+8}(X) \cong KO^n(X)}$

Riemann-Roch theorem for smooth manifolds

X, S	compact C^∞ manifolds, $\dim X - \dim S = n$
$f\colon X \longrightarrow S$	C^∞ map
$w_q(X) = f^* w_q(S) \quad (q = 1, 2)$	“orientation” ~~condition~~ data
$f_!\colon KO^\bullet(X) \longrightarrow KO^{\bullet - n}(S)$	induced *umkehr* map in KO-theory
$\mathrm{ch}\colon KO^\bullet(X) \longrightarrow H(S; \mathbb{Q}[v, v^{-1}])^\bullet$	Chern character, $\deg v = 4$
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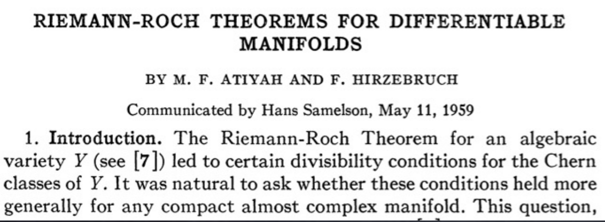
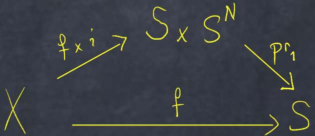


RIEMANN-ROCH THEOREMS FOR DIFFERENTIABLE MANIFOLDS

BY M. F. ATIYAH AND F. HIRZEBRUCH

Communicated by Hans Samelson, May 11, 1959

1. **Introduction.** The Riemann-Roch Theorem for an algebraic variety Y (see [7]) led to certain divisibility conditions for the Chern classes of Y. It was natural to ask whether these conditions held more generally for any compact almost complex manifold. This question,



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Theorem (Atiyah-Hirzebruch): For all $\eta \in KO^\bullet(X)$ we have

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Take $S = \mathrm{pt}$, $\eta = 1$ to deduce the integrality of $\hat{A}(X)[X]$ for a spin manifold X

$\hat{A}$ genus and spin representation

This is done by a universal construction on the classifying space of Spin$(2n)$, using the difference between the two spinor representations Δ^+ and Δ^- of Spin$(2n)$. The formula for chη is a consequence of the character formula:

$$\mathrm{ch}\Delta^+ - \mathrm{ch}\Delta^- = \prod_{i=1}^{n} (e^{x_i/2} - e^{-x_i/2}),$$

Compare:

$$\hat{A}(X) = \prod_{i=1}^{2k} \frac{y_i/2}{\sinh y_i/2}$$

$$\mathrm{ch}\,\mathbb{S}^0 - \mathrm{ch}\,\mathbb{S}^1 = \prod_{i=1}^{2k} \frac{\sinh y_i/2}{1/2}$$

What is the integer $\hat{A}(X)[X]$? (*Analytic* interpretation?)

X compact Riemannian manifold of dimension n

$$\Omega^0(X) \underset{d^*}{\overset{d}{\rightleftarrows}} \Omega^1(X) \underset{d^*}{\overset{d}{\rightleftarrows}} \Omega^2(X) \underset{d^*}{\overset{d}{\rightleftarrows}} \cdots \underset{d^*}{\overset{d}{\rightleftarrows}} \Omega^n(X)$$

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$\Delta = (dd^* + d^*d)$ Hodge-Laplace operator

$\mathcal{H}^q(X)$ space of solutions to $\Delta\omega = 0$, $\omega \in \Omega^q(X)$ *harmonic forms*

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$\mathcal{H}^q(X)$ space of solutions to $\Delta\omega = 0$, $\omega \in \Omega^q(X)$ *harmonic forms*

$$\text{Euler}(X) = \sum_{q=0}^{n} (-1)^q \dim \mathcal{H}^q(X)$$ Euler number

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$$\chi(X, V) = \text{Todd}(X)\,\text{ch}(V)[X] = \sum_{q=0}^{n} (-1)^q \dim \mathcal{H}^{0,q}(X)$$ X Kähler

What is the integer $\hat{A}(X)[X]$? (*Analytic* interpretation?)

X compact Riemannian manifold of dimension n

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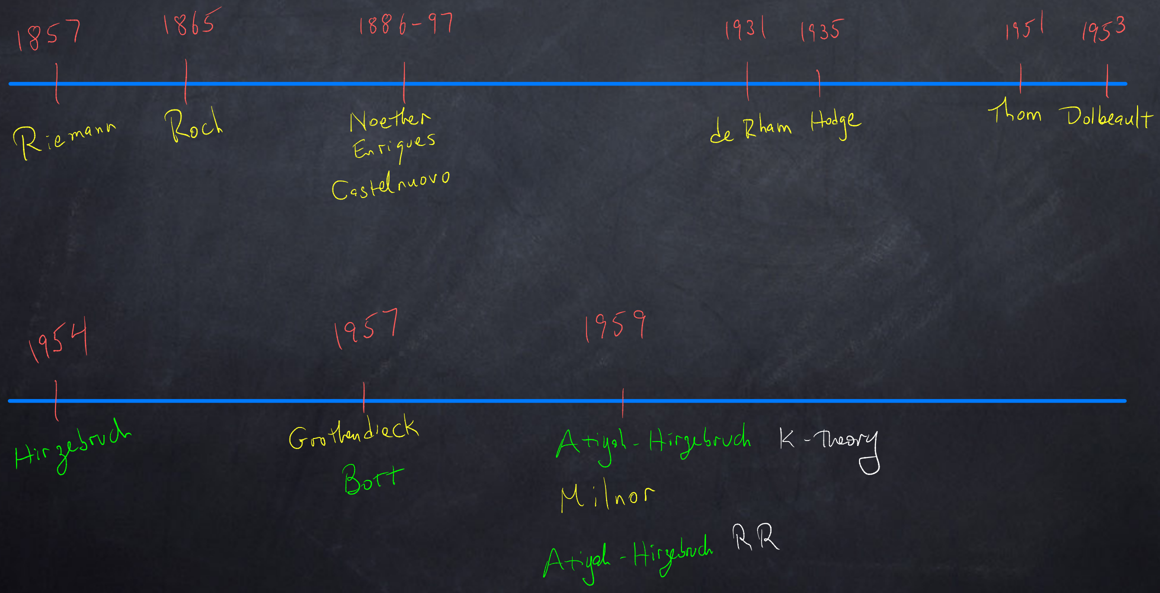
Question: Does $\hat{A}(X)[X]$ count solutions to a differential equation?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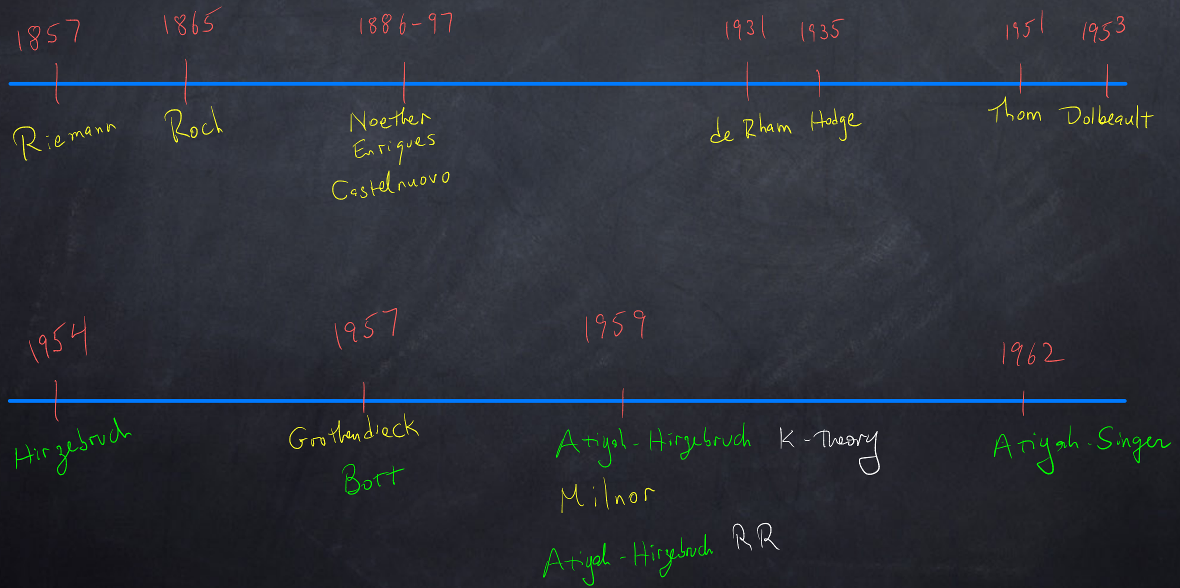
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1962
Atiyah-Singer

