ON DETERMINANT LINE BUNDLES

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Determinant line bundles entered differential geometry in a remarkable paper of Quillen [Q]. He attached a holomorphic line bundle $\mathcal{L}$ to a particular family of Cauchy-Riemann operators over a Riemann surface, constructed a Hermitian metric on $\mathcal{L}$, and calculated its curvature. At about the same time Atiyah and Singer [AS2] made the connection between determinant line bundles and anomalies in physics. Somewhat later, Witten [W1] gave a formula for "global anomalies" in terms of η-invariants. He suggested that it could be interpreted as the holonomy of a connection on the determinant line bundle. These ideas have been developed by workers in both mathematics and physics. Our goal here is to survey some of this work.

We consider arbitrary families of Dirac operators D on a smooth compact manifold X. The associated Laplacian has discrete spectrum, which leads to a patching construction for the determinant line bundle $\mathcal{L}$. The determinant $\det D$ is a section of $\mathcal{L}$. Quillen uses the *analytic torsion* of Ray and Singer [RS1] to define a metric on $\mathcal{L}$. An extension of these ideas produces a unitary connection whose curvature and holonomy can be computed explicitly. The holonomy formula reproduces Witten's global anomaly. Section 1 represents joint work with Jean-Michel Bismut, whose proof of the index theorem for families [B] is a crucial ingredient in the curvature formula (1.30).

These basic themes allow many variations, two of which we play out in §2 and §3. Suppose X is a complex manifold and the family of Dirac operators (or Cauchy-Riemann operators) varies holomorphically. Then $\mathcal{L}$ carries a natural complex structure, and under appropriate restrictions on the geometry the canonical connection is compatible with the holomorphic structure. The proper geometric hypothesis, that the total space swept out by X be Kähler, at least locally in the parameter space, also ensures that the operators vary holomorphically. Many special cases of this result can be found in the literature; the version we prove is due to Bismut, Gillet, and Soulé [BGS].

One novelty here is the observation that $\det D$ has a natural square root[1] if the dimension of X is congruent to 2 modulo 8. On topological grounds one can argue the existence of $\mathcal{L}^{1/2}$ using Rohlin's theorem, which is linked to real K-theory. However, one needs the differential geometry to see that $\det D$ also admits a square root. There is an extension of the holonomy theorem to $\mathcal{L}^{1/2}$.

In §4 we study Riemann surfaces. This is the case originally considered by Quillen. Faltings [Fa] considered determinants on Riemann surfaces in an arithmetic context. These determinants also form the

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[1]Lee and Miller also construct this square root.

cornerstone of Polyakov's formulation of string theory [P]. For elliptic curves there are explicit formulas for the determinant, which illustrate the general theory beautifully. A recent paper of Atiyah [A2] studies the transformation law for Dedekind's η-function in this context. On higher genus surfaces the determinants are related to ϑ-functions. Precise formulas have been derived in the physics literature, where they are applied to "bosonization." In a mathematical vein they clarify the relationship between Quillen's metric and the metric constructed by Faltings. Here and elsewhere in this section we have benefited from Bost [Bo].

The determinant line bundle carries topological information. In §5 we explore a particular example related to Witten's global anomalies (in string theory). One could speculate that this anomaly is related to orientation in some generalized cohomology theory. Of more technical interest is our use of Sullivan's $\mathbb{Z}/k$-manifolds to pass from K-theory to integral cohomology.

Although this paper is rather lengthy, by necessity we have omitted many important topics. For example, the determinant of the Dirac operator can be studied as an energy function on appropriate configuration spaces. Donaldson [D] illustrates this idea in his construction of Hermitian-Einstein metrics on stable bundles over projective varieties. Osgood, Phillips, and Sarnak [OPS] treat the uniformization theorem in Riemann surface theory from a similar point of view. We have not mentioned the approach to determinants using Selberg ζ-functions. Cheeger [C] and Singer [S] give other geometric interpretations of Witten's global anomaly that we leave untouched. We have not given justice to work of Beilinson and Manin, nor to work of Deligne [De], among others. Such omissions are inevitable, though regrettable.

My understanding of determinants owes much to my collaborators Jean-Michel Bismut and Cumrun Vafa. I have also benefited greatly from the insight of Michael Atiyah, Raoul Bott, Gunnar Carlsson, Jeff Cheeger, Pierre Deligne, Simon Donaldson, Richard Melrose, Ed Miller, Haynes Miller, Greg Moore, John Morgan, Phil Nelson, Dan Quillen, Isadore Singer, and Ed Witten on the various topics covered here, and I take this opportunity to thank them all. I am grateful to Peter Landweber for his careful reading of an earlier version of this paper.

§1 The Determinant Line Bundle

Suppose X is a compact spin manifold. Then the chiral Dirac operator maps positive (right-handed) spinor fields to negative (left-handed) spinor fields. Symbolically, we write $D\colon \mathcal{H}_+ \to \mathcal{H}_-$. If $\mathcal{H}_\pm$ were finite dimensional and of equal dimension, then on the highest exterior power there would be an induced map $\det D\colon \det \mathcal{H}_+ \to \det \mathcal{H}_-$. The determinant of D would be regarded as an element of the complex line $(\det \mathcal{H}_+)^* \otimes (\det \mathcal{H}_-)$. As the Dirac operator acts on infinite dimensional spaces, some care must be exercised to define its determinant. In this section we recall Quillen's construction of the metrized determinant line associated to a Dirac operator [Q]. The ζ-functions of Ray and Singer [RS1] play a crucial role. For a family of Dirac operators a patching construction then produces a metrized line bundle over the parameter space. We review in detail the construction of a natural connection on the determinant line bundle [BF1]. At the end of this section we briefly recall the formulas for its curvature and holonomy [BF2]. The holonomy formula is one possible interpretation of Witten's global anomaly [W1]. As stated in the introduction, this section is based on joint work with J.-M. Bismut. The reader may wish to refer to [F1,§1] for related expository material.

The basic setup for our work is given by the

Geometric Data 1.1.

(1) *A smooth fibration of manifolds* $\pi\colon Z \xrightarrow{X} Y$. *To define a family of Dirac operators we need to add the topological hypothesis that the tangent bundle along the fibers* $T(Z/Y) \to Z$ *has a fixed Spin (or* Spin^c*) structure.*
(2) *A metric along the fibers, that is, a metric* $g^{(Z/Y)}$ *on* $T(Z/Y)$.
(3) *A projection* $P\colon TZ \to T(Z/Y)$.
(4) *A complex representation* ρ *of* $\mathrm{Spin}(n)$.
(5) *A complex vector bundle* $E \to Z$ *with a hermitian metric* $g^{(E)}$ *and compatible connection* $\nabla^{(E)}$.

Remarks.

(1) To say that π is a smooth fibration of manifolds simply means that Z is a smooth manifold, π is a smooth map, and for small open sets $U \subset Y$ the inverse image $\pi^{-1}(U)$ is diffeomorphic to $U \times X$. Here we glue local products $U \times X$ using diffeomorphisms of X. The identification of X with the fiber X_y at $y \in Y$ is only up to an element of $\mathrm{Diff}(X)$. Equivalently, $Z \to Y$ is associated to a smooth principal $\mathrm{Diff}(X)$-bundle $\mathcal{P} \to Y$. A point in the fiber of $\mathcal{P}$ at y is an identification $X_y \cong X$. There are technical difficulties in constructing a smooth Lie group out of the space of smooth diffeomorphisms of X. Here we understand that the transition function $\varphi\colon U \to \mathrm{Diff}(X)$ is smooth if and only if $\langle y, x\rangle \mapsto \varphi(y)x$ is smooth. The topological hypothesis amounts to a reduction of this bundle to the subgroup of diffeomorphisms fixing a given orientation and spin structure on X.
(2) The kernel of the projection P defines a distribution $\ker P$ of horizontal subspaces on Z. Then $\ker P \cong \pi^* TY$ and $TZ = T(Z/Y) \oplus \ker P$ splits as a direct sum. Therefore, we can lift a tangent vector $\xi \in T_y Y$ to a horizontal vector field $\tilde{\xi}$ along the fiber Z_y. The projection P is equivalent to a connection on the bundle $\mathcal{P} \to Y$.
(3) We may, of course, choose ρ to be the trivial representation. Also, we allow virtual representations, that is, differences of representations. The data in (1) and (2) determine a $\mathrm{Spin}(n)$ bundle of frames $\mathrm{Spin}(Z) \to Z$ consisting of "spin frames" of the vertical tangent space, and the representation ρ gives

an associated vector bundle $V_\rho \to Z$. For example, if ρ is the n-dimensional representation of $\mathrm{Spin}(n)$, then V_ρ is simply $T(Z/Y)$. The Dirac operator we construct is coupled to V_ρ; this is an intrinsic coupling in that it only involves bundles associated to the tangent bundle. In case $V_\rho = T(Z/Y)$ we obtain the Rarita-Schwinger operator. If ρ is the total spin representation $\sigma_+ \oplus \sigma_-$ we obtain the signature operator.

(4) The vector bundle E determines the extrinsic coupling of the Dirac operator.

The family of Dirac operators $\{D_y\}$ corresponding to the data in (1.1) is constructed fiberwise. For each $y \in Y$ the fiber Z_y is a smooth spin manifold and so has a Levi-Civita connection. There is an induced connection on $V_\rho|_{Z_y}$. The connection $\nabla^{(E)}$ restricts to a connection on $E|_{Z_y}$. Let $S_\pm \to Z$ be the bundles associated to $\mathrm{Spin}(Z)$ via the half-spin representations. Then the ordinary chiral Dirac operator on Z_y, which maps positive spinor fields to negative spinor fields, couples to the bundles $V_\rho|_{Z_y}$ and $E|_{Z_y}$ via the connections. Thus we obtain an operator

$$D_y\colon C^\infty\big((S_+ \otimes V_\rho \otimes E)\big|_{Z_y}\big) \to C^\infty\big((S_- \otimes V_\rho \otimes E)\big|_{Z_y}\big). \tag{1.2}$$

The metrics on $S_\pm$, V_ρ, and E, together with the volume form on Z_y, induce L^2 completions $\big(\mathcal{H}_\pm\big)_y$ of the C^∞ spaces in (1.2). Then D_y extends to an unbounded operator on these L^2 spaces. Furthermore, as y varies over Y these spaces fit together to form *continuous* Hilbert bundles $\mathcal{H}_\pm \to Y$. (These bundles are not smooth since the composition $L^2 \times C^\infty \to L^2$ is not differentiable.) Thus we can view D_y as a bundle map $\mathcal{H}_+ \xrightarrow{D} \mathcal{H}_-$. Notice that the Hilbert bundles $\mathcal{H}_\pm$ carry L^2 metrics by definition. Our constructions below only use finite dimensional subbundles of the Hilbert bundles, and so the technicalities associated with infinite dimensional bundles are not a problem here.

The geometric data (1.1) determine a connection $\nabla^{(Z/Y)}$ on $T(Z/Y)$ as follows. Fix an arbitrary Riemannian metric $g^{(Y)}$ on Y. The metric $g^{(Z/Y)}$ along the fibers together with the lift of $g^{(Y)}$ to the horizontal subspaces (the kernel of the projection P) combine to form a metric $g^{(Z)}$ on Z. Let $\nabla^{(Z)}$ denote its Levi-Civita connection.

Lemma 1.3. *The projection of $\nabla^{(Z)}$ to a connection $\nabla^{(Z/Y)}$ on $T(Z/Y)$ is independent of the choice of metric on Y.*

We omit the easy proof. The connection $\nabla^{(Z/Y)}$ restricts to the Levi-Civita connection on each fiber Z_y.

