Topological symmetry in field theory

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Global Categorical Symmetry
Symmetry in quantum field theory

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Our framework includes “homotopical symmetries”, such as higher groups, 2-groups, . . .

It leads to a calculus of topological defects which takes full advantage of well-developed theorems and techniques in topological field theory.
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Let’s begin with some motivation from representation theory of Lie groups and Lie algebras.
Computations in... $\mathfrak{sl}_2(\mathbb{R})$

Set

\[ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]
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h &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & e &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & f &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\end{align*}
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Simple matrix manipulations verify the identity

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\frac{1}{2}h^2 + ef + fe = \frac{1}{2}h^2 + h + 2fe
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Namely, both sides equal

\[
\begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}
\]
In the 3-dimensional representation of $\mathfrak{sl}_2(\mathbb{R})$ we have

\[
\begin{align*}
    h' &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \\
    e' &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \\
    f' &= \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
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\]

Now slightly less simple matrix manipulations verify the identity

\[
\frac{1}{2} (h')^2 + e' f' + f' e' = \frac{1}{2} (h')^2 + h' + 2 f' e'
\]

Namely, both sides equal

\[
\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}
\]
The Lie group $\text{SL}_2(\mathbb{R})$ acts on the projective line $\mathbb{RP}^1$ as fractional linear transformations.
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\[
\begin{align*}
\tilde{h}: \phi & \mapsto -2x\phi' - 2\lambda\phi \\
\tilde{e}: \phi & \mapsto -\phi' \\
\tilde{f}: \phi & \mapsto x^2\phi' + 2\lambda x\phi
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Some calculus manipulations verify the identity

\[
\frac{1}{2} \tilde{h}^2 + \tilde{e}\tilde{f} + \tilde{f}\tilde{e} = \frac{1}{2} \tilde{h}^2 + \tilde{h} + 2\tilde{f}\tilde{e}
\]

Both sides act as multiplication by $4\lambda^2 - 2\lambda$. 

Instead of each separate computation, we compute \textit{universally} in an abstract algebra.
Instead of each separate computation, we compute *universally* in an abstract algebra. Each representation defines a module over the *universal enveloping algebra* $A = U(\mathfrak{sl}_2(\mathbb{R}))$. Many recent results about extended notions of symmetry in QFT: Apruzzi, Bah, Benini, Bhardwaj, Bonetti, Bullimore, C´ordova, Choi, Cvetiˇc, Del Zotto, Dumitrescu, Fr¨olich, Fuchs, Gaiotto, Garc´ıa Etxebarria, Gould, Gukov, Heckman, Heidenreich, Hopkins, Hosseini, Hsin, H¨ubner, Intriligator, Ji, Jian, Johnson-Freyd, Jordan, Kaidi, Kapustin, Komargodski, Lake, Lam, McNamara, Minasian, Montero, Ohmari, Pantev, Pei, Plavnik, Reece, Robbins, Roumpedakis, Rudelius, Runkel, Sch¨afer-Nameki, Scheimbauer, Schweigert, Seiberg, Seifnashri, Shao, Sharpe, Tachikawa, Thorngren, Torres, Vandermeulen, Wang, Wen, Willett, ...
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holds in $A$, since $[e, f] = ef - fe = h$, hence it holds in every $A$-module.
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**Main idea:** Make analogous universal computations with symmetries in QFT
The word ‘symmetry’ in mathematics usually refers to *groups* (‘invertible symmetries’) rather than algebras (‘noninvertible symmetries’), but in modern QFT-speak the term ‘symmetry’ is also used for the latter. Algebras of operators, including those that commute with a Hamiltonian, date from the earliest days of quantum mechanics.
Motivation: algebras

Abstract symmetry data (for algebras) is a pair \((A, R)\):

- \(A\) algebra
- \(R\) right regular module

**Definition:** Let \(V\) be a vector space. An \((A, R)\)-action on \(V\) is a pair \((L, \theta)\) consisting of a left \(A\)-module \(L\) together with an isomorphism of vector spaces

\[
\theta: R \otimes_A L \xrightarrow{\cong} V
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Analogy: algebra "B topological field theory element of algebra "B defect in TFT"
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Elements of \(A\) act on all modules; relations in \(A\) apply (e.g. Casimirs in \(U(\mathfrak{sl}_2(\mathbb{R}))\))
**Motivation: algebras**

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**Analogy:**

- algebra \(\rightsquigarrow\) topological field theory
- element of algebra \(\rightsquigarrow\) defect in TFT
Example: Let $G$ be a finite group. Its group algebra is

$$\mathbb{C}[G] = \left\{ \sum_{g \in G} \lambda_g g \right\}, \quad \lambda_g \in \mathbb{C}$$