The Dirac operator

The Quantum Theory of the Electron.

By P. A. M. Dirac, St. John's College, Cambridge.

(Communicated by R. H. Fowler, F.R.S.—Received January 2, 1928.)

The symmetry between p_0 and p_1, p_2, p_3 required by relativity shows that, since the Hamiltonian we want is linear in p_0, it must also be linear in p_1, p_2 and p_3. Our wave equation is therefore of the form

$$(p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta)\,\psi = 0, \qquad (4)$$

where for the present all that is known about the dynamical variables or operators $\alpha_1, \alpha_2, \alpha_3, \beta$ is that they are independent of p_0, p_1, p_2, p_3, *i.e.*, that they commute with t, x_1, x_2, x_3. Since we are considering the case of a particle

$$\left.\begin{array}{ll} \alpha_r^2 = 1, & \alpha_r\alpha_s + \alpha_s\alpha_r = 0 \quad (r \neq s) \\ \beta^2 = m^2c^2, & \alpha_r\beta + \beta\alpha_r = 0 \end{array}\right\} \quad r, s = 1, 2, 3.$$

If we put $\beta = \alpha_4 mc$, these conditions become

$$\alpha_\mu^2 = 1 \qquad \alpha_\mu\alpha_\nu + \alpha_\nu\alpha_\mu = 0 \ (\mu \neq \nu) \qquad \mu, \nu = 1, 2, 3, 4. \qquad (6)$$

We can suppose the α_μ's to be expressed as matrices in some matrix scheme,

In $\mathbb{E}^n$:

$$D = \gamma^1 \frac{\partial}{\partial x^1} + \cdots + \gamma^n \frac{\partial}{\partial x^n}$$

$$\Delta = -\left\{ \left(\frac{\partial}{\partial x^1}\right)^2 + \cdots + \left(\frac{\partial}{\partial x^n}\right)^2 \right\}$$

$$D^2 = \Delta \qquad \Longleftrightarrow \qquad \gamma^i\gamma^j + \gamma^j\gamma^i = -2\delta^{ij} = \begin{cases} -2, & i = j; \\ 0, & i \neq j, \end{cases} \qquad 1 \leqslant i, j \leqslant n$$

Atiyah-Bott-Shapiro, *Clifford modules* (1963)

The Clifford algebras: $\mathrm{Cliff}_{\pm n}$: $\gamma^i\gamma^j + \gamma^j\gamma^i = \pm 2\delta^{ij}$ $(\mathrm{Cliff}_{\pm n} = \mathrm{Cliff}^0_{\pm n} \oplus \mathrm{Cliff}^1_{\pm n})$

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The spin group: $\mathrm{Spin}_n \subset \mathrm{Cliff}^0_{\pm n}$ Compare: $O_n \subset M_n\mathbb{R}$

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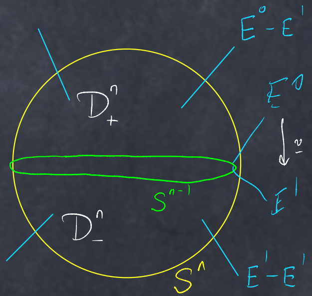
- $\{\text{left } \mathrm{Cliff}_{-n}\text{-modules}\} \longrightarrow KO^{-n}(\mathrm{pt}) \cong \widetilde{KO}(S^n)$

$E^0 \oplus E^1$

$$\sum_{i=1}^{n} x^i\gamma^i \longrightarrow \mathrm{Iso}(E^0, E^1), \qquad (x^1, \dots, x^n) \in S^{n-1}$$



The Atiyah-Singer Dirac operator (1962)

X	Riemannian spin manifold
$\mathrm{O}(X) \longrightarrow X$	bundle of orthonormal frames
$\partial_1, \dots, \partial_n$	tautological horizontal vector fields
$\mathrm{Spin}(X) \longrightarrow \mathrm{O}(X) \longrightarrow X$	lift to principal Spin_n-bundle
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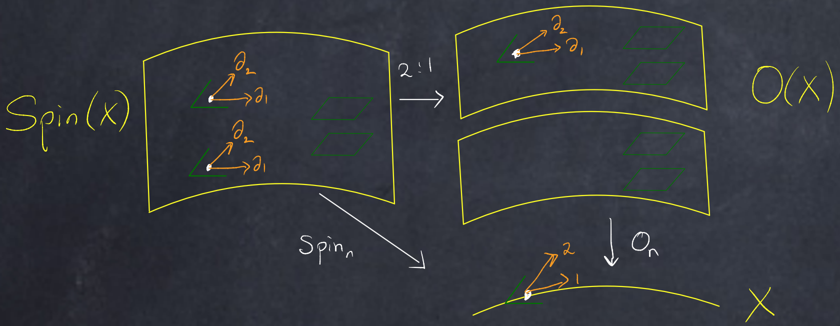

