Topological symmetry in field theory

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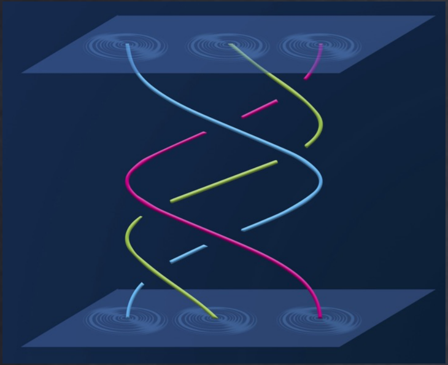
University of Texas at Austin

November 8, 2022

Joint work with Greg Moore and Constantin Teleman

arXiv:2209.07471

Simons Collaboration (https://scgcs.berkeley.edu)



Global Categorical Symmetry

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It leads to a calculus of topological defects which takes full advantage of well-developed theorems and techniques in topological field theory

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Let's begin with some motivation from representation theory of Lie groups and Lie algebras

Computations in... $\mathfrak{sl}_2(\mathbb{R})$

Set

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

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Namely, both sides equal

$$\begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$$

In the 3-dimensional representation of $\mathfrak{sl}_2(\mathbb{R})$ we have

$$h' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad e' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \qquad f' = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

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Now slightly less simple matrix manipulations verify the identity

$$\boxed{\frac{1}{2}(h')^2 + e'f' + f'e' = \frac{1}{2}(h')^2 + h' + 2f'e'}$$

Namely, both sides equal

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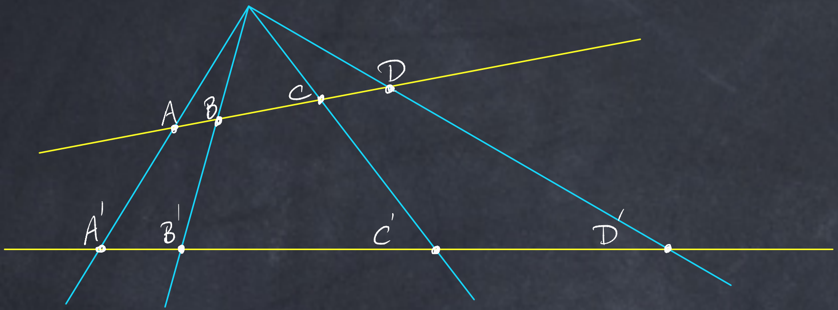
The Lie group $SL_2(\mathbb{R})$ acts on the projective line $\mathbb{RP}^1$ as fractional linear transformations



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Both sides act as multiplication by $4\lambda^2 - 2\lambda$

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Main idea: Make analogous universal computations with symmetries in QFT

Warning

The word ‘symmetry’ in mathematics usually refers to *groups* (“invertible symmetries”) rather than algebras (“noninvertible symmetries”), but in modern QFT-speak the term ‘symmetry’ is also used for the latter. Algebras of operators, including those that commute with a Hamiltonian, date from the earliest days of quantum mechanics

Motivation: algebras

Abstract symmetry data (for algebras) is a pair (A, R):

A	algebra
R	right regular module

Definition: Let V be a vector space. An (A, R)-*action on* V is a pair (L, θ) consisting of a left A-module L together with an isomorphism of vector spaces

$$\theta\colon R \otimes_A L \xrightarrow{\cong} V$$

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Analogy:

algebra $\rightsquigarrow$ topological field theory

element of algebra $\rightsquigarrow$ defect in TFT

Example: Let G be a finite group. Its group algebra is

$$\mathbb{C}[G] = \left\{ \sum_{g \in G} \lambda_g \, g \right\}, \qquad \lambda_g \in \mathbb{C}$$

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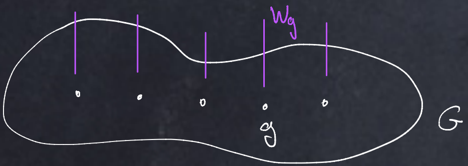
Higher Example: Vect = category of finite dimensional complex vector spaces. Define Vect$[G]$ as the linear category (Vect-module) of vector bundles over G with tensor product pushforward under mult. It is a *fusion category*



$$(W_1 * W_2)_g = \bigoplus_{g_1 g_2 = g} (W_1)_{g_1} \otimes (W_2)_{g_2}$$

Field theory

Analogy: field theory $\sim$ module over an algebra OR $\sim$ representation of a Lie group

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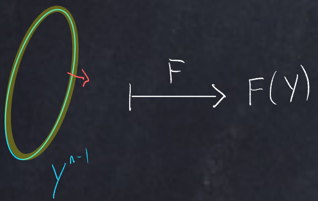
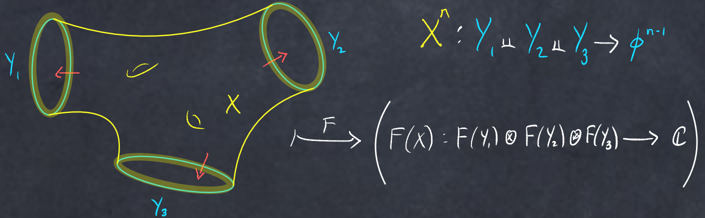
n	dimension of spacetime
$\mathcal{F}$	background fields (orientation, Riemannian metric, . . .)