We use this connection to induce connections on $\mathcal{H}_\pm$. Since these Hilbert bundles are only continuous, the "connections" we define will operate only on smooth sections. To obtain *unitary* connections we must modify $\nabla^{(Z/Y)}$ slightly, since the volume form $\mathbf{vol}$ of $g^{(Z/Y)}$ changes from fiber to fiber. Now although this volume originally comes as a section of $\Lambda^n(T(Z/Y))^*$, the projection P allows us to regard $\mathbf{vol}$ as an n-form on Z. Fix a tangent vector η to Z, and use the splitting $TZ \cong T(Z/Y) \oplus \ker P$ to define a 1-form $\gamma \in \Omega^1(Z)$:

$$\iota(\eta)d(\mathbf{vol}) = 2\gamma(\eta)\,\mathbf{vol} + \text{nonvertical terms}. \tag{1.4}$$

Finally, set

$$\tilde\nabla^{(Z/Y)} = \nabla^{(Z/Y)} + \gamma. \tag{1.5}$$

This defines a new connection on $T(Z/Y)$ and its associated bundles. The tilde serves as a reminder of the correction term. Hence we obtain connections

$$\tilde{\nabla}^{(\pm)} = \dot{\sigma}_{\pm}\big(\tilde{\nabla}^{(Z/Y)}\big) \oplus \dot{\rho}\big(\nabla^{(Z/Y)}\big) \oplus \nabla^{(E)} \tag{1.6}$$

on the bundles $S_\pm \otimes V_\rho \otimes E$. (Here $\dot{\rho}$ denotes the Lie algebra representation induced by ρ, and $\sigma_\pm$ are the half spin representations.) Of course, the metrics $g^{(Z/Y)}$ and $g^{(E)}$ induce metrics $(\cdot\,,\cdot)$ on these bundles. The modification (1.5) was introduced so that if s_1, s_2 are smooth sections of one of these bundles (over Z), then

$$d\big\{(s_1,s_2)\,\mathbf{vol}\big\} = (\tilde{\nabla}^{(\pm)}s_1, s_2)\,\mathbf{vol} + (s_1, \tilde{\nabla}^{(\pm)}s_2)\,\mathbf{vol} + \text{nonvertical terms}. \tag{1.7}$$

We can consider s_i as sections of the Hilbert bundle $\mathcal{H}_\pm \longrightarrow Y$. The L^2 metric on $\mathcal{H}_\pm$ is defined by

$$(s_1,s_2)_{\mathcal{H}_\pm} = \int_{Z/Y} (s_1,s_2)\,\mathbf{vol}. \tag{1.8}$$

Let ξ be a tangent vector to Y and $\tilde{\xi}$ its horizontal lift to a fiber in Z. Then for a section s of $\mathcal{H}_\pm$ we define

$$\tilde{\nabla}^{(\mathcal{H}_\pm)}_{\xi} s = \tilde{\nabla}^{(\pm)}_{\tilde{\xi}} s \tag{1.9}$$

acting pointwise. We view $\tilde{\nabla}^{(\mathcal{H}_\pm)}$ as "connections" on $\mathcal{H}_\pm$, which act only on the dense subspace of smooth sections $C^\infty(S_\pm \otimes V_\rho \otimes E) \subset L^2(S_\pm \otimes V_\rho \otimes E)$. Since integration along the fiber commutes with d, and nonvertical terms integrate to zero, equation (1.7) implies that $\tilde{\nabla}^{(\mathcal{H}_\pm)}$ preserves the L^2 metric (1.8).

Let ξ_1, ξ_2 be vector fields on Y, denote by $\tilde{\xi}_1, \tilde{\xi}_2$ their horizontal lifts to Z, and set

$$T(\xi_1,\xi_2) = [\tilde{\xi}_1, \tilde{\xi}_2] - \widetilde{[\xi_1,\xi_2]}. \tag{1.10}$$

It is easy to check that $T(\xi_1,\xi_2)$ is a vertical vector field on Z which is tensorial in ξ_i. In fact, T is essentially the curvature of the connection on the $\mathrm{Diff}_0(X)$-bundle $\mathcal{P} \longrightarrow Y$ (c.f., remarks (1) and (2) above). The next result will enter our discussion of holomorphic families in §2.

Proposition 1.11. *The curvature $\Omega^{(\mathcal{H}_\pm)}(\xi_1,\xi_2)$ is the first order differential operator*

$$\Omega^{(\mathcal{H}_\pm)}(\xi_1,\xi_2) = \tilde{\nabla}^{(\pm)}_{T(\xi_1,\xi_2)} + \tilde{\Omega}^{\pm}(\tilde{\xi}_1,\tilde{\xi}_2),$$

where $\tilde{\Omega}^\pm$ is the curvature of $\tilde{\nabla}^{(\pm)}$.

The determinant line of the single Dirac operator D_y is

$$\mathcal{L}_y = (\det\,\ker D_y)^* \otimes (\det\operatorname{coker} D_y).$$

(Recall that $\det(V)$ is the highest exterior power of the vector space V.) Since $\dim\,\ker D_y$ may jump as y varies, we need a construction to patch these lines together.

Theorem 1.12 [BF1]**.** *The Geometric Data* (1.1) *determine a smooth line bundle $\mathcal{L} \longrightarrow Y$ along with a natural Hermitian metric $g^{(\mathcal{L})}$ (the Quillen metric) and unitary connection $\nabla^{(\mathcal{L})}$. There is a canonical section* $\det D$ *of $\mathcal{L}$ over components of Y where D has numerical index zero.*

Let us first recall some basic consequences of ellipticity. Each fiber $\mathcal{H}_y$ of the Hilbert bundle $\mathcal{H}_+$ (resp. $\mathcal{H}_-$) decomposes into a direct sum of eigenspaces of the nonnegative elliptic operator D^*D (resp. $D\,D^*$). We delete y from the notation for convenience. The spectra of these operators are discrete, the nonzero eigenvalues $\{\lambda\}$ of D^*D and DD^* agree, and D is an isomorphism between the corresponding eigenspaces. For $a > 0$ not in the spectrum of D^*D, let $\mathcal{H}_\pm^{(a)}$ be the sum of the eigenspaces for eigenvalues less than a. Then $\mathcal{H}_\pm^{(a)}$ is finite dimensional and consists of smooth fields. There is an exact sequence

$$0 \longrightarrow \ker D^*D \longrightarrow \mathcal{H}_+^{(a)} \xrightarrow{D} \mathcal{H}_-^{(a)} \longrightarrow \ker DD^* \longrightarrow 0. \tag{1.13}$$

If $b > a$ we set $\mathcal{H}_\pm^{(a,b)} = \mathcal{H}_\pm^{(b)} \ominus \mathcal{H}_\pm^{(a)}$; then

$$D\big|_{\mathcal{H}_+^{(a,b)}} = D^{(a,b)} \colon \mathcal{H}_+^{(a,b)} \longrightarrow \mathcal{H}_-^{(a,b)}$$

is an isomorphism. (Here we could use the quotient $\mathcal{H}_\pm^{(b)}/\mathcal{H}_\pm^{(a)}$ instead of the Hilbert space difference.) We represent the situation schematically in Figure 1.

Figure 1

The spaces $\mathcal{H}_\pm^{(a)}$ fit together to form *smooth* finite dimensional vector bundles (of locally constant rank) over the open subset $U^{(a)} \subset Y$ in which $a \notin \operatorname{spec}(D^*D)$. Smoothness follows since $\mathcal{H}_\pm^{(a)}$ consists of C^∞ fields, which transform smoothly under diffeomorphisms of X. Therefore, the bundles $\mathcal{H}_\pm^{(a)}$ can be smoothly patched together using the patching in the fibration of manifolds $Z \longrightarrow Y$. Since the spectrum of an elliptic operator is discrete, it follows that $\{U^{(a)}\}$ forms an open cover of Y. Define a line bundle $\mathcal{L}^{(a)} \longrightarrow U^{(a)}$ by

$$\mathcal{L}^{(a)} = \left(\det \mathcal{H}_+^{(a)}\right)^* \otimes \left(\det \mathcal{H}_-^{(a)}\right). \tag{1.14}$$

The sequence (1.13) implies that for each $y \in Y$ there is a canonical isomorphism

$$\mathcal{L}_y^{(a)} \cong (\det \ker D_y)^* \otimes \left(\det \ker D_y^*\right).$$

We emphasize that the expression on the right hand side does not define a line bundle, even locally, because of the possibly jumping dimension of the kernels. However, the line bundle $\mathcal{L}^{(a)}$ is well-defined and smooth since $\mathcal{H}_\pm^{(a)}$ are smooth bundles over $U^{(a)}$. Now we must see how to patch together $\mathcal{L}^{(a)}$ and $\mathcal{L}^{(b)}$ over $U^{(a)} \cap U^{(b)}$. On that intersection

$$\mathcal{L}^{(b)} \cong \mathcal{L}^{(a)} \otimes \mathcal{L}^{(a,b)} \tag{1.15}$$

where

$$\mathcal{L}^{(a,b)} = \left(\det \mathcal{H}_+^{(a,b)}\right)^* \otimes \left(\det \mathcal{H}_-^{(a,b)}\right).$$

Since $D^{(a,b)}\colon \mathcal{H}_+^{(a,b)} \to \mathcal{H}_-^{(a,b)}$ is an isomorphism, it induces an isomorphism

$$\det D^{(a,b)}\colon\ \det \mathcal{H}_+^{(a,b)} \to \det \mathcal{H}_-^{(a,b)},$$

which we regard as a *nonzero* section of $\mathcal{L}^{(a,b)}$. This induces a canonical smooth isomorphism

$$\begin{aligned} \mathcal{L}^{(a)} &\longrightarrow \mathcal{L}^{(a)} \otimes \mathcal{L}^{(a,b)} = \mathcal{L}^{(b)} \\ s &\longmapsto s \otimes \det D^{(a,b)} \end{aligned} \tag{1.16}$$

over $U^{(a)} \cap U^{(b)}$. Finally, a differentiable line bundle $\mathcal{L} \to Y$ is defined by patching the $\mathcal{L}^{(a)}$ using the isomorphisms (1.16).

Over components of Y where the numerical index of D is zero, we have $\dim \mathcal{H}_+^{(a)} = \dim \mathcal{H}_-^{(a)}$. In this case each $\mathcal{L}^{(a)}$ has a canonical section $\det D^{(a)}\colon \det \mathcal{H}_+^{(a)} \to \det \mathcal{H}_-^{(a)}$, and the multiplicative property of determinants shows that $\det D^{(a)}$ and $\det D^{(b)}$ correspond under the isomorphism (1.16). Therefore, a global section $\det D$ of $\mathcal{L} \to Y$ is defined. This section is nonzero exactly where D is invertible. Over components where the index is nonzero, $\det D$ is (by fiat) identically zero.

We now proceed to describe the Quillen metric on $\mathcal{L}$. Fix $a > 0$. Then the subbundles $\mathcal{H}_\pm^{(a)}$ inherit metrics from $\mathcal{H}_\pm$. By linear algebra, metrics are induced on determinants, duals, and tensor products, so that $\mathcal{L}^{(a)}$ inherits a natural metric $g^{(a)}(\cdot,\cdot)$. If $b > a$ then under the isomorphism (1.16) we have two metrics on $\mathcal{L}^{(b)}$, and their ratio is a real number $\|\det D^{(a,b)}\|^2$. Let $\psi_1, \dots, \psi_N$ be a basis of $\mathcal{H}_+^{(a)}$ consisting of eigenfunctions, and $\psi_1^*, \dots, \psi_N^*$ the dual basis; then

$$\det D^{(a,b)} = (\psi_1^* \wedge \dots \wedge \psi_N^*) \otimes (D\psi_1 \wedge \dots \wedge D\psi_N). \tag{1.17}$$

Thus

$$\begin{aligned} \|\det D^{(a,b)}\|^2 &= \prod_{i=1}^N \|\psi_i^*\|^2 \|D\psi_i\|^2 \\ &= \prod_{i=1}^N \|\psi_i^*\|^2 (D^*D\psi_i, \psi_i) \\ &= \prod_{a<\lambda<b} \lambda. \end{aligned}$$

In other words, on $U^{(a)} \cap U^{(b)}$

$$g^{(b)} = g^{(a)} \left(\prod_{a<\lambda<b} \lambda \right). \tag{1.18}$$

To correct for this discrepancy, set

$$\bar{g}^{(a)} = g^{(a)} \cdot \det \left(D^*D\big|_{\lambda>a} \right), \tag{1.19}$$

where we define

$$\det \left(D^*D\big|_{\lambda>a} \right) = \prod_{\lambda>a} \lambda$$

using ζ-functions [RS1].