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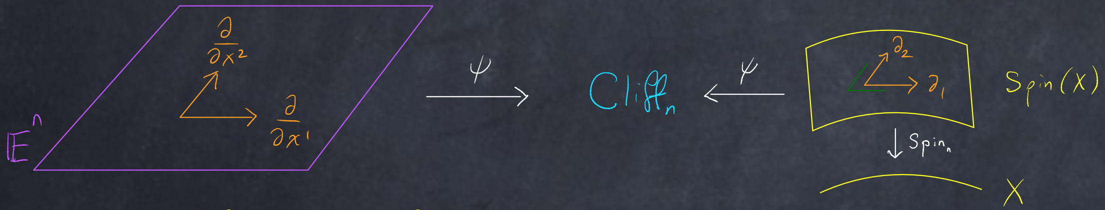

$\mathbb{E}^n$: $\quad D = \gamma^1 \frac{\partial}{\partial x^1} + \cdots + \gamma^n \frac{\partial}{\partial x^n} \circlearrowright \left(\psi\colon \mathbb{E}^n \longrightarrow \mathrm{Cliff}_{+n}\right) \circlearrowleft \mathrm{Cliff}_{+n}$

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Conjecture: $$\hat{A}(X)[X] = \dim_{\mathrm{Cliff}_{+n}} \mathcal{HS}^0(X) - \dim_{\mathrm{Cliff}_{+n}} \mathcal{HS}^1(X)$$

Fredholm operators

H^0, H^1 — Hilbert spaces

$\mathrm{Fred}(H^0, H^1) \subset \mathrm{Hom}(H^0, H^1)$ — Fredholm operators $T\colon H^0 \longrightarrow H^1$

$\mathrm{ind}\colon \pi_0 \,\mathrm{Fred}(H^0, H^1) \xrightarrow{\cong} \mathbb{Z}$ — $\mathrm{ind}\, T = \dim\ker T - \dim\operatorname{coker} T$

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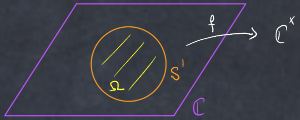
$\Omega \subset \mathbb{C}$ — unit disk



$S^1 = \partial \overline{\Omega}$

$H = L^2_{\mathrm{Hol}}(\overline{\Omega}, \mathbb{C}) \underset{\pi}{\overset{i}{\rightleftarrows}} \widetilde{H} = L^2(S^1, \mathbb{C})$ — L^2 holomorphic functions $\subset L^2$ functions

$M_f\colon \widetilde{H} \longrightarrow \widetilde{H}$ — multiplication by $f \in C^\infty(S^1, \mathbb{C}^\times)$

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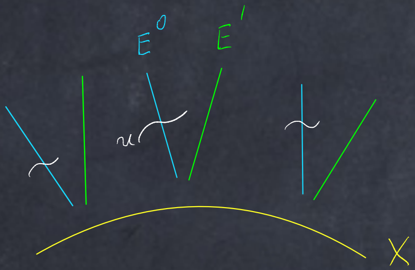
Fritz Noether (1920): T_f is Fredholm and $\mathrm{ind}\, T_f$ equals minus the winding number of f

Elliptic differential operators

An *elliptic differential operator* $P\colon C^\infty(X,E^0)\to C^\infty(X,E^1)$ has the local form

$$Pu = a^{i_1 i_2 \dots i_m}\frac{\partial^m u}{\partial x^{i_1}\partial x^{i_2}\dots\partial x^{i_m}} + \text{lower order terms}$$



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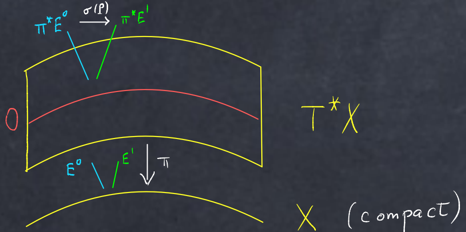
The highest order term is a global tensor field, the *symbol*,

$$\sigma(P)\colon \operatorname{Sym}^m(T^*X) \otimes E^0 \longrightarrow E^1$$

which is an isomorphism $\sigma(P)(\theta, \dots, \theta)\colon E^0_x \to E^1_x$ for all $\theta^{\neq 0} \in T^*_x X$



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ON ELLIPTIC EQUATIONS

I. M. GEL'FAND

The main idea of the paper is contained in § 2, where we pose the problem of describing linear elliptic equations and their boundary problems in topological terms. The most important of the properties in the large of the solutions of these equations and problems are preserved under small deformations of the problem and must therefore be, in some sense, homotopy invariants. The discovery and study of these invariants is the right way to sort out the whole multiplicity of boundary problems for elliptic equations and to classify these problems.

Thus there are two important questions here: firstly *to find all homotopy invariants of elliptic problems* (i.e. equations with boundary conditions) and, secondly, *to discover what these invariants mean in terms of the solutions of the equations.*

The Atiyah-Singer index theorem (1963)

THE INDEX OF ELLIPTIC OPERATORS ON COMPACT MANIFOLDS

BY M. F. ATIYAH AND I. M. SINGER[1]

Communicated by Raoul Bott, February 1, 1963

Introduction. In his paper [16] Gel'fand posed the general problem of investigating the relationship between topological and analytical invariants of elliptic differential operators. In particular he suggested that it should be possible to express the *index* of an elliptic operator (see §1 for the definition) in topological terms. This problem has been taken up by Agranovic [2; 3], Dynin [3; 14; 15], Seeley [20; 21] and Vol'pert [22] who have solved it in special cases. The purpose of this paper is to give a general formula for the index of an elliptic operator on any compact oriented differentiable manifold (Theorem