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Kontsevich-Segal: Axioms for 2-tier nontopological theory $F\colon\ \mathrm{Bord}_{\langle n-1,n\rangle}(\mathcal{F}) \to t\,\mathrm{Vect}$

Domain walls, boundary theories, defects

$\sigma, \sigma_1, \sigma_2$	$(n+1)$-dimensional theories
$\delta\colon\ \sigma_1 \to \sigma_2$	domain wall
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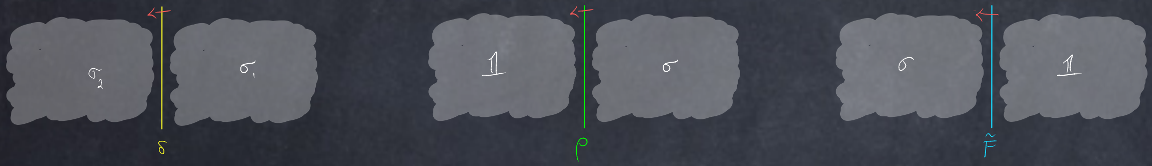

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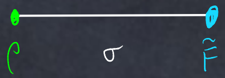

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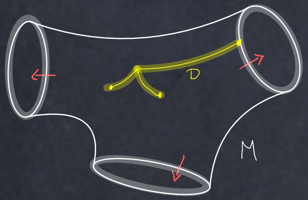
More generally, one can have *defects* supported on any (stratified) manifold $D \subset M$



Composition laws; invertibility

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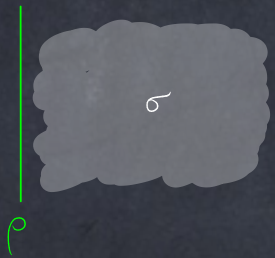
So a notion of *invertible* field theory and *invertible* defect

Main definition: abstract symmetry data

Fix a dimension n and background fields $\mathcal{F}$ (which we keep implicit)

Definition: A *quiche* is a pair (σ, ρ) in which $\sigma\colon \mathrm{Bord}_{n+1}(\mathcal{F}) \to \mathcal{C}$ is an $(n+1)$-dimensional topological field theory and ρ is a right topological σ-module.



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A regular boundary theory is also sometimes called *Dirichlet*

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In this talk we do not pursue these ideas further

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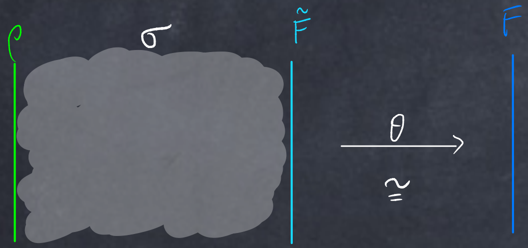

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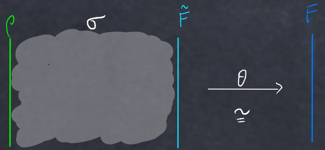

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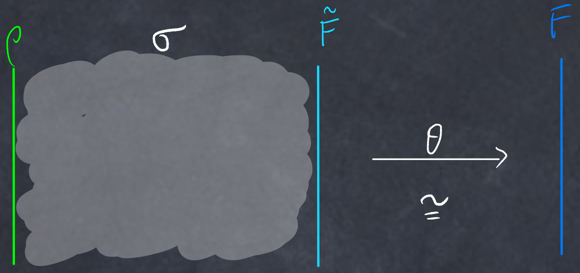
- Symmetry persists under renormalization group flow, hence a low energy approximation to F should also be an (σ, ρ)-module. If F is gapped, then we can bring to bear powerful methods and theorems in topological field theory to investigate *topological* left σ-modules. This leads to dynamical predictions

Example: quantum mechanics with G-symmetry

$n = 1$

$\mathcal{F}$	{orientation, Riemannian metric} for F and $\widetilde{F}$
$\mathcal{H}$	Hilbert space
H	Hamiltonian
$G \circlearrowright \mathcal{H}$	finite group
$S\colon G \to \mathrm{Aut}(\mathcal{H})$	action on $\mathcal{H}$
$\sigma(\mathrm{pt})$	$\mathbb{C}[G]$
$F(\mathrm{pt})$	$\mathcal{H}$
$\widetilde{F}(\mathrm{pt})$	${}_{\mathbb{C}[G]}\mathcal{H}$ (left module)