Recall this procedure. Let

$$\zeta^{(a)}(s) = \sum_{\lambda>a} \frac{1}{\lambda^s} = \operatorname{Tr} \left(\left(D^*D\big|_{\lambda>a} \right)^{-s} \right). \tag{1.20}$$

Then $\zeta^{(a)}(s)$ is holomorphic for $\operatorname{Re} s > n/2$ and has a meromorphic continuation to $\mathbb{C}$ which is holomorphic at $s = 0$ [Se]. The regularized determinant is defined to be

$$\det \left(D^*D\big|_{\lambda>a} \right) = \exp \left(-\zeta^{(a)\prime}(0) \right).$$

The crucial property of this regularization scheme is that it behaves properly with respect to a finite number of eigenvalues:

$$\det \left(D^*D\big|_{\lambda>a} \right) = \det \left(D^*D\big|_{\lambda>b} \right) \left(\prod_{a<\lambda<b} \lambda \right) \tag{1.21}$$

on the intersection $U^{(a)} \cap U^{(b)}$. (In fact, any regularization scheme with this property is adequate.) Equation (1.21) ensures that $\bar{g}^{(a)}$ and $\bar{g}^{(b)}$ agree on the overlap, by (1.18). Thus the $\bar{g}^{(a)}$ patch together to a Hermitian metric $g^{(\mathcal{L})}$ on $\mathcal{L}$, the *Quillen metric*.

Since $\mathcal{H}_\pm^{(a)}$ consists of smooth fields, the "connections" $\tilde{\nabla}^{(\mathcal{H}_\pm)}$ project to honest connections on the smooth bundles $\mathcal{H}_\pm^{(a)} \to U^{(a)}$. (Recall that $\tilde{\nabla}^{(\mathcal{H}_\pm)}$ are defined by pointwise differentiation (1.9).) Furthermore, these projected connections are unitary for the restricted metrics. Connections are induced on determinants (by taking a trace), dual bundles, and tensor products, so that $\mathcal{L}^{(a)}$ inherits a natural connection $\tilde{\nabla}^{(a)}$ which is unitary for the metric $g^{(a)}$. We have two connections on $\mathcal{L}^{(b)}$ over the intersection $U^{(a)} \cap U^{(b)}$, and these differ by

$$\begin{aligned}
\tilde{\nabla} \left(\det D^{(a,b)} \right) &= \tilde{\nabla} \big\{ (\psi_1^* \wedge \cdots \wedge \psi_N^*) \otimes (D\psi_1 \wedge \cdots \wedge D\psi_N) \big\} \\
&= \sum_{i=1}^N \big(\tilde{\nabla}\psi_i^*, \psi_i \big) \det D^{(a,b)} \; + \; \sum_{i=1}^N \big((D\psi_i)^*, \tilde{\nabla}(D\psi_i) \big) \det D^{(a,b)} \\
&= \sum_{i=1}^N \Big\{ \big(\tilde{\nabla}\psi_i^*, \psi_i \big) + \big(\psi_i^*, \tilde{\nabla}\psi_i \big) + \big((D\psi_i)^*, (\tilde{\nabla}D\, D^{-1}) D\psi_i \big) \Big\} \det D^{(a,b)} \\
&= \operatorname{Tr} \left(\tilde{\nabla}D\, D^{-1}\big|_{a<\lambda<b} \right) \det D^{(a,b)}.
\end{aligned}$$

In other words,

$$\tilde\nabla^{(b)} = \tilde\nabla^{(a)} + \operatorname{Tr}\left(\tilde\nabla D\, D^{-1}\big|_{a<\lambda<b}\right). \tag{1.22}$$

By analogy with the Quillen metric, set

$$\bar\nabla^{(a)} = \tilde\nabla^{(a)} + \operatorname{Tr}\left(\tilde\nabla D\, D^{-1}\big|_{\lambda>a}\right), \tag{1.23}$$

where the trace of the infinite operator must be regularized.

Again we use a ζ-function to define this trace. Let

$$\omega^{(a)}(s) = \operatorname{Tr}\left((DD^*)^{-s}\, \tilde\nabla D\, D^{-1}\big|_{\lambda>a}\right). \tag{1.24}$$

The analytic properties of $\omega^{(a)}(s)$ are somewhat more complicated than those of $\zeta^{(a)}(s)$.

Proposition 1.25. *The 1-form $\omega^{(a)}(s)$ is holomorphic for $\operatorname{Re} s > n/2$ and has a meromorphic continuation to $\mathbb{C}$. There is a simple pole at $s=0$. The real part of the residue at $s=0$ is $-\frac{1}{2}d\big(\zeta^{(a)}(0)\big)$.*

Proof. Omitting '$\lambda > a$' from the notation for convenience, set

$$\tau_u^{(a)}(s) = \operatorname{Tr}\left((DD^* + u\tilde\nabla D\, D^*)^{-s}\right).$$

Since the operator in parentheses is elliptic for small u, the function $\tau_u^{(a)}$ has a well-defined meromorphic continuation to $\mathbb{C}$ which is regular at $s=0$. By [APSIII,Proposition 2.9] we can differentiate in u to obtain

$$\begin{aligned} \frac{d}{du}\Big|_{u=0} \tau_0^{(a)}(s) &= -s\operatorname{Tr}\left((DD^*)^{-(s+1)}\tilde\nabla D\, D^*\right) \\ &= -s\,\omega^{(a)}(s). \end{aligned}$$

Thus $\omega^{(a)}(s)$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s=0$. Then differentiating (1.20), we obtain

$$\begin{aligned} d\big(\zeta^{(a)}(s)\big) &= d\ \operatorname{Tr}\big((DD^*)^{-s}\big) \\ &= -s\ \operatorname{Tr}\left(\big\{(DD^*)^{-(s+1)}\big\}\big\{\tilde\nabla D\, D^* + D\,\tilde\nabla D^*\big\}\right) \\ &= -s\ \operatorname{Tr}\left(\big\{(DD^*)^{-s}\big\}\big\{\tilde\nabla D\, D^{-1} + (D^*)^{-1}\,\tilde\nabla D^*\big\}\right) \\ &= -2s\ \operatorname{Re}\omega^{(a)}(s), \end{aligned} \tag{1.26}$$

which proves the final assertion.

Equation (1.26) shows that $\operatorname{Re} s\,\omega^{(a)}(s)$ is holomorphic at $s=0$, and

$$\operatorname{Re}\left(s\,\omega^{(a)}(s)\right)'(0) = -\frac{1}{2}d\left(\zeta^{(a)\prime}(0)\right). \tag{1.27}$$

Define

$$\mathrm{Tr}\left(\tilde{\nabla} D\, D^{-1}\big|_{\lambda>a}\right) = \left(s\,\omega^{(a)}(s)\right)'(0). \tag{1.28}$$

This is the finite part of $\omega^{(a)}(s)$ at $s=0$. Again the ζ-function behaves well with respect to a finite number of eigenvalues: For $a<b$

$$\mathrm{Tr}\left(\tilde{\nabla} D\, D^{-1}\big|_{\lambda>a}\right) = \mathrm{Tr}\left(\tilde{\nabla} D\, D^{-1}\big|_{\lambda>b}\right) + \mathrm{Tr}\left(\tilde{\nabla} D\, D^{-1}\big|_{a<\lambda<b}\right). \tag{1.29}$$

The crucial point is that the pole in $\omega^{(a)}(s)$ is independent of a. Equation (1.29) ensures that $\bar{\nabla}^{(a)}$ and $\bar{\nabla}^{(b)}$ agree on the overlap $U^{(a)}\cap U^{(b)}$ (c.f. (1.22)), and so patch together to a connection $\nabla^{(\mathcal{L})}$ on $\mathcal{L}$. Furthermore, equation (1.27), which governs the real part of the correction term, guarantees that $\nabla^{(\mathcal{L})}$ is unitary for the Quillen metric.

This completes the constructions proving Theorem 1.12. The functorial reader can formulate and prove the naturality of $\mathcal{L}, g^{(\mathcal{L})}, \nabla^{(\mathcal{L})}$ under mappings of geometric families, although this should be clear.

The connection on $\mathcal{L}$ is closely related to the "Levi-Civita superconnection" constructed by Bismut in his heat equation proof of the Atiyah-Singer index theorem for families of Dirac operators [B]. Therefore, it is not surprising that the curvature of $\mathcal{L}$ is the 2-form in the curvature of Bismut's superconnection.

Theorem 1.30 [BF2]**.** *The curvature of the determinant line bundle $\mathcal{L}\to Y$ is the 2-form component of*

$$\Omega^{(\mathcal{L})} = 2\pi i \int_{Z/Y} \hat{A}(\Omega^{(Z/Y)})\,\mathrm{ch}(\dot{\rho}\Omega^{(Z/Y)})\,\mathrm{ch}(\Omega^{(E)}).$$

Here $\hat{A}$ and ch are the usual polynomials

$$\begin{aligned}\hat{A}(\Omega) &= \sqrt{\det\left(\frac{\Omega/4\pi}{\sinh\,\Omega/4\pi}\right)}\\ \mathrm{ch}(\Omega) &= \mathrm{Tr}\; e^{i\Omega/2\pi}\end{aligned}$$

The integrand is a differential form of mixed degree on Z which is integrated over the fibers of $Z\to Y$. The product of the first two terms in the integrand is the index polynomial for the appropriate Dirac-type operator. Thus if we consider a family of signature operators, this product is the Hirzebruch L polynomial; for the $\bar{\partial}$ complex it is the Todd polynomial.

The holonomy formula is more complicated to state. Let $\gamma\colon S^1\to Y$ be a loop. By pullback we obtain a geometric family of Dirac operators parametrized by S^1. Thus there is an $(n+1)$-manifold P fibered over S^1, a metric and spin structure along the fibers, etc. Introduce an arbitrary metric $g^{(S^1)}$ and endow S^1 with its bounding spin structure.[2] Then P acquires a metric and spin structure, and so a self-adjoint Dirac

[2]The bundle of spin frames is the nontrivial double cover of the circle. In [BF2] we use the other spin structure and correspondingly obtain a formula which differs from (1.31) by a sign. The two formulas are easily reconciled by the flat index theorem of Atiyah, Patodi, and Singer [APSIII].

operator A (coupled to $V_\rho \otimes E$). Since our constructions are independent of the metric on the base, we must scale away the choice of metric on the circle. Replace $g^{(S^1)}$ in the preceding by $g^{(S^1)}/\epsilon^2$ for a parameter ϵ, and let A_ϵ denote the Dirac operator for the scaled metric. Set

$$\begin{aligned} \eta_\epsilon &= \eta\text{-invariant of } A_\epsilon \\ h_\epsilon &= \dim \ker A_\epsilon \\ \xi_\epsilon &= \frac{1}{2}(\eta_\epsilon + h_\epsilon). \end{aligned}$$

The η-invariant is a spectral invariant defined by analytic continuation as the value of

$$\sum_{\lambda \in \operatorname{spec}(A_\epsilon)\setminus\{0\}} \frac{\operatorname{sgn} \lambda}{|\lambda|^s}$$

at $s = 0$ [APS]. An easy argument shows $\xi_\epsilon \pmod 1$ is continuous in ϵ.