THEOREM 1. *For any elliptic differential operator D on a compact oriented differentiable manifold X the index $\gamma(D)$ is given by the formula*

$$\gamma(D) = \{\mathrm{ch}\,(D)\cdot\mathfrak{J}(X)\}[X].$$

$$\mathfrak{J}(X) = \prod_j \frac{y_j}{1-e^{-y_j}}\cdot\frac{-y_j}{1-e^{y_j}}$$

Analytic index: $[\sigma(P)] \longmapsto \mathrm{ind}\, P$

Topological index: $[\sigma(P)] \in K^0(T^*X, T^*X\backslash 0) \xrightarrow{\mathrm{ind}} \mathbb{Z}$

The index of elliptic operators: I*

By M. F. ATIYAH and I. M. SINGER

Introduction

This is the first of a series of papers which will be devoted to a study of the index of elliptic operators on compact manifolds. The main result was announced in [6] and, for manifolds with boundary[1], in [5]. The long delay between these announcements and the present paper is due to several factors. On the one hand, a fairly detailed exposition has already appeared in [14]. On the other hand, our original proof, reproduced with minor modifications in [14], had a number of drawbacks. In the first place the use of cobordism, and the computational checking associated with this, were not very enlightening. More seriously, however, the method of proof did not lend itself to certain natural generalizations of the problem where appropriate cobordism groups were not known. The reader who is familiar with the Riemann-Roch theorem will realize that our original proof of the index theorem was modelled closely on Hirzebruch's proof of the Riemann-Roch theorem. Naturally enough we were led to look for a proof modelled more on that of Grothendieck.

THEOREM (6.7). *The analytical index and the topological index coicide as homomorphisms $K_G(TX) \to R(G)$.*

Remarks

- A key analytic ingredient in the first proof is an elliptic boundary value problem with local boundary conditions to prove the bordism invariance of the index

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- *Psuedodifferential* elliptic operators play a crucial role throughout the theory

- If X is an n-dimensional *spin* manifold, Bott periodicity implies that every elliptic symbol class is represented by a Dirac operator twisted by a real vector bundle $E \longrightarrow X$, and the topological index reduces to $f_![E]$, where $f\colon X \longrightarrow \mathrm{pt}$ and

$$f_!\colon KO^0(X) \longrightarrow KO^{-n}(\mathrm{pt})$$

  is the *umkehr* map

- 1952–1963: Hirzebruch Riemann-Roch, Bott periodicity, Atiyah-Hirzebruch K-theory, Atiyah-Singer index theorem

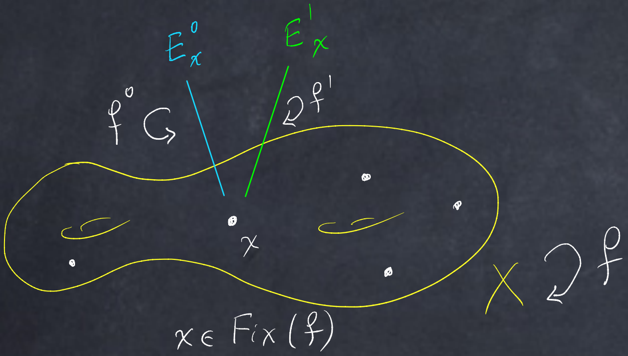
- Variations on the theme

- Global topological invariants $\rightsquigarrow$ local geometric invariants (of Dirac operators)

- An application to physics

Atiyah-Bott fixed point theorem

$f\colon X \longrightarrow X$	diffeomorphism with isolated fixed points
$E^0, E^1 \longrightarrow X$	vector bundles
$f^i\colon E^i \longrightarrow E^i$	linear lifts of f
$P\colon C^\infty(X, E^0) \longrightarrow C^\infty(X, E^1)$	elliptic differential operator



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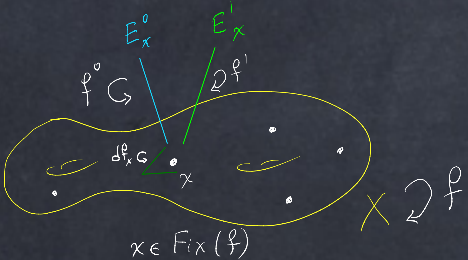
Theorem: $$\mathrm{Tr}\left(f^0|_{\ker P}\right) - \mathrm{Tr}\left(f^1|_{\mathrm{coker}\, P}\right) = \sum_{x \in \mathrm{Fix}(f)} \frac{\mathrm{Tr}\left(f^0|_{E^0_x}\right) - \mathrm{Tr}\left(f^1|_{E^1_x}\right)}{\left|\det(1 - df_x)\right|}$$



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- Weyl character formula for representations of compact Lie groups
- Let X be a connected closed complex manifold with $H^q(X; \mathcal{O}_X) = 0$ for $q > 0$; then any holomorphic map $f\colon X \to X$ has a fixed point

Atiyah-Bott fixed point theorem

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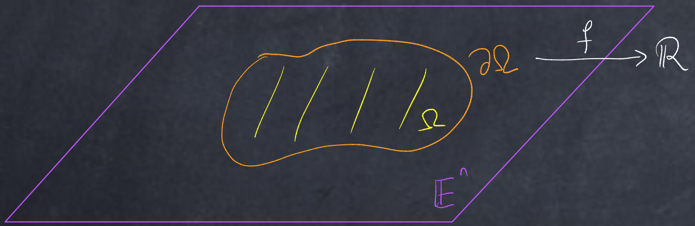
Theorem: $$\mathrm{Tr}\left(f^0\big|_{\ker P}\right) - \mathrm{Tr}\left(f^1\big|_{\operatorname{coker} P}\right) = \sum_{x \in \mathrm{Fix}(f)} \frac{\mathrm{Tr}\left(f^0\big|_{E^0_x}\right) - \mathrm{Tr}\left(f^1\big|_{E^1_x}\right)}{\left|\det(1 - df_x)\right|}$$

- Weyl character formula for representations of compact Lie groups
- Let X be a connected closed complex manifold with $H^q(X; \mathcal{O}_X) = 0$ for $q > 0$; then any holomorphic map $f\colon X \to X$ has a fixed point
- Hirzebruch-Zagier: cotangent sums, Dedekind η, modular forms, real quadratic fields by studying lens spaces, projective spaces, Brieskorn varieties, and algebraic surfaces

Atiyah-Bott-Singer index theorem on manifolds with boundary

Classical Dirichlet problem: $\Delta u = 0$ on $\Omega \subset \mathbb{E}^n$

$u\big|_{\partial\Omega} = f$ for prescribed $f\colon \partial\Omega \to \mathbb{R}$