Example: quantum mechanics with G-symmetry

$n = 1$	
$\mathcal{F}$	{orientation, Riemannian metric} for F and $\widetilde{F}$
$\mathcal{H}$	Hilbert space
H	Hamiltonian
$G \circlearrowright \mathcal{H}$	finite group
$S\colon G \to \mathrm{Aut}(\mathcal{H})$	action on $\mathcal{H}$
$\sigma(\mathrm{pt})$	$\mathbb{C}[G]$
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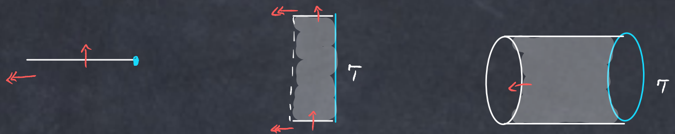


(a) (b) (c)

Evaluation of some bordisms:

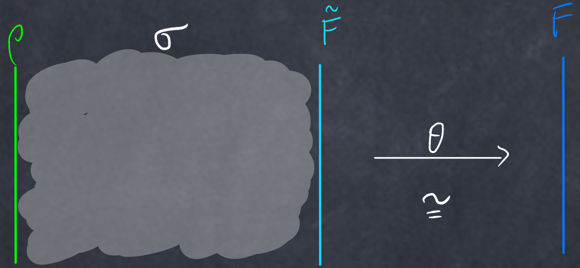
(a) the left module ${}_{\mathbb{C}[G]}\mathcal{H}$

(b) $e^{-\tau H/\hbar}\colon {}_{\mathbb{C}[G]}\mathcal{H} \longrightarrow {}_{\mathbb{C}[G]}\mathcal{H}$

(c) the central function $g \longmapsto \mathrm{Tr}_{\mathcal{H}}\big(S(g)e^{-\tau H/\hbar}\big)$ on G

Example: gauge theory with BA-symmetry

n	any dimension
A	finite abelian group $A = \mu_2$
BA	a homotopical/shifted A ("1-form A-symmetry")
H	Lie group with $A \subset Z(H)$ $H = \mathrm{SU}_2$
$\overline{H} = H/A$	$\overline{H} = \mathrm{SO}_3$
F	H-gauge theory
$\widetilde{F}$	$\overline{H}$-gauge theory



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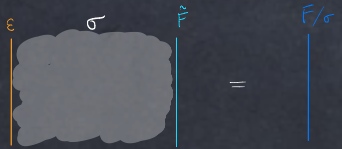
A quotient construction allows to recover absolute $\overline{H}$-gauge theory as a sandwich (later)



Defects: quantum mechanics

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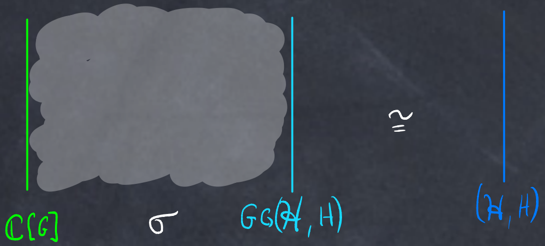
$G \circlearrowright \mathcal{H}$	finite group



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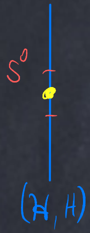

Consider a point defect in F. The link of a point in a 1-manifold (imaginary time) is S^0, a 0-sphere of radius ϵ, and the vector space of defects is

$$\varprojlim_{\epsilon \to 0} \mathrm{Hom}\big(1, F(S^0_\epsilon)\big)$$

which is a space of singular operators on $\mathcal{H}$. To focus on formal aspects we write '$\mathrm{End}(\mathcal{H})$'

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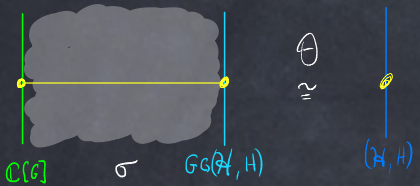

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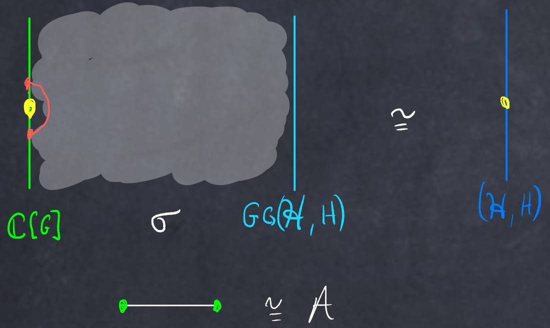
We now consider defects in $(\rho, \sigma, \widetilde{F})$ which transport to point defects in F

Point ρ-defects

The link is a closed interval with ρ-colored boundary. It evaluates under (σ, ρ) to the *vector space* $A = \mathbb{C}[G]$. The "label" of the defect is therefore an element of A. Note $G \subset A$ labels invertible defects.