Theorem 1.31 [W1],[BF2]**.** *The holonomy of $\nabla^{(\mathcal{L})}$ around γ is*

$$\operatorname{hol}_{\mathcal{L}}(\gamma) = \lim_{\epsilon \to 0} e^{-2\pi i \xi_\epsilon}.$$

§2 Holomorphic Families

For applications of determinants to complex geometry it is crucial to understand when the purely Riemannian considerations of §1 are compatible with holomorphic structures. The prototypical result in this direction is a characterization of Kähler metrics: A Hermitian metric on a complex manifold is Kähler if and only if its Levi-Civita connection and Hermitian connection coincide. Special cases of Theorem 2.1 have appeared in the recent literature [AW], [Bo], [BF1], [BJ], [D], [F1]. Theorem 2.1 is due to Bismut, Gillet, and Soulé, who go much further than we do here. They construct a canonical smooth isomorphism of the determinant line bundle with the Knudsen-Mumford [KM] determinant such that the canonical connection is compatible with the induced complex structure.[3]

Theorem 2.1 [BGS]. *Let $\pi\colon Z \xrightarrow{X} Y$ be a holomorphic fibration with smooth fibers. Suppose Z admits a closed (1,1)-form τ which restricts to a Kähler form on each fiber. Let $E \to Z$ be a holomorphic Hermitian bundle with its Hermitian connection. Then the determinant line bundle $\mathcal{L} \to Y$ of the relative $\bar\partial$ complex (coupled to E) admits a holomorphic structure. The canonical connection on $\mathcal{L}$ is the Hermitian connection for the Quillen metric. Finally, the section $\det \bar\partial_E$ of $\mathcal{L}$ is holomorphic.*

The geometric data (1.1) used to define the canonical connection on $\mathcal{L}$ comes from τ. The metric $g^{(Z/Y)}$ along the fibers is Kähler—the Kähler form is the restriction of τ to the fibers. The horizontal distribution on Z is the annihilator of $T(Z/Y)$ relative to τ. So $\xi \in TZ$ is horizontal if and only if $\tau(\xi,\zeta) = 0$ for all vertical vectors $\zeta \in T(Z/Y)$. The existence of τ implies that Z is Kähler, at least over small regions in Y. (If σ is a sufficiently large (local) Kähler form on Y, then $\pi^*\sigma + \tau$ is a Kähler form on Z.) Of course, Theorem 2.1 extends to other operators (e.g. Dirac operators) which are expressed in terms of the $\bar\partial$ complex. In this section we sketch a proof of Theorem 2.1. For notational simplicity we omit the bundle E from our treatment.

The hypothesis of Theorem 2.1 is satisfied in many interesting cases. For example, if Y is the Siegel upper half-plane, parametrizing marked abelian varieties, then the universal abelian variety Z carries a 2-form τ which is the curvature of a suitable metric on the Θ-line bundle. As another example consider a holomorphic fibration $\pi\colon Z \to Y$ with Kähler-Einstein metrics along the fibers. Suppose that the ratio of the Ricci curvature to the Kähler form in each fiber Z_g is a positive constant c. Let $\Omega^{(Z/Y)}$ be the curvature of the Hermitian connection on $T(Z/Y)$ for this family of Kähler-Einstein metrics. Then we choose $\tau = \dfrac{i}{2\pi c}\operatorname{Tr}\Omega^{(Z/Y)}$. This applies to the universal curve over Teichmüller space (or moduli space) with the family of hyperbolic metrics of constant curvature -1. In general we can also construct suitable closed (1,1)-forms τ from relative projective embeddings of $Z \to Y$. In this case, τ is the curvature of a relatively ample line bundle over Z.

The following lemma states the consequences of the equation $d\tau = 0$ on the local geometry of Z. Recall the tensor T of (1.10); it is the curvature of the horizontal distribution on Z.

Lemma 2.2. *Under the hypotheses of Theorem 2.1, the connection $\nabla^{(Z/Y)}$ is the Hermitian connection on $T(Z/Y)$, the tensor T is of type (1,1), and the 1-form γ in* (1.4) *vanishes.*

Proof. As remarked above, we can lift a Kähler metric on Y to obtain a Kähler metric on Z. Its Levi-Civita connection is the Hermitian connection on TZ. Since $T(Z/Y)$ is a holomorphic subbundle of TZ,

[3] I am grateful to Henri Gillet for explaining this work.

the projected connection $\nabla^{(Z/Y)}$ is the Hermitian connection. (Notice a difference with Lemma 1.3. There we lifted a metric on Y to construct a *Riemannian submersion*. The Kähler metric on Z does not lead to a Riemannian submersion in our present context, but the projected connection $\nabla^{(Z/Y)}$ is insensitive to the horizontal metric in any case.)

Suppose ξ_1, ξ_2 are lifts of horizontal vector fields on Y, and let ζ be any vertical vector field. Then $[\xi_i, \zeta]$ are vertical vector fields, and the 6-term formula for $d\tau$ yields $\tau([\xi_1, \xi_2], \zeta) = \zeta \cdot \tau(\xi_1, \xi_2)$. This implies that $[\xi_1, \xi_2]$ is horizontal if both ξ_i are of type (1,0) or (0,1). Hence T is of type (1,1).

Finally, the volume form $\mathbf{vol} = \tau^n$, where n is the complex dimension of the fibers. Hence $d\,\mathbf{vol} = 0$, and $\gamma = 0$ follows from (1.4).

Suppose momentarily that the cohomology H^i of the relative $\bar\partial$ complex

$$\Omega^{0,0}_{Z/Y} \xrightarrow{\bar\partial^{(Z/Y)}} \Omega^{0,1}_{Z/Y} \xrightarrow{\bar\partial^{(Z/Y)}} \cdots \xrightarrow{\bar\partial^{(Z/Y)}} \Omega^{0,n}_{Z/Y} \tag{2.3}$$

has (locally) constant rank. Then $H^i \to Y$ is a vector bundle, and it admits a natural holomorphic structure (without using the metric on Z). Working locally in Y, suppose $\alpha \in \Omega^{0,i}_{Z/Y}$ restricts to be holomorphic on each fiber. Let $\bar\xi$ be a (0,1)-vector field in Y. Lift to *any* (0,1)-vector field $\tilde{\bar\xi}$ along the fiber. Define $\bar\partial^{(H^i)}_{\bar\xi}\alpha = \bar\partial^{(Z)}_{\tilde{\bar\xi}}\alpha$ acting pointwise. Let $\bar\zeta$ be a vertical vector field of type (0,1). Then (setting $\bar\partial = \bar\partial^{(Z)}$)

$$\bar\partial_{\bar\zeta}\bar\partial_{\tilde{\bar\xi}}\alpha = \bar\partial_{\tilde{\bar\xi}}\bar\partial_{\bar\zeta}\alpha - \bar\partial_{[\tilde{\bar\xi}, \bar\zeta]}\alpha.$$

But $[\tilde{\bar\xi}, \bar\zeta]$ is vertical since $\tilde{\bar\xi}$ is the lift of a vector field from the base. Since α is holomorphic along the fiber, the right hand side vanishes. Furthermore, it is easily seen that the definition of $\bar\partial^{(H^i)}$ is independent of the lift. Hence $\bar\partial^{(H^i)}$ is well-defined. Its square is zero, since $\left(\bar\partial^{(Z)}\right)^2 = 0$, so it defines a holomorphic structure on $H^i \to Y$. Then the determinant of the cohomology

$$\bigotimes_i \left(\det H^i\right)^{(-1)^{i+1}} \tag{2.4}$$

also inherits a holomorphic structure.

In general, the cohomology does not have locally constant rank, and we use the Kähler metric to construct a holomorphic structure on the determinant line bundle. Denote the i^{th} term $\Omega^{0,i}_{Z/Y}$ of (2.3) by $\mathcal{H}_i \to Y$, and denote $\bar\partial^{(Z/Y)}$ by D. The $\mathcal{H}_i$ are *continuous* Hilbert bundles over Y. Eventually we work with *smooth* finite dimensional subbundles consisting of smooth forms, so we treat the $\mathcal{H}_i$ formally as if they were smooth and finite dimensional. Let $\tilde\nabla_i$ be the unitary connection on $\mathcal{H}_i$ defined in (1.9).

Proposition 2.5. *The (0,1) part of the connection $\tilde\nabla_i = \nabla_i$ defines a complex structure on $\mathcal{H}_i$. The relative $\bar\partial$ complex*

$$\mathcal{H}_0 \xrightarrow{D} \mathcal{H}_1 \xrightarrow{D} \cdots \xrightarrow{D} \mathcal{H}_n \tag{2.6}$$

varies holomorphically over the parameter space Y.

Proof. By Lemma 2.2 and (1.5) we have $\tilde{\nabla}^{(Z/Y)} = \nabla^{(Z/Y)}$, hence also $\tilde{\nabla}_i = \nabla_i$. Furthermore, $\nabla^{(Z/Y)}$ is the Hermitian connection, so its curvature $\Omega^{(Z/Y)}$ is of type (1,1). Since T is of type (1,1), it follows from Proposition 1.11 that $\Omega^{(\mathcal{H}_i)}$ is of type (1,1). Thus the (0,1) component $\bar{\nabla}_i$ defines a $\bar{\partial}$ operator on $\mathcal{H}_i$, since $\left(\bar{\nabla}_i\right)^2 = 0$.

To check that D varies holomorphically, we introduce local frames. Let σ be a Kähler from on Y such that $\pi^*\sigma + \tau$ is a Kähler form on Z. Fix ξ_α a local unitary basis of vector fields of type (1,0) on Y, and lift to horizontal vector fields ξ_α on Z. Note that ξ_α (on Z) are *not* unitary. Choose vertical vector fields ζ_i of type (1,0) which form a unitary basis on each fiber. Let θ^α, ϕ^i be the dual 1-forms. Then $\tau = \sum_i \phi^i \wedge \bar{\phi}^i$, and

$$d\left(\sum_i \phi^i \wedge \bar{\phi}^i\right) = 0. \tag{2.7}$$

As a preliminary step we show

$$\phi^k(\nabla_{\zeta_j}\xi_\alpha) = 0. \tag{2.8}$$

For this evaluate (2.7) on vectors $\zeta_j, \xi_\alpha, \bar{\zeta}_k$. Then since the Kähler connection $\nabla^{(Z)}$ is torsionfree, we obtain

$$\bar{\phi}^j(\nabla_{\xi_\alpha}\bar{\zeta}_k) + \phi^k(\nabla_{\xi_a}\zeta_j) - \phi^k(\nabla_{\zeta_j}\xi_\alpha) = 0.$$

The first two terms cancel since ∇ preserves the metric along the fibers:

$$\begin{aligned} 0 &= \xi_\alpha\langle \zeta_j, \bar{\xi}_k\rangle \\ &= \langle \nabla_{\xi_\alpha}\zeta_j, \bar{\zeta}_k\rangle + \langle \zeta_j, \nabla_{\xi_\alpha}\bar{\zeta}_k\rangle \\ &= \phi^k(\nabla_{\xi_\alpha}\zeta_j) + \bar{\phi}^j(\nabla_{\xi_\alpha}\bar{\zeta}_k). \end{aligned}$$

The desired equation (2.8) follows.