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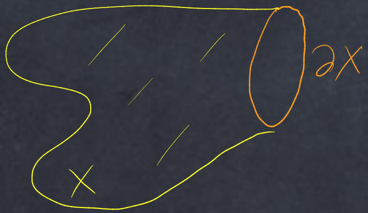

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The index is the *umkehr* map applied to the relative symbol

Atiyah-Singer index theorem for families

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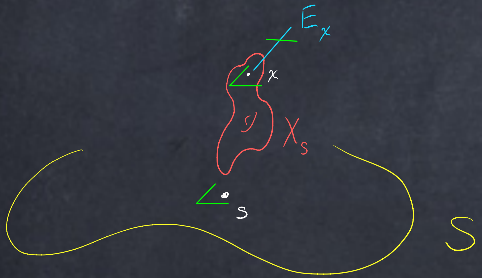

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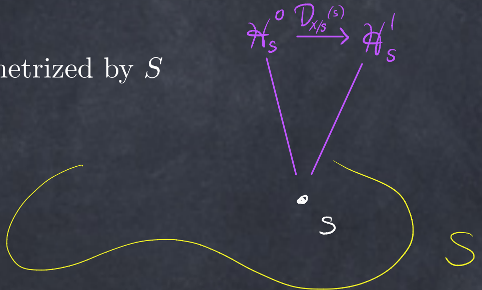
Topological index: $f_!: KO^0(X) \longrightarrow KO^{-n}(S)$



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Theorem: $\operatorname{ind} D_{X/S} = f_!([E])$

- 1952–1963: Hirzebruch Riemann-Roch, Bott periodicity, Atiyah-Hirzebruch K-theory, Atiyah-Singer index theorem

- Variations on the theme

- Global topological invariants $\rightsquigarrow$ local geometric invariants (of Dirac operators)

- An application to physics

Zeta functions and heat kernels

- kernel only $\rightsquigarrow$ complete spectrum

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$\Delta \geqslant 0$ self-adjoint second-order elliptic operator on sections of $E \longrightarrow X$:

$$H_t = e^{-t\Delta}, \qquad (t \in \mathbb{R}^{>0}) \qquad \textit{heat operator}$$

$$\zeta_\Delta(s) = \operatorname{Tr} \Delta^{-s}, \qquad (s \in \mathbb{C},\ \operatorname{Re}(s) \gg 0) \qquad \textit{zeta function}$$

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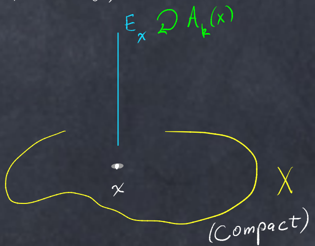
Asymptotic expansion of the heat kernel (Minakshisundarum-Pleijel, Seeley):

$$h_t(x, y) = \left(e^{-t\Delta}\delta_y\right)(x), \qquad x, y \in X,$$

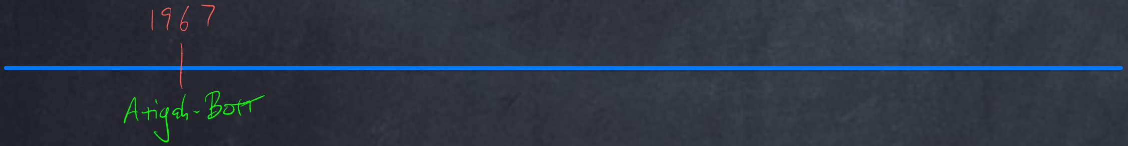
$$\boxed{h_t(x, x) \sim t^{-n/2} \sum_{k=0}^{\infty} A_k(x) t^i} \qquad \text{as } t \to 0$$



Equivalent to meromorphic continuation of $\zeta_\Delta(s)$ to $s \in \mathbb{C}$

The Atiyah-Bott formula

X	closed n-dimensional Riemannian manifold
$E^0, E^1 \longrightarrow X$	vector bundles
$P\colon C^\infty(X, E^0) \longrightarrow C^\infty(X, E^1)$	first-order elliptic operator
$\mathcal{E}^i_\lambda \subset C^\infty(X, E^i)$	λ-eigenspace of P^*P $(i = 0)$ and PP^* $(i = 1)$



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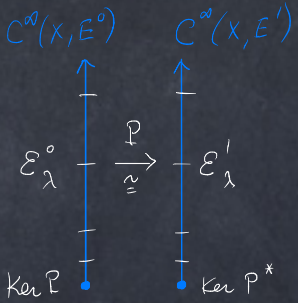
$$P\big|_{\mathcal{E}^0_\lambda} : \mathcal{E}^0_\lambda \longrightarrow \mathcal{E}^1_\lambda$$



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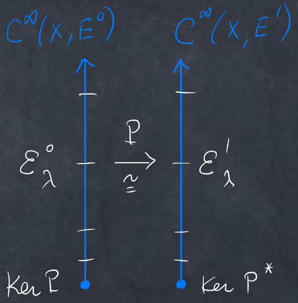
Index formulas:

$$\begin{aligned} \operatorname{ind} P &= \operatorname{Tr} \zeta_{P^*P}(s) - \operatorname{Tr} \zeta_{PP^*}(s) \\ &= \operatorname{Tr} e^{-tP^*P} - \operatorname{Tr} e^{-tPP^*} \\ &= \int_X \operatorname{tr}\left[A^0_{n/2}(x) - A^1_{n/2}(x)\right] |dx| \end{aligned}$$



The local index theorem

Mark Kac: What do the eigenvalues of $\Delta^{(q)} \circlearrowleft \Omega^q_X$ determine of Riemannian n-manifold X?