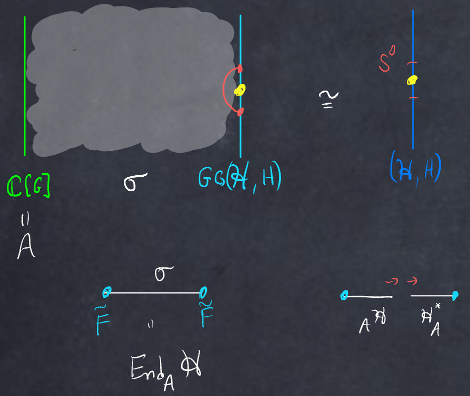
ρ-defects are topological



Point $\widetilde{F}$-defects

The link is again a closed interval, but now with $\widetilde{F}$-colored boundary. The value under $(\sigma, \widetilde{F})$ is $\mathrm{End}_A(\mathcal{H})$, the space of observables that commute with the G-action

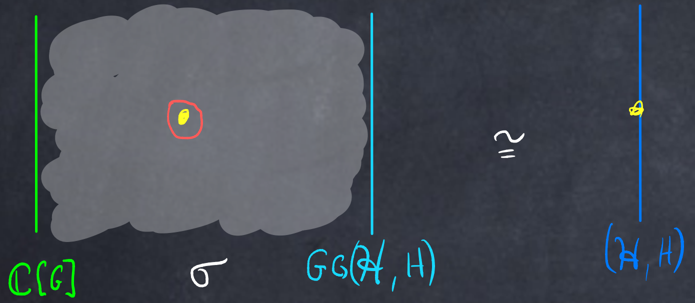
$\widetilde{F}$-defects are typically not topological



Point σ-defects: central defects

The link is S^1, and the value under σ is the vector space which is the center of the group algebra $A = \mathbb{C}[G]$.

σ-defects are topological

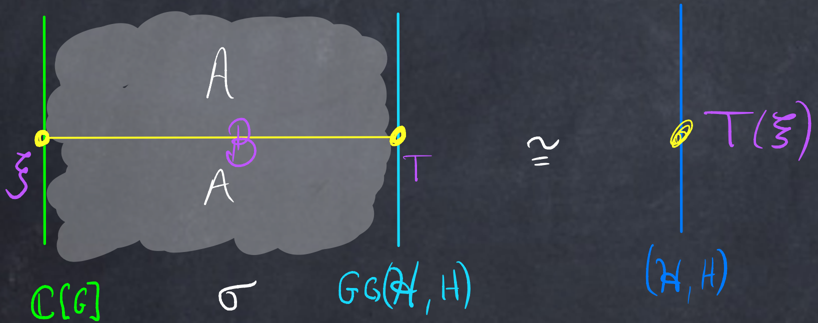


The general point defect

A general point defect in F can be realized by a line defect in $(\rho, \sigma, \widetilde{F})$.

Label the defect beginning with the highest dimensional strata and work down in dimension

B	(A, A)-bimodule
ξ	vector in B
T	(A, A)-bimodule map $B \longrightarrow \mathrm{End}(\mathcal{H})$

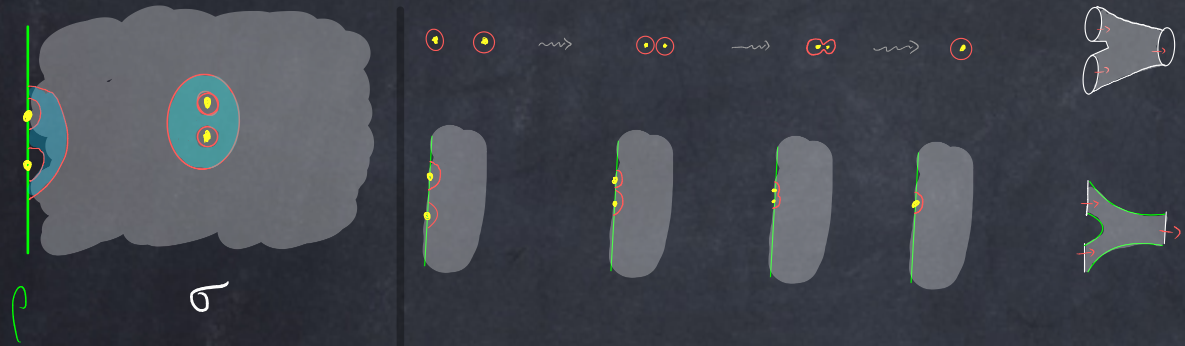


Composition law on defects

Compute using the links of the defects—2 incoming and 1 outgoing

σ-defects: pair of pants

ρ-defects: pair of chaps



Commutation relations among defects

The sandwich realization makes clear that

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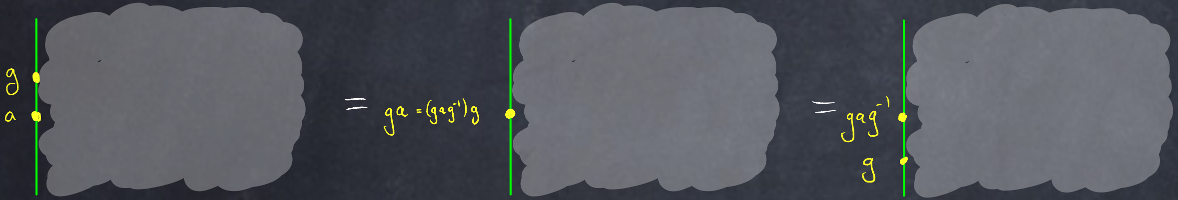