Now

$$\begin{aligned} D &= \bar{\phi}^j \wedge \nabla_{\bar{\zeta}_j}, \\ \overline{\nabla} &= \bar{\theta}^\alpha \otimes \nabla_{\bar{\xi}_\alpha}, \end{aligned} \tag{2.9}$$

where $\nabla = \nabla^{(Z/Y)}$ on the right hand side of (2.9), and repeated indices are summed. We compute

$$\begin{aligned} \overline{\nabla}D &= \bar{\theta}^\alpha \otimes \left(\bar{\phi}^j \wedge \nabla_{\bar{\xi}_\alpha}\nabla_{\bar{\zeta}_j} + \nabla_{\bar{\xi}_\alpha}\bar{\phi}^j \wedge \nabla_{\bar{\zeta}_j}\right), \\ D\overline{\nabla} &= \bar{\theta}^\alpha \otimes \left(\bar{\phi}^j \wedge \nabla_{\bar{\zeta}_j}\nabla_{\bar{\xi}_\alpha}\right). \end{aligned} \tag{2.10}$$

Since the curvature $\Omega^{(Z/Y)}$ has type (1,1),

$$\nabla_{\bar{\xi}_\alpha}\nabla_{\bar{\zeta}_j} - \nabla_{\bar{\zeta}_j}\nabla_{\bar{\xi}_\alpha} = \nabla_{[\bar{\xi}_\alpha, \bar{\zeta}_j]}. \tag{2.11}$$

Now $[\bar\xi_\alpha, \bar\zeta_j]$ is vertical, so using (2.8) and the fact that ∇ is torsionfree,

$$[\bar\xi_\alpha, \bar\zeta_j] = \nabla_{\bar\xi_\alpha}\bar\zeta_j = \bar\phi^k(\nabla_{\bar\xi_\alpha}\bar\zeta_j)\bar\zeta_k. \tag{2.12}$$

Combining (2.10)–(2.12), and writing $\nabla_{\bar\xi_\alpha}\bar\phi^j = (\nabla_{\bar\xi_\alpha}\bar\phi^j)(\bar\zeta_k)\bar\phi^k$, we obtain

$$\overline{\nabla}D - D\overline{\nabla} = \left\{(\nabla_{\bar\xi_\alpha}\bar\phi^j)(\bar\zeta_k) + \bar\phi^j(\nabla_{\bar\xi_\alpha}\bar\zeta_k)\right\}\bar\theta^\alpha \otimes (\bar\phi^k \wedge \nabla_{\bar\zeta_j}).$$

The expression in braces vanishes, and therefore $[\overline{\nabla}, D] = 0$.

Next, we construct the holomorphic determinant line bundle $\mathcal{L}_{\text{hol}} \to Y$ of the complex (2.6). Each fiber of $\mathcal{L}_{\text{hol}}$ is isomorphic to (2.4), but as in §1 we must use a patching argument to construct the bundle. To grasp the ideas it is best to work first in a general setting, as in Quillen [Q]. Let $\text{Fred}(\mathbf{H})$ be the space of *Fredholm complexes*

$$0 \longrightarrow H_0 \xrightarrow{T_1} H_1 \xrightarrow{T_2} \cdots \xrightarrow{T_n} H_n \longrightarrow 0. \tag{2.13}$$

Each T_i has closed range, $\operatorname{im} T_i \subset \ker T_{i+1}$, and the cohomology $\ker T_{i+1}/\operatorname{im} T_i$ is finite dimensional. Over open sets $\mathcal{U} \subset \text{Fred}(\mathbf{H})$ there exist finite dimensional subcomplexes

$$0 \longrightarrow V_0 \xrightarrow{T_1} V_1 \xrightarrow{T_2} \cdots \xrightarrow{T_n} V_n \longrightarrow 0 \tag{2.14}$$

such that $V_i \subset \mathcal{U} \times H_i$ is a holomorphic subbundle, and the inclusion of (2.14) into (2.13) induces an isomorphism on cohomology. To construct a local model (2.14), choose a finite dimensional subspace $V_n \subset H_n$ such that V_n maps onto $\operatorname{coker} T_n(y)$ at some fixed $y \in \text{Fred}(\mathbf{H})$; then V_n surjects onto $\operatorname{coker} T_n$ in a neighborhood of y. For each $i = 0, 1, \dots, n-1$ choose an (infinite dimensional) subspace $U_i \subset H_i$ complementary to $\ker T_{i+1}(y)$; then $U_i \oplus \ker T_i = H_i$ in a neighborhood of y. Also, choose a finite dimensional holomorphic subbundle $W_i \subset \ker T_{i+1}$ which surjects onto the cohomology. This is possible since $\ker T_{i+1}$ is holomorphic and the dimension of the cohomology is locally bounded. Then define V_i (using downward induction) via the formula

$$V_i = W_i + \left(T_{i+1}^{-1}(V_{i+1}) \cap U_i\right). \tag{2.15}$$

To each local model (2.14) corresponds a holomorphic line bundle

$$\mathcal{L}_V = \bigotimes_{i=1}^{n} (\det V_i)^{(-1)^{i+1}} \tag{2.16}$$

over $\mathcal{U}$. If the Fredholm complex (2.13) has index zero, then $\det T$ is a holomorphic section of $\mathcal{L}_{\text{hol}}$. To patch with another local model $\{V_i'\}$, we may as well assume $V_i \subset V_i'$. Then in the commutative diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & 0 & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & V_0 & \longrightarrow & V_1 & \longrightarrow & \cdots & \longrightarrow & V_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & V_0' & \longrightarrow & V_1' & \longrightarrow & \cdots & \longrightarrow & V_n' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & V_0'/V_0 & \longrightarrow & V_1'/V_1 & \longrightarrow & \cdots & \longrightarrow & V_n'/V_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & & 0 & & & & 0 & &
\end{array}$$

the columns and the bottom row are exact. Hence there is a canonical isomorphism

$$\mathcal{L}_V \cong \mathcal{L}_V \otimes \mathcal{L}_{V'/V} = \mathcal{L}_{V'} \tag{2.17}$$

via the nonzero holomorphic section $\det T$ of $\mathcal{L}_{V'/V}$ determined by the action of T on the quotient complex. This patching produces a holomorphic determinant line bundle $\mathcal{L}_{\text{hol}} \to \text{Fred}(\mathbf{H})$ and a canonical holomorphic section over the index zero Fredholm complexes.

For the relative $\bar{\partial}$ complex (2.6), the Hilbert spaces change as we vary the parameter. But since the complexes vary holomorphically, by Proposition 2.5, the preceding constructions extend easily to this case. Therefore, we obtain a holomorphic determinant bundle $\mathcal{L}_{\text{hol}} \to Y$ and a holomorphic section $\det D$ (if $\operatorname{index} D$ vanishes).

Next[4] we define a smooth map $j\colon \mathcal{L}_{\text{hol}} \to \mathcal{L}$, where $\mathcal{L}$ is the determinant line bundle (constructed in §1) for the collapsed complex

$$D + D^*\colon \bigoplus_{i \text{ even}} \mathcal{H}_i \longrightarrow \bigoplus_{i \text{ odd}} \mathcal{H}_i. \tag{2.18}$$

As a preliminary step, consider an exact sequence of inner product spaces

$$0 \to E_0 \xrightarrow{T} E_1 \xrightarrow{T} \cdots \xrightarrow{T} E_n \to 0. \tag{2.19}$$

The determinant

$$\det T \in \bigotimes_{i=0}^{n} (\det E_i)^{(-1)^{i+1}} \tag{2.20}$$

[4]The relationship between $\mathcal{L}_{\text{hol}}$ and $\mathcal{L}$ was elucidated in Donaldson [D], and we follow his arguments closely.

is nonzero. Form the linear transformation

$$T + T^* \colon \bigoplus_{i \text{ even}} E_i \longrightarrow \bigoplus_{i \text{ odd}} E_i. \tag{2.21}$$

It is invertible, and its determinant

$$\det(T + T^*) \in \bigotimes_{i \text{ even}} (\det E_i)^{-1} \bigotimes_{i \text{ odd}} (\det E_i) \tag{2.22}$$

is nonzero. However, the determinants do *not* correspond under the natural isomorphism of the complex lines in (2.20) and (2.22). Write

$$E_i = E_i' \oplus E_i'', \qquad E_i' = \operatorname{im} T, \quad E_i'' = \ker T^*. \tag{2.23}$$

Then a short calculation shows[5]

$$\det(T + T^*) = (\det T)\biggl(\prod_{i \text{ even}} \det\bigl(T\,T^*\bigr|_{E_i'}\bigr)\biggr). \tag{2.24}$$

Fix $a > 0$. Recall the local model (1.14) for the determinant line bundle $\mathcal{L}$. Choose a local model (2.14) for $\mathcal{L}_{\mathrm{hol}}$ as follows. Fix $y \in Y$ such that a is not an eigenvalue of the Laplacian at y, and set $V_i = \mathcal{H}_i^{(a)}$ at the point y. (As before, $\mathcal{H}_i^{(a)}$ is the sum of the eigenspaces for eigenvalues $< a$ of the i^{th} Laplacian $\triangle_i$.) Extend to a local model in a neighborhood of y, as described above, and denote the determinant of the resulting complex by $\mathcal{L}_{\mathrm{hol}}^{(a)}$. Then, possibly in a smaller neighborhood of y, the projections $P^{(a)}$ onto $\mathcal{H}_i^{(a)}$ define an isomorphism of complexes

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & V_0 & \xrightarrow{D} & V_1 & \xrightarrow{D} & \cdots & \xrightarrow{D} & V_n & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle P^{(a)}} & & \big\downarrow{\scriptstyle P^{(a)}} & & & & \big\downarrow{\scriptstyle P^{(a)}} & & \\ 0 & \longrightarrow & \mathcal{H}_0^{(a)} & \xrightarrow{D} & \mathcal{H}_1^{(a)} & \xrightarrow{D} & \cdots & \xrightarrow{D} & \mathcal{H}_n^{(a)} & \longrightarrow & 0 \end{array} \tag{2.25}$$

There is an induced isomorphism $\det P^{(a)}$ on the determinants. We introduce a correction factor to pass to the determinant of the collapsed complex (2.18). The Hodge decomposition of $\mathcal{H}_i$ is

$$\mathcal{H}_i = H_i \oplus \mathcal{H}_i' \oplus \mathcal{H}_i'', \qquad H_i = \ker \triangle_i, \quad \mathcal{H}_i' = \operatorname{im} D, \quad \mathcal{H}_i'' = \operatorname{im} D^*. \tag{2.26}$$

The nonzero eigenvalues of the Laplacian $\triangle_i$ split accordingly into $\{\lambda_i'\}, \{\lambda_i''\}$. Define the regularized determinant

$$\mu^{(a)} = \biggl(\prod_{\substack{i \text{ even} \\ \lambda_i' > a}} \lambda_i'\biggr)^{-1} \tag{2.27}$$

[5]Observe that $\det(T + T^*) = \det T$ for complexes of length 2. So this extra factor does not enter for Riemann surfaces.

using ζ-functions. Then by (2.24) the maps

$$\mu^{(a)}\cdot\det P^{(a)}\colon \mathcal{L}^{(a)}_{\text{hol}}\longrightarrow\mathcal{L}^{(a)} \tag{2.28}$$

patch together into a smooth isomorphism

$$j\colon \mathcal{L}_{\text{hol}}\longrightarrow\mathcal{L}. \tag{2.29}$$

We use j to pull back the metric and connection from $\mathcal{L}$.

It remains to prove that $j^*\nabla^{(\mathcal{L})}$ is the Hermitian connection for $j^*g^{(\mathcal{L})}$. Since $j^*\nabla^{(\mathcal{L})}$ is unitary, we need only check that it is compatible with the complex structure on $\mathcal{L}_{\text{hol}}$. We work in the local model (2.25) for j. Recall that on $\mathcal{L}^{(a)}$ the connection $\nabla^{(\mathcal{L})}$ is a sum of two terms (1.23). The first is the determinant of the projected connections $\nabla^{(a)}_i = P^{(a)}\nabla_i$ on $\mathcal{H}^{(a)}_i\subset\mathcal{H}_i$. We claim that each $P^{(a)*}\nabla^{(a)}_i$ is compatible with the complex structure on V_i. For if s is a local section of V_i, then since (2.25) is a map of complexes, and $\overline{\nabla}$ commutes with D (Proposition 2.5), we have (omitting the index i)

$$\begin{aligned} P^{(a)}D\left(P^{(a)*}\overline{\nabla}^{(a)}\right)s &= DP^{(a)}\overline{\nabla}P^{(a)}s\\ &= P^{(a)}D\overline{\nabla}P^{(a)}s\\ &= P^{(a)}\overline{\nabla}DP^{(a)}s\\ &= P^{(a)}\left(P^{(a)*}\overline{\nabla}^{(a)}\right)Ds.\end{aligned} \tag{2.30}$$

Hence $j^*\overline{\nabla}^{(a)}$ commutes with D, as claimed.