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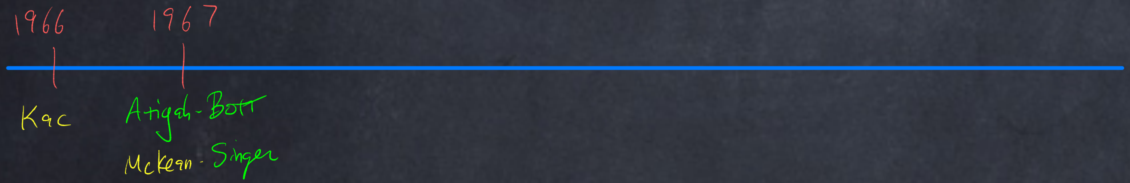
$$h_t^{(q)}(x,x) \sim t^{-n/2} \sum_{k=0}^{\infty} A_k(x) t^i$$

Weyl: $\int_X A_0^{(0)}(x)\,|dx| = (4\pi)^{-n/2}\,\mathrm{Vol}(X)$

McKean-Singer: $A_1^{(0)}(x) = (4\pi)^{-n/2} R(x)/3$



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For $n = 2$ they proved and conjectured in general

$$\lim_{t\to 0} \sum_{q=0}^{n} (-1)^q \operatorname{tr} h_t^{(q)}(x,x) = \sum_{q=0}^{n} (-1)^q \operatorname{tr} A^{(q)}_{n/2}(x)$$

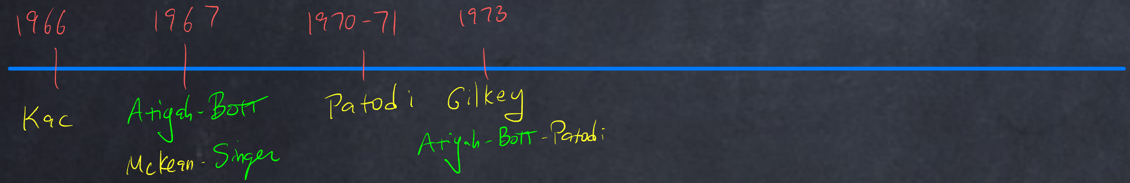
exists and equals the Gauss-Bonnet-Chern integrand for the Euler number of X

Existence of limit $\iff \sum_{q=0}^{n} (-1)^q \operatorname{tr} A_k^{(q)}(x) = 0, \quad k < \frac{n}{2}$ (cancellation for all $x \in X$)

Patodi proved the conjecture and analog for Riemann-Roch on Kähler manifolds

Gilkey thesis: same for twisted signature operators

Atiyah-Bott-Patodi: exposition of Gilkey and general local index theorem



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ω *homogeneous of weight* k: $\omega(\lambda^2 g) = \lambda^k \omega(g)$

ω *regular*: $$\omega(g)(x) = \sum_I \sum_\alpha^{\text{finite}} \sum_{i,j=1}^n \omega_{I,\alpha}^{i,j}(x) \frac{\partial^{|\alpha|} g_{ij}}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_n}}\, dx^{i_1} \wedge \cdots \wedge dx^{i_q}$$

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Theorem: A natural differential form which is regular and homogeneous of nonnegative weight is a polynomial in the Chern-Weil forms of the Pontrjagin classes

Analytic insights into $\hat{A}$ genus

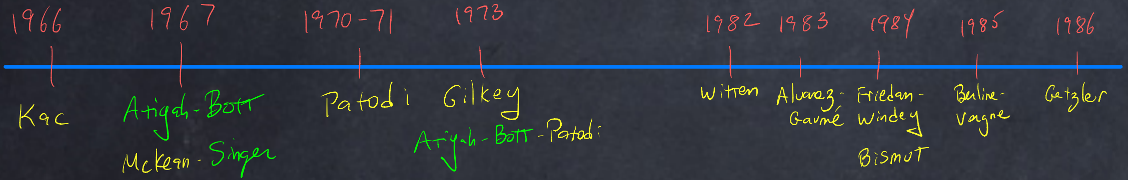
- Dirac operators; cancellation using Clifford algebra symmetry

Getzler: scaling argument, $\hat{A}$ from heat kernel of the harmonic oscillator (Mehler's formula)

Witten, Alvarez-Gaumé, Friedan-Windey, Atiyah: supersymmetric quantum mechanics, $\hat{A}$ from infinite product and Duistermaat-Heckman formula

Bismut: Wiener measure and Malliavin calculus, $\hat{A}$ from Lévy formula

Berline-Vergne: heat kernel on frame bundle, $\hat{A}$ from differential of exponential map on O_n

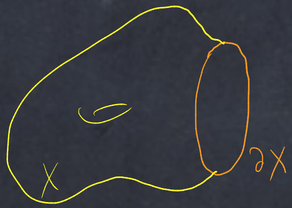


The signature defect

Gauss-Bonnet: X compact Riemannian 2-manifold

$$\mathrm{Euler}(X) = \int_X \frac{K}{2\pi}\, d\mu_X \qquad (X \text{ closed, } K \text{ Gauss curvature})$$

$$= \int_X \frac{K}{2\pi}\, d\mu_X + \int_{\partial X} \frac{\kappa}{2\pi}\, d\mu_{\partial X} \qquad (\kappa \text{ geodesic curvature of } \partial X)$$



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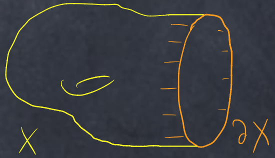
X compact with boundary, product metric near boundary

$$\alpha(Y) = \mathrm{Sign}(X) - \int_X \omega \qquad (\text{signature defect})$$



Hirzebruch (1973): Hilbert modular surfaces, Shimizu L-functions

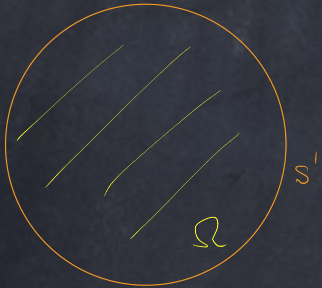
Atiyah-Patodi-Singer global boundary conditions

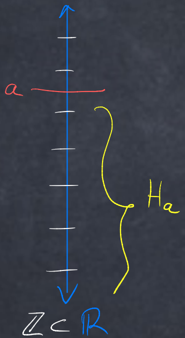
$\Omega \subset \mathbb{C}, \quad S^1 = \partial\overline{\Omega}$ unit disk

$\mathrm{Hol}(\overline{\Omega}, \mathbb{C}) \quad \subset \; C^\infty(S^1, \mathbb{C}) \; \circlearrowright \dfrac{\partial}{\partial \bar{z}}$ infinite dimensional kernel of $\bar\partial$ operator

$\mathrm{span}\{z^n\}_{n \in \mathbb{Z}^{\geq 0}} \subset \; \mathrm{span}\{z^n\}_{n \in \mathbb{Z}}$ Fourier series

For $a \in \mathbb{R}\backslash\mathbb{Z}$ let $H_a \subset C^\infty(S^1, \mathbb{C})$ be the $f\colon S^1 \to \mathbb{C}$ with vanishing Fourier coef of z^n, $n > a$





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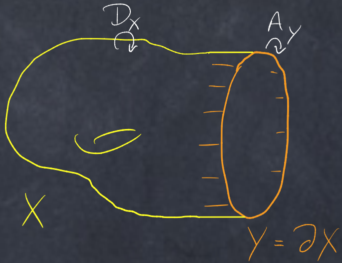
$D_X = \gamma(dt)\frac{\partial}{\partial t} + D_Y$ — Dirac on X