Identify $\mathbb{C}[G] = \text{Fun}(G)$; convolution product is pushforward under

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**Higher Example:** $\text{Vect} = \text{category of finite dimensional complex vector spaces}$. Define $\text{Vect}[G]$ as the linear category (Vect-module) of vector bundles over $G$ with tensor product pushforward under mult. It is a *fusion category*

$$
(w_1 \ast w_2)_g = \bigoplus_{g_1 g_2 = g} (w_1)_{g_1} \otimes (w_2)_{g_2}
$$
Field theory

**Analogy:** field theory $\sim$ module over an algebra OR $\sim$ representation of a Lie group

**Warning:** This analogy is quite limited
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**Segal Axiom System:** A (Wick-rotated) field theory $F$ is a linear representation of a bordism (multi)category $\text{Bord}_n(\mathcal{F})$

- $n$ dimension of spacetime
- $\mathcal{F}$ background fields (orientation, Riemannian metric, ...)

\[ F: \gamma^{n-1} \rightarrow F(\gamma) \]
\[ X: \gamma_1 \cup \gamma_2 \cup \gamma_3 \rightarrow \phi^{n-1} \]
\[ F(X): F(\gamma_1) \otimes F(\gamma_2) \otimes F(\gamma_3) \rightarrow \mathcal{C} \]
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**Kontsevich-Segal:** Axioms for 2-tier nontopological theory $F: \text{Bord}_{n-1,n}(\mathcal{F}) \to t\text{Vect}$
Domain walls, boundary theories, defects

\( \sigma, \sigma_1, \sigma_2 \) \hspace{1cm} (\( n + 1 \))-dimensional theories

\( \delta : \sigma_1 \rightarrow \sigma_2 \) \hspace{1cm} domain wall

\( \rho : \sigma \rightarrow \mathbb{1} \) \hspace{1cm} right boundary theory

\( \tilde{F} : \mathbb{1} \rightarrow \sigma \) \hspace{1cm} left boundary theory

The "sandwich" \( \tilde{F} \) is an (absolute) \( n \)-dimensional theory

More generally, one can put defects on any (stratified) manifold \( D \hookrightarrow M \).
Domain walls, boundary theories, defects

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\[ \delta: \sigma_1 \rightarrow \sigma_2 \] \quad domain wall \quad \quad \quad \quad \quad (\sigma_2, \sigma_1)-bimodule

\[ \rho: \sigma \rightarrow \mathbb{1} \] \quad right boundary theory \quad \quad \quad \quad \quad right \ \sigma\text{-module}

\[ \tilde{F}: \mathbb{1} \rightarrow \sigma \] \quad left boundary theory \quad \quad \quad \quad \quad left \ \sigma\text{-module}

\begin{align*}
\sigma_2 & \quad \delta \quad \sigma_1 \\
\mathbb{1} & \quad \rho \\
\sigma & \quad \tilde{F} \\
\mathbb{1} &
\end{align*}
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![Diagram](image_url)
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More generally, one can have defects supported on any (stratified) manifold \( D \subset M \)
Composition laws; invertibility

- Given two field theories $F_1, F_2$ on the same domain $\text{Bord}_n(\mathcal{F})$, there is a composition $F_1 \otimes F_2$. The composition law is sometimes called \textit{stacking}. There is a unit $\mathbf{1}$ for the composition law.
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- There is also a composition law on parallel defects, for example the OPE on point defects. In a topological theory one obtains a higher algebra of defects.
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- There is also a composition law on parallel defects, for example the OPE on point defects. In a topological theory one obtains a higher algebra of defects.

So a notion of invertible field theory and invertible defect.
Main definition: abstract symmetry data

Fix a dimension $n$ and background fields $\mathcal{F}$ (which we keep implicit)

**Definition:** A *quiche* is a pair $(\sigma, \rho)$ in which $\sigma: \text{Bord}_{n+1}(\mathcal{F}) \to \mathcal{C}$ is an $(n + 1)$-dimensional topological field theory and $\rho$ is a right topological $\sigma$-module.
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**Example:** Let $G$ be a finite group. Then for a $G$-symmetry we let $\sigma$ be finite gauge theory in dimension $n + 1$. Note this is the *quantum* theory which sums over principal $G$-bundles.
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**Regular $\rho$:** Suppose $\mathcal{C}'$ is a symmetric monoidal $n$-category and $\sigma$ is an $(n + 1)$-dimensional topological field theory with codomain $\mathcal{C} = \text{Alg}(\mathcal{C}')$. Let $A = \sigma(\text{pt})$. Then $A$ is an algebra in $\mathcal{C}'$ which, as an object in $\mathcal{C}$, is $(n + 1)$-dualizable. Assume that the right regular module $A_A$ is $n$-dualizable as a 1-morphism in $\mathcal{C}$. Then the boundary theory $\rho$ determined by $A_A$ is the *right regular boundary theory* of $\sigma$, or the *right regular $\sigma$-module*. 
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A regular boundary theory is also sometimes called *Dirichlet*
An important generalization