The second contribution to the connection on $\mathcal{L}^{(a)}$ is the regularized 1-form (1.28). Relative to the Hodge decomposition (2.26) we write (c.f. (1.24))

$$\omega^{(a)}(s)=\sum_{i\text{ even}}\operatorname{Tr}\left(\triangle_i^{-s}\,\nabla D\,D^{-1}\big|_{\lambda'>a}\right)+\sum_{i\text{ even}}\operatorname{Tr}\left(\triangle_i^{-s}\,\nabla D^*(D^*)^{-1}\big|_{\lambda''>a}\right). \tag{2.31}$$

Since D varies holomorphically (Proposition 2.5), equation (2.31) is the decomposition of $\omega^{(a)}(s)$ into forms of type (1,0) and (0,1). The compatibility of (1.28) with the holomorphic structure amounts to the assertion

$$\bar\partial\left(\log\mu^{(a)}\right)+(0,1)\text{ part of }\left(s\,\omega^{(a)}(s)\right)'(0)=0. \tag{2.32}$$

We verify (2.32) by a calculation similar to (1.26). Set

$$\zeta^{(a)}_i(s)=\operatorname{Tr}\left(\triangle_i^{-s}\big|_{\lambda>a}\right). \tag{2.33}$$

(Henceforth we omit reference to 'a' in the notation.) Choose numbers n_i satisfying

$$n_{i-1}+n_i=\begin{cases}1, & i\text{ even},\\ 0, & i\text{ odd}.\end{cases}$$

Then $\log \mu = \sum_{i=0}^{n} n_i\, \zeta_i'(0)$. Since $\overline{\nabla} D = 0$, differentiating (2.33) we obtain

$$\bar\partial \zeta_i(s) = -s \ \mathrm{Tr}\left(\triangle_i^{-s}\left\{\overline{\nabla} D^*(D^*)^{-1} + (D^*)^{-1}\overline{\nabla} D^*\right\}\right).$$

Therefore,

$$\begin{aligned}\bar\partial\left(\sum_{i=0}^{n} n_i\, \zeta_i(s)\right) &= -s\sum_{i=0}^{n} \mathrm{Tr}\left(\triangle_i^{-s}\left\{n_i\, \overline{\nabla} D^*(D^*)^{-1} + n_{i-1}\overline{\nabla} D^*(D^*)^{-1}\right\}\right)\\ &= -s\sum_{i \text{ even}} \mathrm{Tr}\left(\triangle_i^{-s}\, \overline{\nabla} D^*(D^*)^{-1}\right)\\ &= -s\,\omega(s)_{(0,1)}.\end{aligned} \tag{2.34}$$

The desired equation (2.32) follows by differentiating (2.34) and setting $s = 0$.

This completes the proof of Theorem 2.1. As a final note, we display the metric $j^* g^{(\mathcal{L})}$ on $\mathcal{L}_{\mathrm{hol}}$ explicitly. At a fixed point in the parameter space we take the sequence of cohomology groups as a model for $\mathcal{L}_{\mathrm{hol}}$. Its determinant (2.4) carries the L^2 metric, and the metric $j^* g^{(\mathcal{L})}$ is this L^2 metric multiplied by the *analytic torsion* [RS1]

$$\left(\prod_{i \text{ even}} \lambda_i'\right)^{-1}\left(\prod_{i \text{ even}} \lambda_i''\right).$$

§3 A Square Root

There are several possible variations on the basic determinant line bundle construction of §1. In this section we consider families of Dirac operators on a spin manifold X of dimension $8k+2$. Riemann surfaces, 10 dimensional manifolds, and 26 dimensional manifolds all fit into this framework. These arise in string theory and in its low energy field theory limits. In $8k+2$ dimensions the determinant line bundle $\mathcal{L}$ of the complex chiral Dirac operator admits a canonical square root $\mathcal{L}^{1/2}$, and $\sqrt{\det D}$ makes sense as a section of $\mathcal{L}^{1/2}$. (The Dirac operator has numerical index zero in these dimensions.) This construction extends to Dirac operators coupled to *real* vector bundles (e.g., the real Rarita-Schwinger operator). The relevance of this square root in physics is found in a discrepancy between Clifford algebras in Euclidean space and Lorentz space. Relative to the Euclidean signature the even two dimensional Clifford algebra is $\mathbb{C}$, whereas for the Lorentz signature it is $\mathbb{R}\oplus\mathbb{R}$. By periodicity, in dimension $8k+2$ these are replaced by appropriate matrix algebras. Hence in Lorentz geometry there is a real chiral Dirac operator—there exist Majorana-Weyl spinors. These are the fermions of the physical theories. But when continued to imaginary time, i.e., to Riemannian geometry, there are only *complex* chiral spinors. We view the square root of the complex Dirac determinant as the Euclidean analog of the real Lorentz determinant. It accounts for a mysterious factor of 2 that crops up in the physics literature [AgW], [ASZ], [W1]. This square root plays a significant role in §4 and §5. We remark that the complex Pfaffian also enters the work of Pressley and Segal [PS], who use it to construct the spin representation.

Topologically, the existence of this square root can be attributed to Rohlin's theorem. One generalization of Rohlin's theorem asserts that the Dirac operator on a closed spin $(8k+4)$-manifold has even index. This follows from the fact that the even Clifford algebra in $8k+4$ dimensions is a direct sum of two quaternion algebras. The chiral Dirac operator is quaternionic, its kernel and cokernel are quaternionic vector spaces, and so have even complex dimension. Suppose $\pi\colon Z\to Y$ is a family of $8k+2$ dimensional spin manifolds, and $\mathcal{L}\to Y$ is the determinant line bundle for the family of complex Dirac operators. The rational Chern class of $\mathcal{L}$ is evaluated on closed 2-manifolds $\Sigma\to Y$. The fibration π pulls back to $Q\to Y$, where Q is a closed $(8k+4)$-manifold, and $c_1(\mathcal{L})[\Sigma]=\operatorname{index} D_Q$ is even (by Rohlin's theorem). To detect the integral Chern class we must also use $\mathbb{Z}/\ell$-surfaces $\bar\Sigma$ as in [F2] (c.f. §5). Then if $\bar\Sigma\to Y$ pulls back π to a fibration $\bar Q\to\bar\Sigma$, we see that $c_1(\mathcal{L})[\bar\Sigma]=\operatorname{index} D_{\bar Q} \pmod{\ell}$ is the mod ℓ index of the Atiyah-Patodi-Singer boundary value problem on an $8k+4$ dimensional manifold with boundary [FM]. Again this is even because spinors are quaternionic. Thus $c_1(\mathcal{L})$ is divisible by 2, and $\mathcal{L}$ admits a topological square root. The import of Theorem 3.1 is the existence of a canonical square root of the section $\det D$, which is not guaranteed by these topological considerations.

Theorem 3.1. *In the situation of* (1.1) *assume that X is an $(8k+2)$-dimensional spin manifold, ρ is a real representation, and $E\to Z$ is a real vector bundle. Then there is a canonical complex line bundle $\mathcal{L}^{1/2}\to Y$ together with a canonical isomorphism $\mathcal{L}^{1/2}\otimes\mathcal{L}^{1/2}\cong\mathcal{L}$, where $\mathcal{L}$ is the determinant line bundle constructed in Theorem 1.12. If* $\operatorname{index} D_E=0$ *there is a section* $\operatorname{Pfaff}(D)$ *of $\mathcal{L}^{1/2}$ with* $\operatorname{Pfaff}(D)^{\otimes 2}=\det D$. *The square root $\mathcal{L}^{1/2}$ inherits a metric and connection. Its curvature is*

$$\Omega^{(\mathcal{L}^{1/2})}=\frac{1}{2}\cdot 2\pi i\int_{Z/Y}\hat A(\Omega^{(Z/Y)})\,\operatorname{ch}(\dot\rho\Omega^{(Z/Y)})\,\operatorname{ch}(\Omega^{(E)}). \tag{3.2}$$

The holonomy around a loop $\gamma\colon S^1 \to Y$ is (c.f. Theorem 1.31 for notation)

$$\mathrm{hol}_{\mathcal{L}^{1/2}}(\gamma) = \lim_{\epsilon\to 0} e^{-\pi i \xi_\epsilon}. \tag{3.3}$$

For holomorphic families (as in Theorem 2.1) the bundle $\mathcal{L}^{1/2} \to Y$ and section Pfaff(D) *are holomorphic.*

The limit in (3.3) makes sense as the self-adjoint Dirac operator in $8k+3$ dimensions is quaternionic, and so $\xi(\mathrm{mod}\ 2)$ is continuous as a function of the operator. For simplicity we will only consider the pure Dirac operator ($\rho = 1$, E omitted) for the rest of this section.

The construction of $\mathcal{L}^{1/2}$ relies on the *complex Pfaffian*. Let V be a finite dimensional complex vector space and $T\colon V \to V^*$ a skew-symmetric linear map. Then T can be identified with an element $\omega_T \in \bigwedge^2 V^*$. If T is nonsingular, then $\dim V = 2r$ is even, and

$$\mathrm{Pfaff}(T) = \frac{1}{r!}\,\omega_T^r \in \bigwedge\nolimits^{2r} V^* = \det V^*. \tag{3.4}$$

On the other hand, regarding $T \in V^* \otimes V^*$ we have

$$\det(T) \in (\det V^*) \otimes (\det V^*)\,. \tag{3.5}$$

An easy calculation using a normal form for T shows

$$\mathrm{Pfaff}(T)^{\otimes 2} = \det(T) \tag{3.6}$$

in $(\det V^*) \otimes (\det V^*) = (\det V^*)^{\otimes 2}$. The Pfaffian obeys the multiplicative law

$$\mathrm{Pfaff}(T_1 \oplus T_2) = \mathrm{Pfaff}(T_1) \otimes \mathrm{Pfaff}(T_2). \tag{3.7}$$

The real Dirac operator in $n = 8k+2$ dimensions is an infinite dimensional example of a skew-symmetric complex map [H,§4.2] (c.f. [AS1]). Let $\mathrm{Spin}(X) \to X$ denote the spin bundle of frames on X, and set $W = \mathrm{Spin}(X) \times_{\mathrm{Spin}(n)} C_n$, where $\mathrm{Spin}(n)$ acts on the real Clifford algebra C_n by left multiplication. Then $W = W_+ \oplus W_-$ according to the splitting of C_n into even and odd elements, and the space $\mathcal{H}$ of sections of W is a $\mathbb{Z}/2\mathbb{Z}$-graded Clifford module, since C_n acts on W by right multiplication. The real Dirac operator $D\colon \mathcal{H} \to \mathcal{H}$ is defined as usual by the covariant derivative and left exterior multiplication by covectors; it is an odd endomorphism of the Clifford module $\mathcal{H}$. Fix a basis $e_n, \dots, e_n$ of $\mathbb{R}^n$, and consider the operator

$$A = e_n D\colon \mathcal{H}_+ \to \mathcal{H}_+. \tag{3.8}$$

Here e_n acts by right Clifford multiplication. The operator A is skew-symmetric and anticommutes with right multiplication by $J = e_1 \cdots e_n$. Now $J^2 = -1$, so that J is a complex structure on $\mathcal{H}_\pm$. Since A anticommutes with J, it is a map from $\mathcal{H}_+$ to its complex dual $\mathcal{H}_+^*$. Finally, $A\colon \mathcal{H}_+ \to \mathcal{H}_+^*$ is complex skew-adjoint. We use e_n to identify $\mathcal{H}_+^*$ and $\mathcal{H}_-$, and so identify A and D.