$A_Y = \gamma(dt)^{-1} D_Y$ — self-adjoint Dirac on Y

$\bigoplus_{\lambda\in\mathrm{spec}(A_Y)} E_\lambda$ — spectral decomposition of A_Y



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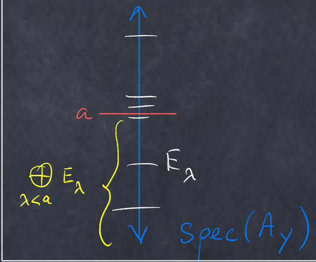
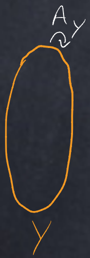


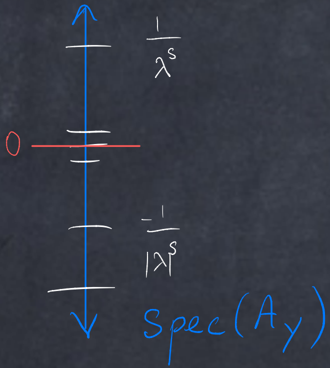
APS boundary condition for $a \in \mathbb{R} \backslash \mathrm{spec}(A_Y)$: $\{\psi$ spinor field on $X : \psi|_Y \in \oplus_{\lambda<a} E_\lambda\}$

Atiyah-Patodi-Singer η-invariant

Split $\mathrm{spec}(A_Y)$ at $a = 0$ and use meromorphic continuation to define

$$\eta(s) = \sum_{\lambda \in \mathrm{spec}(A_Y) \setminus \{0\}} (\mathrm{sign}\,\lambda)\,|\lambda|^{-s}, \qquad \eta_Y = \eta(0)$$





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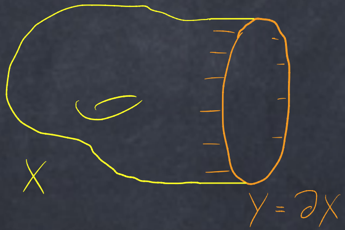

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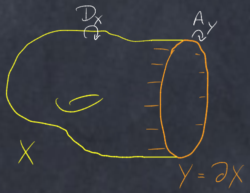
This is a special case of a general index theorem for Dirac operators:

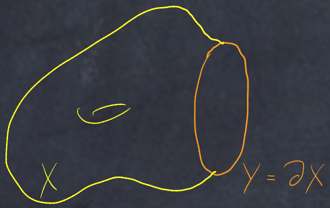
$$\mathrm{ind}\, D_X = \int_X \hat{A}(\Omega_X) \; - \; \xi_Y, \qquad \xi_Y = \frac{\eta_Y + \dim\ker A_Y}{2}$$



Secondary geometric invariants

$$\int_X \frac{K}{2\pi}\, d\mu_X = -\int_Y \frac{\kappa}{2\pi}\, d\mu_Y + \mathrm{Euler}(X)$$

$$= -\int_Y \frac{\kappa}{2\pi}\, d\mu_Y \qquad (\mathrm{mod}\ 1)$$



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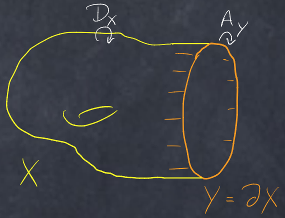
Chern-Simons invariants are secondary invariants of Chern-Weil invariants

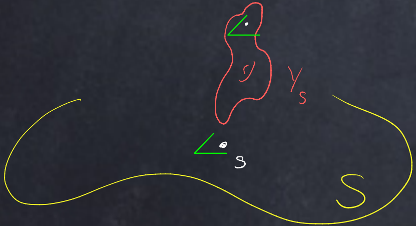


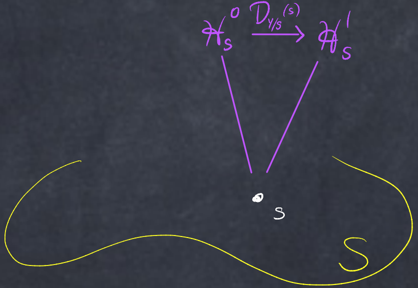
η-invariants are secondary invariants in K-theory:

$$\int_X \hat{A}(\Omega_X) = \xi_Y = \frac{\eta_Y + \dim\ker A_Y}{2} \qquad \pmod 1$$

Secondary invariants in families

$Y \longrightarrow S$	proper Riemannian spin fiber bundle of *odd* relative dimension n
$D_{Y/S}$	family of Dirac operators parametrized by S
$\operatorname{ind} D_{Y/S} \in K^{-n}(S)$	index in complex K-theory
$K^{\mathrm{odd}}(S) \longrightarrow H^1(S;\mathbb{Z})$	"lowest" piece of K-theory: *homotopy class* of maps $S \longrightarrow \mathbb{R}/\mathbb{Z}$





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$X \longrightarrow S$ of *even* relative dimension: $K^{\mathrm{even}}(S) \longrightarrow H^0(S;\mathbb{Z})$ numerical index

$K^{\mathrm{even}}(S) \longrightarrow H^2(S;\mathbb{Z})$ determinant line bundle

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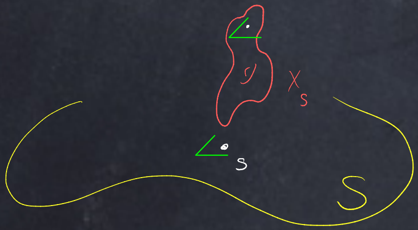

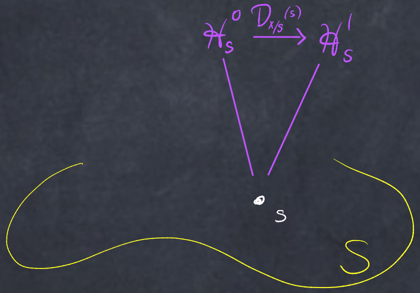

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$$\operatorname{curv} \operatorname{Det} D_{X/S}(V) = \left[2\pi i \int_{X/S} \hat{A}(\Omega_{X/S}) \right]_{(2)}$$ curvature

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Inspired by Witten's global anomaly formula (1985)