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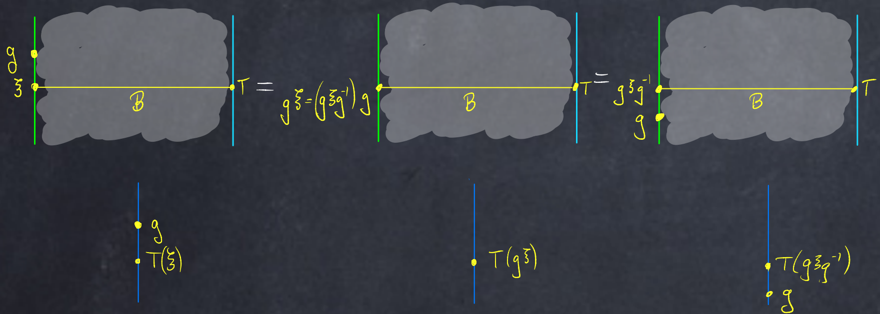
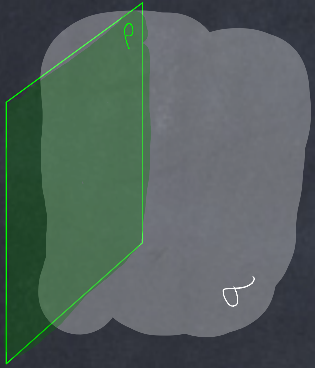


Finite group symmetries of an $(n = 2)$-dimensional theory

Let G be a finite group, and let σ be the 3-dimensional finite G-gauge theory

$$\sigma\colon \operatorname{Bord}_3 \longrightarrow \operatorname{Alg}(\operatorname{Cat})$$

with $\sigma(\mathrm{pt}) = \operatorname{Vect}[G]$, and let ρ be the regular right σ-module with $\rho(\mathrm{pt}) = \operatorname{Vect}[G]_{\operatorname{Vect}[G]}$



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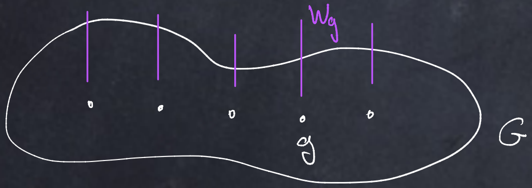

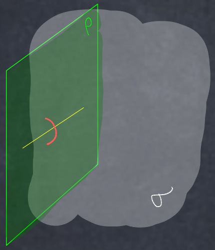

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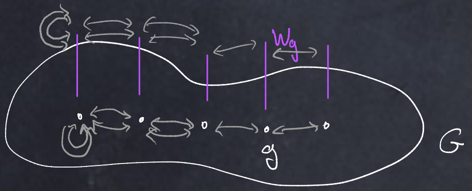

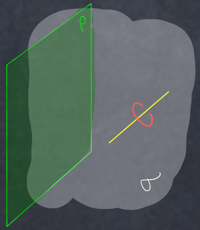

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As opposed to G-symmetry in $n = 1$, here the center is “bigger”

Line defects in 4-dimensional gauge theory

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Finite homotopy theories

Definition: A topological space $\mathcal{X}$ is *π-finite* if (i) $\pi_0\mathcal{X}$ is a finite set, (ii) for all $x \in \mathcal{X}$, the homotopy group $\pi_q(\mathcal{X}, x)$, $q \geqslant 1$, is finite, and (iii) there exists $Q \in \mathbb{Z}^{>0}$ such that $\pi_q(\mathcal{X}, x) = 0$ for all $q > Q$, $x \in \mathcal{X}$.

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Examples: (1) An Eilenberg-MacLane space $K(\pi, q)$ is π-finite if π is a finite group. Denote $K(G, 1)$ by BG for G a finite group, and if $q \geqslant 1$ and A is a finite abelian group, we denote $K(A, q)$ by B^qA.