When X is a Riemann surface ($n = 2$) the Clifford algebra $C_2 \cong \mathbb{H}$ has 4 real dimensions. Thus $W_\pm$ are one dimensional complex spaces, and $\mathcal{H}_+$ ($\mathcal{H}_-$) is the space of positive (negative) complex spinor fields on X. Hence $D\colon \mathcal{H}_+ \to \mathcal{H}_-$ is the usual complex Dirac operator. In terms of the holomorphic geometry on the surface, a spin structure corresponds to a holomorphic square root $K_X^{1/2}$ of the canonical bundle K_X. Then $\bar\partial_{K^{1/2}}\colon \Omega^{0,0}(K^{1/2}) \to \Omega^{0,1}(K^{1/2})$ is the Dirac operator [H,§2.1]. (We assume that the metric on X is Kähler.) The natural pairing $\Omega^{0,0}(K^{1/2}) \otimes \Omega^{0,1}(K^{1/2}) \to \mathbb{C}$ (via integration) is the duality of the previous paragraph. The operator $\bar\partial_{K^{1/2}}$ is skew-adjoint by Leibnitz' rule and Stokes' theorem.

In higher dimensions the Clifford algebra C_n contains several copies of the complex irreducible half-spin representations. The simultaneous eigenspace decomposition of the commuting operators e_1e_2, e_3e_4, $\dots$, $e_{n-1}e_n$ (acting by right Clifford multiplication) gives the irreducible pieces. Since J commutes with these operators, each irreducible representation admits a complex structure. For our geometric constructions we replace $\mathcal{H}_\pm$ by fields in some irreducible representation. Then $D\colon \mathcal{H}_+ \to \mathcal{H}_-$ is the complex Dirac operator of §1.

Proof of Theorem 3.1. For ease of notation we denote $\mathcal{L}^{1/2}$ by $\mathcal{K}$. The construction of $\mathcal{K}$ proceeds by patching, as in §1. Following the notation used there, set

$$\mathcal{K}^{(a)} = \left(\det \mathcal{H}_+^{(a)}\right)^*. \tag{3.9}$$

On the overlap $U^{(a)} \cap U^{(b)}$ we have the invertible skew-symmetric operator $D^{(a,b)}$, and

$$\begin{aligned} \mathcal{K}^{(a)} &\longrightarrow \mathcal{K}^{(a)} \otimes \mathcal{K}^{(a,b)} \cong \mathcal{K}^{(b)} \\ s &\longmapsto s \otimes \operatorname{Pfaff}\left(D^{(a,b)}\right) \end{aligned} \tag{3.10}$$

are the patching maps. The natural isomorphisms $\mathcal{K}^{(a)} \otimes \mathcal{K}^{(a)} \cong \mathcal{L}^{(a)}$ correspond on overlaps—compare (1.16) and (3.10) via (3.6)—thus giving $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{L}$. Also, the sections $\operatorname{Pfaff}\left(D^{(a)}\right)$ of $\mathcal{K}^{(a)}$ fit together into a section $\operatorname{Pfaff}(D)$ of $\mathcal{K}$ by (3.7). (The Pfaffian of a singular operator is defined to be zero.) We have $\operatorname{Pfaff}(D)^{\otimes 2} = \det D$ by (3.6).

The metric and connection lift to the square root. Alternatively, they can be constructed directly as in §1, replacing determinants by Pfaffians. Note in particular that $\omega^{(a)}(s)$ in (1.24) is replaced by $\frac{1}{2}\omega^{(a)}(s)$. The curvature formula (3.2) is a direct consequence of (1.30)—the curvature of the square root of a line bundle is half the original curvature. What is not immediately apparent is the holonomy formula (3.3), which depends on the particular square root (and isomorphism $\mathcal{K}\otimes\mathcal{K} \cong \mathcal{L}$). Now the main step in the proof of Theorem 1.31 is the proof of the holonomy formula for loops γ along which D is invertible; the general case follows by perturbation [BF2]. But when D is invertible there is a canonical trivialization of $\mathcal{L}$ given by $\det D$, and the logarithm of the holonomy around γ is $-\int_\gamma \omega^{(\delta)}$ for δ sufficiently small. The holonomy theorem for γ is the assertion[6]

$$\lim_{\epsilon \to 0} \xi_\epsilon = \frac{1}{2\pi i} \int_\gamma \omega^{(\delta)}. \tag{3.11}$$

[6]Equation (3.11) is the common analytic link between the various geometric interpretations of Witten's global anomaly (c.f. [C], [S]).

The important point is that (3.11) holds over the reals. Therefore, we can divide by 2 to obtain (3.3) for loops of invertible operators. The general case follows as before by perturbation (using the fact that $\frac{1}{2}\xi_\epsilon \pmod 1$ is continuous).

The topological class of $\mathcal{L}^{1/2}$ is determined in real K-theory. Recall the operator A in (3.8). Its symbol is the KO orientation along the fibers of $\pi\colon Z \to Y$, and so

$$\operatorname{index} A = \pi_!(1), \tag{3.12}$$

where

$$\pi_!\colon KO(Z) \longrightarrow KO^{-(8k+2)}(Y)$$

is the direct image map. (See [H,§4.2] for details.) Now $KO^{-(8k+2)} \cong KO^{-2}$ by periodicity, and there is a natural map

$$\operatorname{Pfaff}\colon KO^{-2}(Y) \longrightarrow H^2(Y;\mathbb{Z}), \tag{3.13}$$

which we call the Pfaffian. (One can see this by doing our construction over the space of skew-adjoint complex Fredholm operators, which is a classifying space for KO^{-2} [AS1]. Another model for that classifying space is ΩO, the loop space of the infinite orthogonal group, and $H^2(\Omega O;\mathbb{Z}) \cong \mathbb{Z}$.) Note that the diagram

$$\begin{array}{ccc} KO^{-2}(\Sigma) & & \\ & \searrow^{\operatorname{Pfaff}^{\otimes 2}} & \\ \downarrow & & H^2(\Sigma;\mathbb{Z}) \\ & \nearrow_{\det} & \\ K^{-2}(\Sigma) & & \end{array} \tag{3.14}$$

commutes. The vertical arrow is complexification. Summarizing,

Proposition 3.15. *In the situation of Theorem 3.1 we have*

$$c_1(\mathcal{L}^{1/2}) = \operatorname{Pfaff}\,(\pi_!(1))$$

as a topological line bundle.

§4 Riemann Surfaces

Determinants on Riemann surfaces arise in both mathematics and physics. Here we survey some recent literature, choosing examples which illustrate the geometric principles of §1–§3. The case of elliptic curves is most instructive, as everything can be calculated explicitly. The Pfaffian of the Dirac operator (for the distinguished spin structure) is essentially the Dedekind η-function. Atiyah [A2] made a very detailed study of its transformation law, relating it to Witten's formula (Theorem 1.31) on the one hand,[7] and to various topological expressions on the other. The determinant of the Dirac operator with coefficients in a flat bundle is computed by Kronecker's second limit formula [RS2]. The result (4.11) is ubiquitous in the string theory literature. For surfaces of arbitrary genus this determinant is essentially a ϑ-function. There is a general formula (4.16) expressing the variation of the determinant under a scale change—the conformal anomaly. We give a simple derivation due to Bost and Jolicœur. Another result due to Bost relates Quillen's norm to that occurring in Faltings' work (Proposition 4.21). The transformation law for the determinant of the Dirac operator coupled to a flat bundle of finite order can be calculated in K-theory, as we discuss at the end of this section.

We begin by recalling some facts about spin structures on Riemann Surfaces [A1]. Let X be a closed Riemann surface of genus g. In holomorphic terms a spin structure on X is a choice of a *theta characteristic* [ACGH,p.287], i.e., a line bundle $K^{1/2} \in \mathrm{Pic}^{g-1}(X)$ satisfying $\left(K^{1/2}\right)^{\otimes 2} \cong K$. There are 2^{2g} such choices; the ratio of any two is a point of order two on the Jacobian $\mathrm{Pic}^0(X)$. The choice of $K^{1/2}$ produces an isomorphism $\mathrm{Pic}^0(X) \cong \mathrm{Pic}^{g-1}(X)$, by tensor product. The canonical divisor on $\mathrm{Pic}^{g-1}(X)$, consisting of bundles L for which $H^0(X, L) \neq 0$, pulls back to a symmetric divisor on $\mathrm{Pic}^0(X)$, the Θ-divisor. The spin structures split into two classes according as $\dim H^0(X, K^{1/2})$ is even or odd. In fact,

$$K^{1/2} \longmapsto \dim H^0(X, K^{1/2}) \pmod 2 \tag{4.1}$$

is a quadratic function on the space of spin structures. Atiyah proves that

$$\dim H^0(X, K^{1/2}) = \pi_!^X(1) \pmod 2, \tag{4.2}$$

where $\pi_!^X : KO(X) \to KO^{-2}(\text{point}) = \mathbb{Z}/2\mathbb{Z}$ is the direct image map in KO-theory determined by the spin structure. The group of orientation-preserving diffeomorphisms of X acts on the set of spin structures. There are two orbits, and they are distinguished by the invariant (4.2). Spin structures correspond to quadratic forms on $H_1(X; \mathbb{Z}/2\mathbb{Z})$ which refine the intersection pairing. If S is an embedded circle in X, then the form evaluated on the homology class of S is 0 or 1 according as the spin structure restricts trivially or nontrivially to S. (Here 'trivial' means 'bounding.') The Arf invariant of this quadratic form is (4.2).

For an elliptic curve X there are 4 spin structures. The unique odd spin structure—the trivial double cover of the frame bundle—corresponds to the identity element of $\mathrm{Pic}^0(X) \cong X$. The three even spin structures are permuted by $SL(2;\mathbb{Z})$, which acts by diffeomorphisms on X. (Identify X with the differentiable manifold $\mathbb{R}^2/\mathbb{Z}^2$.) The odd spin structure is invariant under $SL(2;\mathbb{Z})$. The even spin structures are invariant

[7] We modify Atiyah's presentation somewhat. He uses the *signature* operator and refines Witten's formula to compute the logarithm of the holonomy (as a real number).

under a subgroup isomorphic to $\Gamma_0(2)$, the congruence subgroup consisting of matrices in $SL(2;\mathbb{Z})$ whose mod 2 reduction is upper triangular. We follow the physics language by terming the odd spin structure 'Periodic-Periodic' (PP) and the even spin structure invariant under $\Gamma_0(2)$ 'Periodic-Antiperiodic' (PA). The quadratic form corresponding to PP is $s + t + st$, where s, t are coordinates on $H_1(X;\mathbb{Z}/2\mathbb{Z})$. The Arf invariant zero forms st, $s + st$, $t + st$ are permuted by $SL(2;\mathbb{Z})$. The form $s + st$ corresponds to PA.

Let H denote the upper half-plane with its usual complex structure. It parametrizes a set of lattices in $\mathbb{C}$; the point $\tau \in H$ corresponds to the lattice L_τ generated by 1 and τ. Equivalently, H parametrizes equivalence classes of marked elliptic curves $X_\tau = \mathbb{C}/L_\tau$, i.e., elliptic curves with a distinguished basis for the first homology. Let $\mathbb{Z}\times\mathbb{Z}$ act on $H\times\mathbb{C}$ by a trivial action on the first factor, and let $\langle n_1, n_2\rangle$ act as translation by $n_1 + n_2\tau$ on $\{\tau\}\times\mathbb{C}$. Set $Z = (H\times\mathbb{C})/(\mathbb{Z}\times\mathbb{Z})$. Then $\pi\colon Z \to H$ is a holomorphic fibration whose fiber over τ is X_τ. Endow Z with the flat metric, normalized so that the fibers have unit volume. There are four possible choices of spin structure along the fibers. The hypotheses of Theorem 2.1 are satisfied, so that the determinant line bundle $\mathcal{L} \to H$ of the Dirac operator is holomorphic. By Theorem 3.1 the determinant of the Dirac operator has a natural holomorphic square root Pfaff(D), a section of $\mathcal{L}^{1/2} \to H$.