The bulk topological theory $\sigma$ need not be defined on $(n + 1)$-manifolds; it can be a once-categorified $n$-dimensional theory.
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Analog of boundary theories: relative field theories (Stolz-Teichner called them twisted field theories).
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Defects are also defined in once-categorified theories; the link is a raviolo or UFO.

In this talk we do not pursue these ideas further.
Main definition: concrete realization of symmetry

**Definition:** Let \((\sigma, \rho)\) be an \(n\)-dimensional quiche. Let \(F\) be an \(n\)-dimensional field theory. A \((\sigma, \rho)\)-module structure on \(F\) is a pair \((\tilde{F}, \theta)\) in which \(\tilde{F}\) is a left \(\sigma\)-module and \(\theta\) is an isomorphism

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\theta : \rho \otimes_{\sigma} \tilde{F} \xrightarrow{\cong} F
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of absolute \(n\)-dimensional theories.
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- The sandwich picture of \(F\) as \(\rho \otimes_\sigma \tilde{F}\) separates out the topological part \((\sigma, \rho)\) of the theory from the potentially nontopological part \(\tilde{F}\) of the theory.
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- The sandwich picture of \(F\) as \(\rho \otimes_\sigma \tilde{F}\) separates out the topological part \((\sigma, \rho)\) of the theory from the potentially nontopological part \(\tilde{F}\) of the theory.
- Symmetry persists under renormalization group flow, hence a low energy approximation to \(F\) should also be an \((\sigma, \rho)\)-module. If \(F\) is gapped, then we can bring to bear powerful methods and theorems in topological field theory to investigate *topological* left \(\sigma\)-modules. This leads to dynamical predictions.
Example: quantum mechanics with $G$-symmetry

$n = 1$

$\mathcal{F}$ \{orientation, Riemannian metric\} for $F$ and $\tilde{F}$

$\mathcal{H}$ Hilbert space

$H$ Hamiltonian

$G \odot \mathcal{H}$ finite group

$S: G \to \text{Aut}(\mathcal{H})$ action on $\mathcal{H}$

$\sigma(\text{pt})$ $\mathbb{C}[G]$

$F(\text{pt})$ $\mathcal{H}$

$\tilde{F}(\text{pt})$ $\mathbb{C}[G] \mathcal{H}$ (left module)
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Evaluation of some bordisms:

(a) the left module $\mathbb{C}[G] \mathcal{H}$

(b) $e^{-\tau H/\hbar} : \mathbb{C}[G] \mathcal{H} \to \mathbb{C}[G] \mathcal{H}$

(c) the central function $g \mapsto \text{Tr}_\mathcal{H}(S(g)e^{-\tau H/\hbar})$ on $G$
Example: gauge theory with $BA$-symmetry

- $n$: any dimension
- $A$: finite abelian group $A = /\mu_2$
- $BA$: a homotopical/shifted $A$ ("1-form $A$-symmetry")
- $H$: Lie group with $A \subset Z(H)$ $H = SU_2$
- $\overline{H} = H/A$: $\overline{H} = SO_3$
- $F$: $H$-gauge theory
- $\tilde{F}$: $\overline{H}$-gauge theory

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A quotient construction allows to recover absolute $\overline{H}$-gauge theory as a sandwich (later).
Consider a point defect in $F$. The link of a point in a 1-manifold (imaginary time) is $S^0$, a 0-sphere of radius $\varepsilon$, and the vector space of defects is $\lim_{\varepsilon \to 0} \text{Hom}_{\text{End} p H q} S^0 \varepsilon$. To focus on formal aspects, we write '$\text{End} p H q$'.