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(2) Let G be a finite group, let A be a finite abelian group, and fix a cocycle k for a cohomology class $[k] \in H^3(G; A)$. (One can also include an action of G on A.) Realize k as a map $k\colon BG \to B^3A$, and form the π-finite space $\mathcal{X}$ as a pullback:

$$\begin{array}{ccccc} B^2A & \longrightarrow & \mathcal{X} & \longrightarrow & BG \\ \Vert & & \downarrow & & \downarrow k \\ B^2A & \longrightarrow & * & \longrightarrow & B^3A \end{array}$$

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Remark: If we drop the π-finiteness assumption, then we can construct a once-categorified theory from *any* topological space

Finite homotopy theories

m	(spacetime) dimension	
$\mathcal{X}$	π-finite space	
λ	cocycle of degree m on $\mathcal{X}$	$[\lambda] \in H^m(\mathcal{X}; \mathbb{C}^\times)$
M	closed manifold	
$\mathcal{X}^M$	$\mathrm{Map}(M, \mathcal{X})$	

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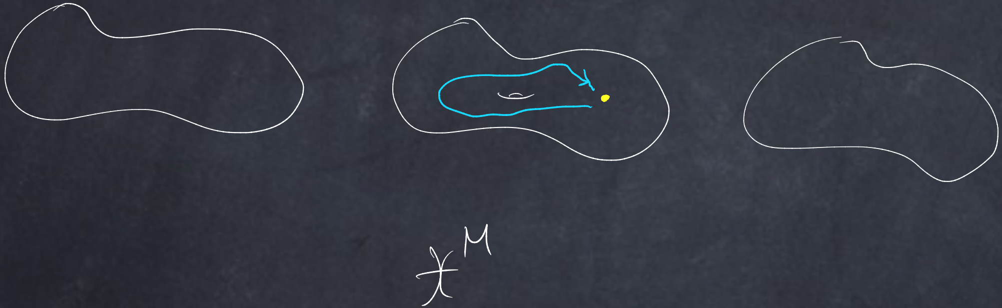
$$m = 5: \qquad \sigma(M) = \sum_{[\phi] \in \pi_0(\mathcal{X}^M)} \frac{\#\pi_2(\mathcal{X}^M, \phi)}{\#\pi_1(\mathcal{X}^M, \phi)} = \frac{\#H^0(M; A)}{\#H^1(M; A)} \, \#H^2(M; A)$$

Finite homotopy theories

codim 1—the vector space of locally constant complex-valued functions on $\mathcal{X}^M$:

$m = 4:$ $$\sigma(M) = \mathrm{Fun}\big(\pi_0(\mathcal{X}^M)\big) = \mathrm{Fun}\big(H^2(M;A)\big)$$



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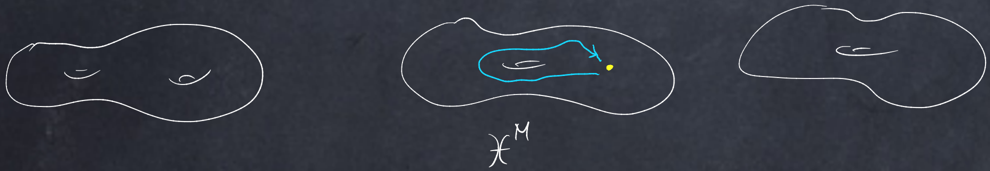

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The quantization of a bordism $M\colon N_0 \to N_1$ uses the correspondence of mapping spaces:

$$\begin{array}{ccccc} & & \mathcal{X}^M & & \\ & \swarrow^{p_0} & & \searrow^{p_1} & \\ \mathcal{X}^{N_0} & & & & \mathcal{X}^{N_1} \end{array}$$

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Semiclassical descriptions of boundaries and defects lead to computable quantizations

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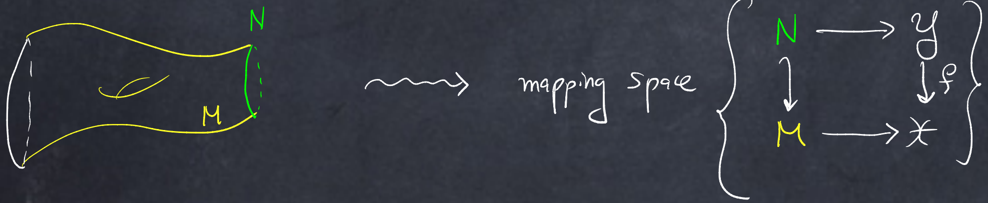
Fix $(\mathcal{X}, \lambda)$ a π-finite space and cocycle

Definition: A *right semiclassical boundary theory* of $(\mathcal{X}, \lambda)$ is a triple $(\mathcal{Y}, f, \mu)$ consisting of a π-finite space $\mathcal{Y}$, a map $f\colon \mathcal{Y} \to \mathcal{X}$, and a trivialization μ of $-f^*\lambda$

Quantization:



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$$\delta\colon \mathcal{Y} \longrightarrow \mathcal{L}^{\ell-1}\mathcal{X}$$

and a trivialization μ of $\delta^*(\tau^{\ell-1}\lambda)$

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Compositions of defects are computed using homotopy fiber products

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Quotients: augmentations

Definition: An *augmentation* of an algebra A is an algebra homomorphism $\epsilon\colon A \to \mathbb{C}$.