Bismut Riemann-Roch formula

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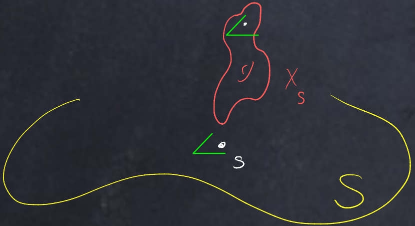
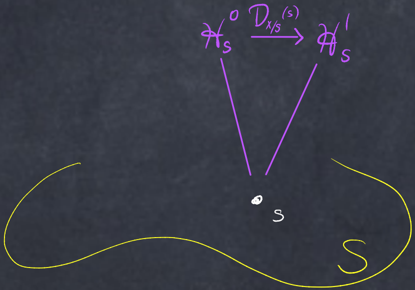
$\operatorname{ch} \operatorname{ind} D_{X/S} = f_*\big[\hat{A}(X)\big]$	Riemann-Roch/index formula





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Geometric refinement: Bismut (after Quillen) *superconnection* ∇_t on $\mathcal{H}^0 \oplus \mathcal{H}^1 \longrightarrow S$

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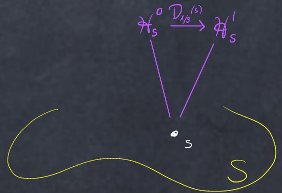

Index theorem in differential K-theory

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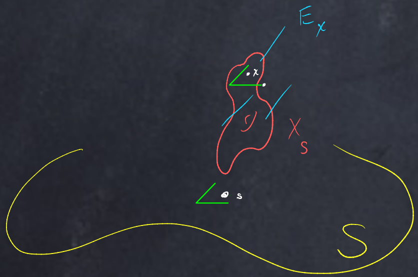

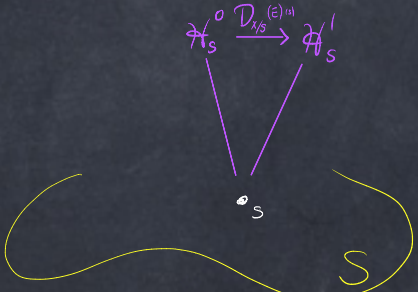

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Theorem (F-Lott): $\mathrm{ind}^{\mathrm{an}} = \mathrm{ind}^{\mathrm{top}}$

- 1952–1963: Hirzebruch Riemann-Roch, Bott periodicity, Atiyah-Hirzebruch K-theory, Atiyah-Singer index theorem
- Variations on the theme
- Global topological invariants $\rightsquigarrow$ local geometric invariants (of Dirac operators)
- An application to physics

Quantum theory is projective

We say a Hilbert space $\mathcal{H}$ is the "state space" of a quantum system, *but*

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- obstruction class in $H^2(G;\mathbb{C}^\times)$ $\leftsquigarrow$ 1-dimensional representation in $H^1(G;\mathbb{C}^\times)$

Projectivity in quantum field theory

Fix a dimension $n \in \mathbb{Z}^{>0}$ of (Wick-rotated) spacetime

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But F is a quantum system, so is projective

Projectivity (*anomaly*) is a 1-dimensional theory in dimension $n+1$:

$$\alpha\colon\ \mathrm{Bord}_{n+1}(\mathcal{F}) \longrightarrow \mathrm{Line}$$

Anomaly of spinor fields

The relationship between anomalies of spinor fields and the index theorem was pioneered in a 1984 paper of Atiyah-Singer

Proc. Natl. Acad. Sci. USA
Vol. 81, pp. 2597–2600, April 1984
Mathematics

Dirac operators coupled to vector potentials

(elliptic operators/index theory/characteristic classes/anomalies/gauge fields)

M. F. Atiyah† and I. M. Singer‡

†Mathematical Institute, University of Oxford, Oxford, England; and ‡Department of Mathematics, University of California, Berkeley, CA 94720

Contributed by I. M. Singer, January 6, 1984

$\mathbf{C}^N$. Each $A \in \mathfrak{A}$ gives a Dirac operator $\partial\!\!\!/_A: C^\infty(S^+ \otimes E) \to C^\infty(S^- \otimes E)$ where $S^\pm$ are the spin bundles over M of positive and negative chirality, respectively. In local coordinates

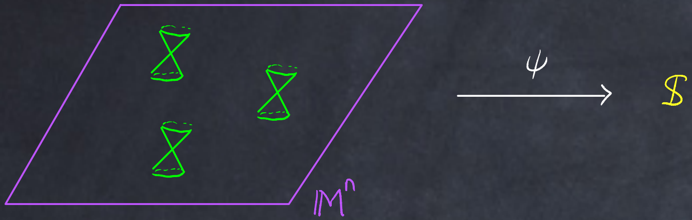
$$\partial\!\!\!/_A = \sum_{\mu=1}^{2n} \gamma_\mu(\partial_\mu + \Gamma_\mu + A_\mu)\left(\frac{1+\gamma_5}{2}\right)$$

where Γ_μ is the Riemannian connection and acts on spinorial indices, while A_μ acts on the scalar indices 1, ..., N. We have the covariance $\partial\!\!\!/_{\phi\cdot A} = \phi^{-1}\partial\!\!\!/_A\phi$.

The analytic index of the Dirac family $\{\partial\!\!\!/_A\}_{A\in\mathfrak{A}}$, which we denote by $\partial\!\!\!/_{\mathfrak{A}/\mathfrak{G}}$ is the formal difference $\{\ker \partial\!\!\!/_A\}_{A\in\mathfrak{A}} - \{\ker \partial\!\!\!/_A^*\}_{A\in\mathfrak{A}}$. Each term is not a vector bundle over $\mathfrak{A}$ because the dimensions of $\ker \partial\!\!\!/_A$ and $\ker \partial\!\!\!/_A^*$ can jump (the same amount) as A varies over $\mathfrak{A}$. Nevertheless, the formal difference is

One interpretation for this anomaly involves determinants. Consider the operator $T_\phi = \partial\!\!\!/_B^*\partial\!\!\!/_{\phi A}: C^\infty(S^+ \otimes E) \to C^\infty(S^+ \otimes E)$, when $\partial\!\!\!/_A$ and $\partial\!\!\!/_B$ have no zero frequency modes. The operator T_ϕ is a Laplacian plus lower-order term. It has pure point spectrum $\{\lambda_j\}$, and all but a finite number of eigenvalues lie inside a wedge about the positive real axis. Hence, $\Sigma\lambda_j^{-s}$ makes sense except for a finite number of eigenvalues lying on the negative real axis.

Let $\mathbb{S}$ be a representation of the Lorentz group $\mathrm{Spin}_{1,n-1}$



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Formula for deformation class of anomaly theory (F-Hopkins):

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