We now consider defects in $p \mapsto F$, which transport to point defects in $F$. Da) $G \subset \mathcal{H}$, \( H \).
Defects: quantum mechanics

\[ n = 1 \]
\[ \mathcal{H} \quad \text{Hilbert space} \]
\[ H \quad \text{Hamiltonian} \]
\[ G \subset \mathcal{H} \quad \text{finite group} \]
\[ (\mathcal{H}, \mathcal{H}) \]

Consider a point defect in \( F \). The link of a point in a 1-manifold (imaginary time) is \( S^0 \), a 0-sphere of radius \( \varepsilon \), and the vector space of defects is

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$$\lim_{\epsilon \to 0} \text{Hom}(1, F(S^0))$$

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We now consider defects in $(\rho, \sigma, \tilde{F})$ which transport to point defects in $F$
Point $\rho$-defects

The link is a closed interval with $\rho$-colored boundary. It evaluates under $(\sigma, \rho)$ to the vector space $A = \mathbb{C}[G]$. The “label” of the defect is therefore an element of $A$. Note $G \subset A$ labels invertible defects.

$\rho$-defects are topological

$$\text{C}(G) \quad \sigma \quad G \mathbb{C} \mathbb{C} \quad \approx \quad (\mathbb{C}, \mathbb{C})$$
Point $\tilde{F}$-defects

The link is again a closed interval, but now with $\tilde{F}$-colored boundary. The value under $(\sigma, \tilde{F})$ is $\text{End}_A(\mathcal{H})$, the space of observables that commute with the $G$-action $\tilde{F}$-defects are typically not topological
The link is $S^1$, and the value under $\sigma$ is the vector space which is the center of the group algebra $A = \mathbb{C}[G]$.

$\sigma$-defects are topological
The general point defect

A general point defect in $F$ can be realized by a line defect in $(\rho, \sigma, \tilde{F})$.

Label the defect beginning with the highest dimensional strata and work down in dimension

$B$  $(A, A)$-bimodule
$\xi$  vector in $B$
$T$  $(A, A)$-bimodule map $B \rightarrow \text{End}(\mathcal{H})$
Composition law on defects

Compute using the links of the defects—2 incoming and 1 outgoing

$\sigma$-defects: pair of pants

$\rho$-defects: pair of chaps
Commutation relations among defects

The sandwich realization makes clear that

- $\rho$-defects (symmetries) commute with $\widetilde{F}$-defects
- $\sigma$-defects (central symmetries) commute with both $\rho$-defects and with $\widetilde{F}$-defects

However, $\rho$-defects do not necessarily commute with each other.

Nor do they commute with the general defect.
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Finite group symmetries of an \((n = 2)\)-dimensional theory

Let \(G\) be a finite group, and let \(\sigma\) be the 3-dimensional finite \(G\)-gauge theory

\[
\sigma : \text{Bord}_3 \longrightarrow \text{Alg}(\text{Cat})
\]

with \(\sigma(\text{pt}) = \text{Vect}[G]\), and let \(\rho\) be the regular right \(\sigma\)-module with \(\rho(\text{pt}) = \text{Vect}[G]_{\text{Vect}[G]}\).
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Line \(\rho\)-defects are labeled by objects in \(\text{Vect}[G]\); elements \(g \in G\) label invertible defects
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As opposed to \(G\)-symmetry in \(n = 1\), here the center is “bigger”
Line defects in 4-dimensional gauge theory

Our goal is to explain the paper *Reading between the lines of four-dimensional gauge theories* (arXiv:1305.0318) of Aharony-Seiberg-Tachikawa in this framework.
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Their paper concerns the gauge theories for compact Lie groups lying between a simply connected group $G$ and its adjoint group $\overline{G}$. In our context this requires a *quotient* construction (*gauging*), which we then describe in this context.

The main point is a *higher Gauss law*, which is the final prerequisite that we discuss.
Definition: A topological space $X$ is $\pi$-finite if (i) $\pi_0 X$ is a finite set, (ii) for all $x \in X$, the homotopy group $\pi_q(X, x)$, $q \geq 1$, is finite, and (iii) there exists $Q \in \mathbb{Z}^{>0}$ such that $\pi_q(X, x) = 0$ for all $q > Q$, $x \in X$. 
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**Definition:** A topological space $\mathcal{X}$ is $\pi$-finite if (i) $\pi_0\mathcal{X}$ is a finite set, (ii) for all $x \in \mathcal{X}$, the homotopy group $\pi_q(\mathcal{X}, x)$, $q \geq 1$, is finite, and (iii) there exists $Q \in \mathbb{Z}^+ \cup \{0\}$ such that $\pi_q(\mathcal{X}, x) = 0$ for all $q > Q$, $x \in \mathcal{X}$.