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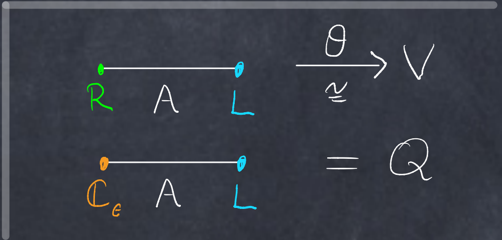

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Augmentations for higher algebras: Φ tensor category $\quad \epsilon\colon \Phi \to \mathrm{Vect}$ *fiber functor*

Quotients and quotient defects

We use the yoga of fully local topological field theory: let $\mathcal{C}'$ be a symmetric monoidal n-category and set $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$, the $(n+1)$-category whose objects are algebras in $\mathcal{C}'$

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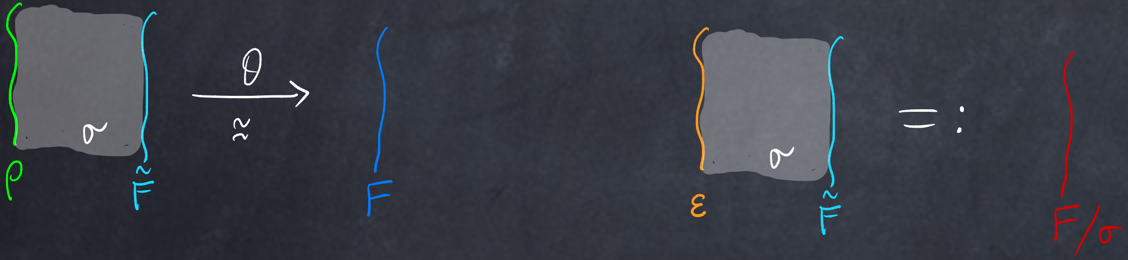
Augmentations do not always exist

Definition: Suppose given finite symmetry data (σ, ρ) and a (σ, ρ)-module structure $(\widetilde{F}, \theta)$ on a quantum field theory F. Suppose ϵ is an augmentation of σ. Then the *quotient* of F by the symmetry σ is

$$F \underset{\epsilon}{/} \sigma = \epsilon \otimes_{\sigma} \widetilde{F}$$



Line defects in 4-dimensional gauge theory

Our goal is to explain the paper *Reading between the lines of four-dimensional gauge theories* (arXiv:1305.1308) of Aharony-Seiberg-Tachikawa in this framework

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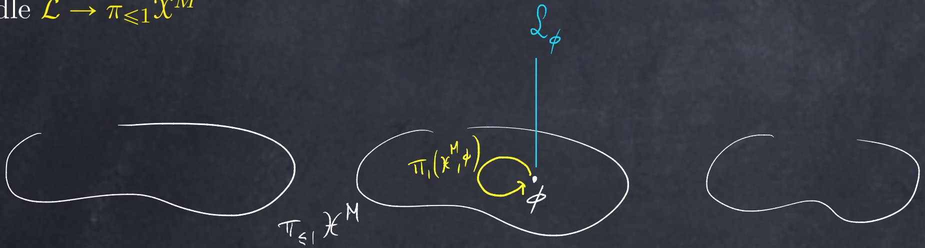

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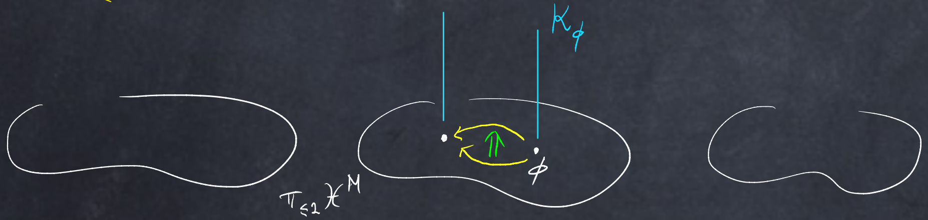

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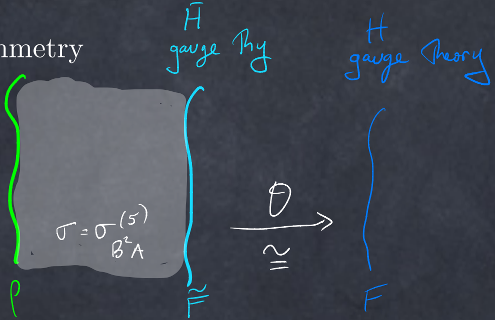
BA symmetry

H	compact Lie group
A	finite subgroup of center(H)
$\overline{H}$	H/A
σ	5-dimensional finite homotopy with $\mathfrak{X} = B^2A$
ρ	right topological boundary theory $* \to B^2A$
F	a 4-dimensional H-gauge theory with BA symmetry
$\widetilde{F}$	the corresponding $\overline{H}$-gauge theory