**Examples:**

1. An Eilenberg-MacLane space $K(\pi, q)$ is $\pi$-finite if $\pi$ is a finite group. Denote $K(G, 1)$ by $BG$ for $G$ a finite group, and if $q \geq 1$ and $A$ is a finite abelian group, we denote $K(A, q)$ by $B^qA$. 


Finite homotopy theories

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(2) Let $G$ be a finite group, let $A$ be a finite abelian group, and fix a cocycle $k$ for a cohomology class $[k] \in H^3(G; A)$. (One can also include an action of $G$ on $A$.) Realize $k$ as a map $k: BG \to B^3A$, and form the $\pi$-finite space $\mathcal{X}$ as a pullback:

\[
\begin{array}{ccc}
B^2A & \longrightarrow & \mathcal{X} & \longrightarrow & BG \\
\| & & \downarrow & & \downarrow k \\
B^2A & \longrightarrow & * & \longrightarrow & B^3A
\end{array}
\]
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Remark: If we drop the $\pi$-finiteness assumption, then we can construct a once-categorified theory from any topological space.
Finite homotopy theories

$m$  (spacetime) dimension
\[X\]  \(\pi\)-finite space
\(\lambda\)  cocycle of degree \(m\) on \(X\)  \([\lambda] \in H^m(X; \mathbb{C}^\times)\)
\(M\)  closed manifold
\(\mathcal{X}^M\)  \(\text{Map}(M, \mathcal{X})\)
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\( m \)  (spacetime) dimension
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For definiteness take \( \mathcal{X} = B^2A \), \( \lambda = 0 \), and \( m = 5 \) ("1-form A-symmetry on a 4d theory")

Denote the resulting topological field theory as \( \sigma \)
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\text{Denote the resulting topological field theory as } \sigma \\
m = 5: \quad \sigma(M) &= \sum_{[\phi] \in \pi_0(X^M)} \frac{\#\pi_2(X^M, \phi)}{\#\pi_1(X^M, \phi)} = \frac{\#H^0(M; A)}{\#H^1(M; A)} \#H^2(M; A)
\end{align*}
\]
Finite homotopy theories

codim 1—the vector space of locally constant complex-valued functions on $\mathcal{X}^M$:

$m = 4 : \quad \sigma(M) = \text{Fun}(\pi_0(\mathcal{X}^M)) = \text{Fun}(H^2(M; A))$
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$$\sigma(M) = \text{Vect}(\pi_{\leq 1}(\mathcal{X}^M)) = \text{Vect}(H^2(M; A)) \times \text{Rep}(H^1(M; A))$$

$$\simeq \text{Vect}(H^2(M; A) \times H^1(M; A)^\vee)$$
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$\cong \text{Vect}(H^2(M; A) \times H^1(M; A) \wedge)$

The quantization of a bordism $M: N_0 \to N_1$ uses the correspondence of mapping spaces:

```
\mathcal{X}^N_0 \xrightarrow{p_0} \mathcal{X}^M \xrightarrow{p_1} \mathcal{X}^N_1
```
Finite homotopy theories

Semiclassical descriptions of boundaries and defects lead to computable quantizations
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Fix \((X, \lambda)\) a \(\pi\)-finite space and cocycle
Finite homotopy theories

Semiclassical descriptions of boundaries and defects lead to computable quantizations

Fix \((\mathcal{X}, \lambda)\) a \(\pi\)-finite space and cocycle

**Definition:** A right semiclassical boundary theory of \((\mathcal{X}, \lambda)\) is a triple \((\mathcal{Y}, f, \mu)\) consisting of a \(\pi\)-finite space \(\mathcal{Y}\), a map \(f: \mathcal{Y} \to \mathcal{X}\), and a trivialization \(\mu\) of \(-f^*\lambda\)

**Quantization:** Compositions of defects are computed using homotopy fiber products
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**Definition:** Fix \(m, \ell \in \mathbb{Z}_{>2}\) with \(\ell \leq m\). A *semiclassical local defect* of codimension \(\ell\) for \((\mathcal{X}, \lambda)\) is a \(\pi\)-finite map

\[
\delta : \mathcal{Y} \to \mathcal{L}^{\ell-1}\mathcal{X}
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Finite homotopy theories

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A regular boundary theory has $\mathcal{Y} = *$

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Compositions of defects are computed using homotopy fiber products
Line defects in 4-dimensional gauge theory

Our goal is to explain the paper *Reading between the lines of four-dimensional gauge theories* (arXiv:1305.1308) of Aharony-Seiberg-Tachikawa in this framework.

The relevant topological theory $\sigma$ can be realized by a semiclassical construction based in topology, so we first introduce such *finite homotopy theories*.

Their paper concerns the gauge theories for compact Lie groups lying between a simply connected group $G$ and its adjoint group $\overline{G}$. In our context this requires a *quotient construction* (gauging), which we then describe in this context.

The main point is a *higher Gauss law*, which is the final prerequisite that we discuss.
**Quotients: augmentations**

**Definition:** An *augmentation* of an algebra $A$ is an algebra homomorphism $\epsilon: A \to \mathbb{C}$.

Use $\epsilon$ to give a right $A$-module structure to $\mathbb{C}$: $\lambda \cdot a = \lambda \epsilon(a)$, $\lambda \in \mathbb{C}$.
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**Example:** $A = \mathbb{C}[G]$:

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$$\sum_{g \in G} \lambda_g g \mapsto \sum_{g \in G} \lambda_g$$
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The “quotient” of a left $A$-module $L$ is the vector space

$$Q = \mathbb{C} \otimes_A L = \mathbb{C} \otimes_\epsilon L$$
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**Example:** $A = \mathbb{C}[G], \, S$ a finite $G$-set, $L = \mathbb{C}\langle S \rangle$: then $Q = \mathbb{C} \otimes_A \mathbb{C}\langle S \rangle \cong \mathbb{C}\langle S/G \rangle$
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**Augmentations for higher algebras:** $\Phi$ tensor category $\epsilon: \Phi \to \text{Vect}$ fiber functor
Quotients and quotient defects

We use the yoga of fully local topological field theory: let $\mathcal{C}'$ be a symmetric monoidal $n$-category and set $\mathcal{C} = \text{Alg}(\mathcal{C}')$, the $(n + 1)$-category whose objects are algebras in $\mathcal{C}'$.

**Definition:** An augmentation $\varepsilon_A : A \to 1$ of an algebra $A \in \text{Alg}(\mathcal{C}')$ is an algebra homomorphism from $A$ to the tensor unit $1 \in \mathcal{C}$.

**Definition:** Let $\mathcal{F}$ be a collection of $(n + 1)$-dimensional fields, and suppose $\sigma : \text{Bord}_{n+1}(\mathcal{F}) \to \mathcal{C}$ is a topological field theory. A right boundary theory $\varepsilon$ for $\sigma$ is an augmentation of $\sigma$ if $\varepsilon(\text{pt})$ is an augmentation of $\sigma(\text{pt})$.

Augmentations are also called Neumann boundary theories.
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**Definition:** Let $\mathcal{F}$ be a collection of $(n + 1)$-dimensional fields, and suppose $\sigma : \text{Bord}_{n+1}(\mathcal{F}) \to \mathcal{C}$ is a topological field theory. A right boundary theory $e$ for $\sigma$ is an augmentation of $\sigma$ if $e(\text{pt})$ is an augmentation of $\sigma(\text{pt})$

Augmentations are also called Neumann boundary theories

Augmentations do not always exist
Definition: Suppose given finite symmetry data \((\sigma, \rho)\) and a \((\sigma, \rho)\)-module structure \((\tilde{F}, \theta)\) on a quantum field theory \(F\). Suppose \(\varepsilon\) is an augmentation of \(\sigma\). Then the quotient of \(F\) by the symmetry \(\sigma\) is

\[
\frac{F}{\sigma} = \varepsilon \otimes_{\sigma} \tilde{F}
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Gauss laws in finite homotopy theories

Begin with the usual Gauss law for quantization in codimension 1

Recall that we have a mapping space $X \rightarrow M$ whose quantization—in an untwisted situation—is the vector space of locally constant functions $\text{Fun}_{\text{flat}}^p X \rightarrow M$. In a twisted situation there is a "flat" complex line bundle $L \rightarrow X \rightarrow M$, or local system, and the quantization is the space of flat sections. More precisely, replace $X \rightarrow M$ by its fundamental groupoid $\pi_1 X \rightarrow M$, and take sections of the line bundle $L \rightarrow \pi_1 X \rightarrow M$. The Gauss law says that sections vanish over components of $X \rightarrow M$ on which $\pi_1$ acts by a non-identity character on $L$. In categorical terms, this is the limit of the map (functor) $L : \pi_1 X \rightarrow M \rightarrow \text{Vect}$.
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Recall that we have a mapping space $\mathcal{X}^M$ whose quantization—in an *untwisted* situation—is the vector space of locally constant functions $\text{Fun}_{\text{flat}}(\mathcal{X}^M)$

In a twisted situation there is a "flat" complex line bundle $\mathcal{L} \overset{\pi}{\to} \mathcal{X}^M$, or local system, and the quantization is the space of flat sections $\text{Flat} \mathcal{L}$