Topological right σ-modules

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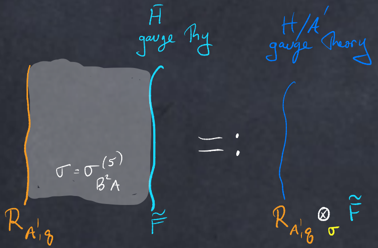

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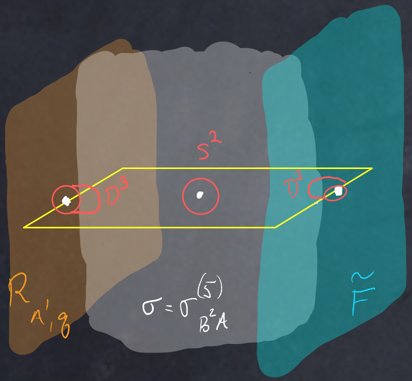
The quadratic form q gives rise to the *Pontrjagin square* cohomology operation

$$\mathcal{P}_q\colon H^2(X;A') \longrightarrow H^4(X;\mathbb{C}^\times)$$

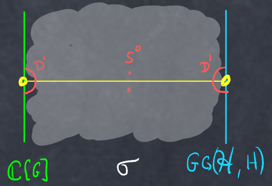
which enters the formula for the partition function in the theory $R_{A',q} \otimes_\sigma \widetilde{F}$, which is an H/A'-gauge theory

Line defects in the H/A'-gauge theory $R_{A',q} \otimes_\sigma \widetilde{F}$

M	4-manifold
$C \subset M$	1-dimensional submanifold
$[0,1] \times C$	2-dimensional submanifold of $[0,1] \times M$



Recall + compare:



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$$\pi_0\big(\mathrm{Map}(S^2, B^2A)\big) = H^2(S^2; A) \cong A$$
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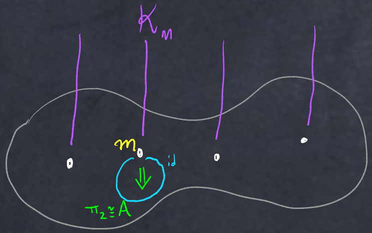

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2-category of local systems of linear categories over the indicated 2-groupoid, so for $m \in H^2(S^2;A) \cong A$ we have a linear category $\mathcal{K}_m$ equipped with an action of $\pi_2 \cong A$ by automorphisms of the identity functor, hence $\mathcal{K}_m$ decomposes as

$$\mathcal{K}_m = \bigoplus_e \mathcal{K}_{m,e} \cdot e, \qquad e \in H^0(S^2;A)^\vee \cong A^\vee$$

The line defect $[0,1) \times C$ in $(\sigma, R_{A',q})$

First, fix a pair $(m_0, e_0) \in A \times A^\vee$ and choose the interior label $\mathcal{K}$ to be the "δ-function" supported at (m_0, e_0):

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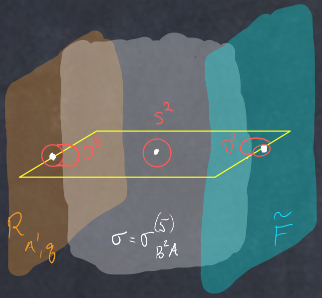

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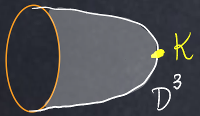
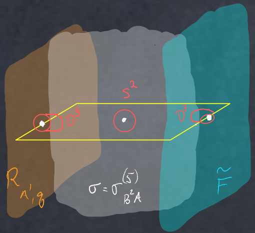
Claim: This 1-category vanishes unless (m_0, e_0) obeys a selection rule





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The selection rule is an assertion in the *topological* field theory $(\sigma, R_{A',q})$

The selection rule

From the quadratic function $q\colon A' \to \mathbb{C}^\times$ we obtain a bihomomorphism

$$b\colon A' \times A' \longrightarrow \mathbb{C}^\times$$

which induces a perfect pairing

$$H^2(S^2; A') \times H^0(S^2; A') \longrightarrow \mathbb{C}^\times$$

and so too an isomorphism

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Selection rule:

$$\boxed{\begin{aligned} m &\in A' \\ e\big|_{A'} &= e'(m)^{-1} \end{aligned}}$$

Sketch proof of the selection rule

Compute the homotopy limit of the diagram:

$$
\begin{array}{ccccccc}
\left(\mathrm{Map}(S^2, B^2A'), \tau^2(\mu_q)\right) & & & \mathrm{Map}(D^3 \backslash B^3, B^2A) & & & (B^2A, e) \\
& \searrow & \swarrow & & \searrow & \overset{m}{\swarrow} & \\
& & \mathrm{Map}(S^2, B^2A) & & \mathrm{Map}(S^2, B^2A) & &
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$m \in A'$

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