More precisely, replace $\mathcal{X}^M$ by its fundamental groupoid $\pi_1(\mathcal{X}^M)$, and take sections of the line bundle $\mathcal{L} \overset{\pi}{\to} \pi_1(\mathcal{X}^M)$

The Gauss law says that sections vanish over components of $\mathcal{X}^M$ on which $\pi_1$ acts by a non-identity character on $\mathcal{L}$

In categorical terms, this is the limit of the map (functor) $\mathcal{L} \overset{\pi}{\to} \text{Vect}$
Gauss laws in finite homotopy theories

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We need the analogous Gauss law for quantization in codimension 2.
Gauss laws in finite homotopy theories

We need the analogous Gauss law for quantization in codimension 2

Untwisted case: the quantization is the linear category $\text{Vect}_{\text{flat}}(\mathcal{X}^M)$ of flat vector bundles

Twisted case: there is a "flat" $\text{Vect}$-line bundle $K \rightarrow X^M$, where $\text{Vect}$ is the linear category of vector spaces, and the quantization is the space of flat sections

More precisely, replace $X^M$ by its fundamental 2-groupoid $\pi_2(X^M)$, and take sections of the $\text{Vect}$-bundle $K \rightarrow \pi_2(X^M)$

In categorical terms, this is the limit of the map (functor) $K : \pi_2(X^M) \rightarrow \text{Cat}$

Higher Gauss law: At a point $P \in X^M$, if $\pi_1(P)$, then $\pi_2(P)$ acts on $K$ by automorphisms of the identity functor via a character, and sections of $K \rightarrow X^M$ vanish on the component which contains $P$ if that character is not the identity
Gauss laws in finite homotopy theories

We need the analogous Gauss law for quantization in codimension 2

Untwisted case: the quantization is the linear category $\text{Vect}_{\text{flat}}(\mathcal{X}^M)$ of flat vector bundles

In a twisted situation there is a “flat” $\mathcal{V}$-line bundle $\mathcal{K} \to \mathcal{X}^M$, where $\mathcal{V} = \text{Vect}$ is the linear category of vector spaces, and the quantization is the space of flat sections

$\text{higher Gauss law: At a point } P \in \mathcal{X}^M, \text{ if } \pi_1^1\pi : 1 \to \mathcal{X}^M, \text{ then } \pi_2^2\pi \text{ acts on } K \text{ by automorphisms of the identity functor via a character, and sections of } K \to \mathcal{X}^M \text{ vanish on the component which contains } P \text{ if that character is not the identity}$
Gauss laws in finite homotopy theories

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More precisely, replace $\mathcal{X}^M$ by its fundamental 2-groupoid $\pi_{\leq 2} \mathcal{X}^M$, and take sections of the $\mathcal{V}$-bundle $\mathcal{K} \to \pi_{\leq 2} \mathcal{X}^M$. 

Gauss laws in finite homotopy theories

We need the analogous Gauss law for quantization in codimension 2

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Higher Gauss law: At a point $\phi \in \mathcal{X}^M$, if $\pi_1(\mathcal{X}^M, \phi) = 0$ then $\pi_2(\mathcal{X}^M, \phi)$ acts on $\mathcal{K}_\phi$ by automorphisms of the identity functor via a character, and sections of $\mathcal{K} \to \mathcal{X}^M$ vanish on the component which contains $\phi$ if that character is not the identity
Our goal is to explain the paper *Reading between the lines of four-dimensional gauge theories* ([arXiv:1305.1308](https://arxiv.org/abs/1305.1308)) of Aharony-Seiberg-Tachikawa in this framework.

The relevant topological theory $\sigma$ can be realized by a semiclassical construction based in topology, so we first introduce such finite homotopy theories.

Their paper concerns the gauge theories for compact Lie groups lying between a simply connected group $G$ and its adjoint group $\overline{G}$. In our context this requires a quotient construction (*gauging*), which we then describe in this context.

The main point is a higher Gauss law, which is the final prerequisite that we discuss.
BA symmetry

- \( H \): compact Lie group
- \( A \): finite subgroup of center(\( H \))
- \( \overline{H} \): \( H/A \)
- \( \sigma \): 5-dimensional finite homotopy with \( X = B^2A \)
- \( \rho \): right topological boundary theory \( \star \rightarrow B^2A \)
- \( F \): a 4-dimensional \( H \)-gauge theory with \( BA \) symmetry
- \( \tilde{F} \): the corresponding \( \overline{H} \)-gauge theory
Topological right $\sigma$-modules

Recall the semiclassical description: $f : Y \to B^2A$ and a trivialization $\mu$ of $-f^*\lambda = 0$, so a 4-cocycle $\mu$ on $Y$ with coefficients in $\mathbb{C}^\times$.
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For any subgroup \( A' \subset A \) there is an induced map \( B^2A' \rightarrow B^2A \)
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Eilenberg-MacLane compute

$$H^4(B^2 A'; \mathbb{C}^\times) \cong \{\text{quadratic functions } q : A' \to \mathbb{C}^\times\}$$
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The pair $(A', q)$ determines the right topological boundary theory $R_{A', q}$.
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The pair $(A', q)$ determines the right topological boundary theory $R_{A', q}$

The quadratic form $q$ gives rise to the Pontrjagin square cohomology operation

$$\mathcal{P}_q: H^2(X; A') \to H^4(X; \mathbb{C}^\times)$$

which enters the formula for the partition function in the theory $R_{A', q} \otimes_{\sigma} \tilde{F}$, which is an $H/A'$-gauge theory
Line defects in the $H/A'$-gauge theory $R_{A',q} \otimes_\sigma \tilde{F}$

$M$ \quad 4-manifold

$C \subset M$ \quad 1-dimensional submanifold

$[0, 1] \times C$ \quad 2-dimensional submanifold of $[0, 1] \times M$

\[
\sigma = \int_{S^2} \hat{F}
\]

Recall + compare:
Line defects in the $H/A'$-gauge theory $R_{A',q} \otimes \sigma \tilde{F}$

$M$ 4-manifold

$C \subset M$ 1-dimensional submanifold

$[0, 1] \times C$ 2-dimensional submanifold of $[0, 1] \times M$

Label in $(0, 1) \times C$ is an object in the 2-category $\text{Hom}(1, \sigma(S^2))$, so we compute $\sigma(S^2)$:

$\pi_0(\text{Map}(S^2, B^2A)) = H^2(S^2; A) \cong A$

$\pi_1(\text{Map}(S^2, B^2A)) = H^1(S^2; A) = 0$

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\end{align*}
$$

2-category of local systems of linear categories over the indicated 2-groupoid, so for $m \in H^2(S^2; A) \cong A$ we have a linear category $\mathcal{K}_m$ equipped with an action of $\pi_2 \cong A$ by automorphisms of the identity functor, hence $\mathcal{K}_m$ decomposes as

$$
\mathcal{K}_m = \bigoplus_e \mathcal{K}_{m,e} \cdot e, \quad e \in H^0(S^2; A)^\vee \cong A^\vee
$$
The line defect $[0, 1) \times C$ in $(\sigma, R_{A', q})$

First, fix a pair $(m_0, e_0) \in A \times A^\vee$ and choose the interior label $\mathcal{K}$ to be the “$\delta$-function” supported at $(m_0, e_0)$:

$$\mathcal{K}_{m,e} = \begin{cases} 
\text{Vect}, & (m, e) = (m_0, e_0); \\
0, & (m, e) \neq (m_0, e_0).
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The quantization of the link $D^3$ at the $R_{A',q}$ boundary is a 1-category

**Claim:** This 1-category vanishes unless $(m_0, e_0)$ obeys a selection rule
The line defect \([0, 1) \times C\) in \((\sigma, R_{A',q})\)

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**Claim:** This 1-category vanishes unless \((m_0, e_0)\) obeys a selection rule

The selection rule is an assertion in the *topological* field theory \((\sigma, R_{A',q})\)
The selection rule

From the quadratic function $q: A' \to \mathbb{C}^\times$ we obtain a bihomomorphism

$$b: A' \times A' \longrightarrow \mathbb{C}^\times$$

which induces a perfect pairing

$$H^2(S^2; A') \times H^0(S^2; A') \longrightarrow \mathbb{C}^\times$$

and so too an isomorphism

$$e': H^2(S^2; A') \longrightarrow H^0(S^2; A')^\vee$$
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Selection rule:

$$\begin{align*}
m &\in A' \\
e|_{A'} &= e'(m)^{-1}
\end{align*}$$
Sketch proof of the selection rule

Compute the homotopy limit of the diagram:

\[(\text{Map}(S^2, B^2A'), \tau^2(\mu_q)) \rightarrow \text{Map}(S^2, B^2A) \rightarrow \text{Map}(D^3 \setminus B^3, B^2A) \rightarrow (B^2A, e)\]

\[m \in A'\]
\[e|_{A'} = e'(m)^{-1}\]