The canonical connection on $\mathcal{L}^{1/2}$ is flat [A1,Proposition 5.14]. For according to (3.2) its curvature is

$$\Omega^{(\mathcal{L}^{1/2})} = \pi i \int_{Z/H} \hat{A}(\Omega^{(Z/H)}). \tag{4.3}$$

But the metric on Z/H lifts to the constant flat metric along the fibers of $H\times\mathbb{C}\to H$, so the curvature $\Omega^{(Z/H)}$ vanishes. Since H is simply connected, there is a global covariant constant nonzero section s_0 of $\mathcal{L}^{1/2}$, which we take to have unit norm. Then s_0 is determined up to an overall constant phase. The section s_0 is holomorphic, by Theorem 2.1, and so we can identify Pfaff(D) as a holomorphic function on H, which depends on the underlying spin structure. For the odd (PP) spin structure it is identically zero, due to the nonzero cohomology. However, the kernel of D is one dimensional for all τ, and we consider instead $\text{Pfaff}'_{PP}(D)$, constructed on the orthogonal complement of the kernel; it is not holomorphic. There are explicit formulæ for these functions. Let

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}\left(1 - e^{2\pi i n\tau}\right) \tag{4.4}$$

be the Dedekind η-function.

Proposition 4.5. *We have*

$$\text{Pfaff}'_{PP}(D)(\tau) = \sqrt{\operatorname{Im}\tau}\cdot\eta(\tau), \tag{4.6}$$

$$\text{Pfaff}_{PA}(D)(\tau) = \sqrt{2}e^{\pi i\tau/12}\prod_{n=1}^{\infty}\left(1 + e^{2\pi i n\tau}\right). \tag{4.7}$$

Proposition 4.5 is a special case of a formula for the determinant of the Dirac operator twisted by a flat bundle (Proposition 4.10). Equation (4.7) follows immediately from the product expansion (4.11). Equation (4.6) is obtained as the limit where the twisting bundle becomes trivial, omitting the eigenvalue which tends to zero. Of course, there are similar formulas for the remaining two spin structures.

The factor of $\sqrt{\operatorname{Im}\tau}$ in (4.6) arises from the kernel of the Dirac operator. Recall that this is the cohomology $H^0(X_\tau, K^{1/2})$, which is one dimensional for all τ. These fit together into a line bundle $L \to H$, and $\omega_\tau = \sqrt{dz}$ is a holomorphic section of L. The norm square of ω_τ is $\operatorname{Im}\tau$.

We introduce the action of the modular group $PSL(2;\mathbb{Z})$ on H. It lifts to an $SL(2;\mathbb{Z})$ action on Z, and the metaplectic double cover $Mp(2;\mathbb{Z})$ acts on the spinors (with the odd spin structure). Hence $Mp(2;\mathbb{Z})$ also acts on the cohomology L and on the Pfaffian line bundle $\mathcal{L}^{1/2}_{PP}$. There is an identification

$$L^{\otimes 4} \cong TH \tag{4.8}$$

which preserves the $SL(2;\mathbb{Z})$ action.[8] To see this we change our point of view and fix the square torus, on which $SL(2;\mathbb{Z})$ acts by diffeomorphisms. Then $\omega_\tau = \sqrt{dx + \tau dy}$ is the holomorphic differential, $dx \wedge dy$ is the relative Kähler form, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{Z})$ acts on the coordinates of $H \times \mathbb{R}^2$ by

$$\begin{aligned} \tau &\longmapsto \frac{a\tau + b}{c\tau + d} \\ x &\longmapsto ax - by \\ y &\longmapsto dy - cx \end{aligned}$$

and so

$$\begin{aligned} (dx + \tau dy) &\longmapsto \frac{1}{c\tau + d}\,(dx + \tau dy) \\ d\tau &\longmapsto \frac{1}{(c\tau + d)^2}\,d\tau. \end{aligned}$$

Thus $\omega_\tau^4/d\tau$ is invariant, which is (4.8). The holomorphic section ω_τ^{-1} of L^{-1} determines a (nonvanishing) holomorphic section s of $\mathcal{L}^{1/2}_{PP}$ by patching with the Dirac operator. Under the identification $\omega_\tau^{-4} \leftrightarrow d\tau$,

$$s^4 = \eta^4(\tau)\,d\tau. \tag{4.9}$$

Hence s is essentially the Dedekind η-function.

Now since the action of $SL(2;\mathbb{Z})$ on Z preserves the given geometric data, the induced action of $Mp(2;\mathbb{Z})$ on $\mathcal{L}^{1/2}_{PP}$ preserves the Quillen metric and Hermitian connection. Therefore, the global flat section s_0 transforms according to a character $\mu\colon Mp(2;\mathbb{Z}) \to \mathbb{Z}/24\mathbb{Z}$. (The abelianization of $Mp(2;\mathbb{Z})$ is $\mathbb{Z}/24\mathbb{Z}$.) One expression for μ follows from the holonomy formula. Namely, given $\tilde{A} \in Mp(2;\mathbb{Z})$ lifting $A \in SL(2;\mathbb{Z})$ we form the torus bundle K_A over the circle with monodromy A. The lift $\tilde{A}$ is used to glue spinors. Endow K_A with the flat metric along the fibers and an arbitrary metric on the base circle. Then by (3.3) we have $\mu(\tilde{A}) = -\xi(K_A)/2 \pmod 1$, where $\xi(K_A)$ is the ξ-invariant of the three dimensional Dirac operator. No

[8] Here we follow some arguments of Atiyah [A2]. $L^{\otimes 4}$ is the cohomology bundle of the *signature* operator, which admits an action of $SL(2;\mathbb{Z})$ (in fact, of $PSL(2;\mathbb{Z})$). Our notation differs from Atiyah's; our ω_τ is the square root of his, and our μ (below) is $-\frac{1}{8}$ times his.

adiabatic limit is necessary since the curvature (4.3) is zero. Atiyah [A2] studies μ in great detail, equating it with many other invariants. In particular, he relates μ to Dedekind's transformation law for $\log \eta$.

Similar remarks apply to formula (4.7). Now the Dirac operator on the elliptic curve has no kernel, which simplifies matters a bit. The double cover $\widetilde{\Gamma_0(2)}$ of the congruence subgroup $\Gamma_0(2)$ acts on $\mathcal{L}_{PA}^{1/2}$. The transformation law is now a character of $\widetilde{\Gamma_0(2)}$.

As mentioned above, we can twist the Dirac operator on X_τ by a flat holomorphic bundle $L \in \operatorname{Pic}^0(X_\tau)$. The square root of the determinant is defined only when L is real, i.e., when L has order 2, in which case D_L is the ordinary Dirac operator with respect to a different spin structure. For definiteness fix the odd spin structure (PP). There is a holomorphic fibration $q\colon Y \to H$, where the fiber over τ is $\operatorname{Pic}^0(X_\tau)$. The pullback $q^*Z \to Y$ is also a holomorphic fibration, and over q^*Z we have a Poincaré line bundle P, whose restriction on the fiber X_τ over $L \in \operatorname{Pic}^0(X_\tau)$ is isomorphic to L (c.f. [GH,p.328]). The union of the identity elements in X_τ forms a subvariety of q^*Z, and we require that P restricts trivially to this subvariety. Then P is determined uniquely, and there is a family of Dirac operators D_P parametrized by Y. By Theorem 2.1 its determinant $\det D_P$ is a holomorphic section of a line bundle $\mathcal{L} \to Y$.

The line bundle $\mathcal{L}$ has curvature along the fibers of q. (This is exactly the situation considered in [Q].) Thus we cannot write $\det D_P$ as a holomorphic *function*. To display a formula we pass to the universal cover $\tilde{Y} = H \times \mathbb{C}$, where the pullback of $\mathcal{L}$ admits a (nonholomorphic) trivialization. Note that on each slice $H \times \{w\}$ the trivialization is holomorphic. Set $q = e^{2\pi i\tau}$ and let $\bar{u}, \bar{v} \in \mathbb{R}/\mathbb{Z}$ be real coordinates on $\mathbb{C}$. The point $\langle \tau, u - \tau v\rangle \in \tilde{Y}$ projects to the bundle $L \in \operatorname{Pic}^0(X_\tau)$ corresponding to the character $m\tau + n \mapsto e^{2\pi i(mu+nv)}$ on $H_1(X_\tau)$.

Proposition 4.10. *We have*

$$\det D_P(q; u, v) = q^{\frac{6v(v-1)+1}{12}} \prod_{n=1}^{\infty} \left(1 - q^{n-v} e^{2\pi i u}\right)\left(1 - q^{n+v-1} e^{-2\pi i u}\right). \tag{4.11}$$

This equality makes sense only up to a constant phase. Proposition 4.10 is well-known in the physics literature (e.g. [AMV], [V])—the translation of the path integral to the operator formulation immediately gives the product expansion. The equality of *norms* in (4.11) is Kronecker's second limit formula [Si]; the calculation is carried out in [RS2,Theorem 4.1]. The curvature of $\mathcal{L}$ is $\frac{i}{2\pi}\bar{\partial}\partial$ applied to the logarithm of the prefactor in (4.11), and then the proposition follows from the equality of norms [Q,§4]. Notice that the right-hand side is a perfect square for $u, v \in \frac{1}{2}\mathbb{Z}$, i.e., for the real points in the Jacobian.

Now fix $L \in \operatorname{Pic}^0(X_\tau)$ of order N. The bundle L_τ is specified topologically by $[L] \in H^1(X; \mathbb{Z}/N\mathbb{Z})$. Suppose $A \in SL(2;\mathbb{Z})$ preserves $[L]$. This happens if A belongs to the level N congruence subgroup $\Gamma(N)$, for example. Since A also preserves the odd spin structure, each of the lifts $\tilde{A} \in Mp(2;\mathbb{Z})$ acts on the determinant line bundle $\mathcal{L}_L \to H$ of D_L. Let $\mathcal{L} \to H$ denote the determinant line bundle of the ordinary Dirac operator; $\tilde{A}$ also acts on $\mathcal{L}$. The ratio of these actions is a complex number $\phi_L(\tilde{A})$ of unit norm. Formula (4.11) is related to Klein forms [L,Chapter XV], and the following result is related to their transformation law.

Proposition 4.12. *Choose a symplectic basis a, b of $H_1(X)$, and let $\tilde{u}, \tilde{v} \in \mathbb{Z}/N\mathbb{Z} \cong \frac{1}{N}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ be the evaluation of $[L]$ on a, b. Fix rational lifts u, v, and suppose*

$$(1 - A^*)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix} \tag{4.13}$$

for integers s,t. Then $\phi_L(\tilde{A})$ depends only on A and is given by

$$\phi_L(A) = (-1)^{s+t+st} e^{\pi i(sv-tu)}. \tag{4.14}$$

Note that $\phi_L(A)$ is a $(2N)^{\text{th}}$ root of unity. We discuss a more general conjecture at the end of the section. Proposition 4.12 is a special case of Proposition 4.24, which we discuss in more detail at the end of this section.

We turn now to closed Riemann surfaces X of arbitrary genus g. Here there are many possible choices of metric, and we first determine the behavior of the determinant under conformal transformations, reproducing an argument of Bost and Jolicœur [BJ], [Bo].[9] Consider the $\bar\partial$ operator on X coupled to K_X^n, the n^{th} power of the canonical bundle.

Proposition 4.15. *Fix a metric $g^{(X)}$ and a function f. Let L be the determinant line associated to $\bar\partial_{K^n}$ and $g_0^{(L)}$, $g_f^{(L)}$ the Quillen metrics associated to $g^{(X)}$, $e^{2f}\cdot g^{(X)}$, respectively. Then*

$$\frac{g_f^{(L)}}{g_0^{(L)}} = \exp\left\{\frac{6n(n-1)+1}{6\pi i}\int_X (f\Omega^{(X)} + \partial f\wedge\bar\partial f)\right\}, \tag{4.16}$$

where $\Omega^{(X)}$ is the curvature of the holomorphic line bundle TX with the metric $g^{(X)}$.

Proof [Bo]. Consider the trivial holomorphic fibration $\pi\colon \mathbb{C}\times X\to\mathbb{C}$. Fix a real function $F(y,\cdot)$ on $\mathbb{C}\times X$ which depends only on $|y|$, with $F(0,\cdot)\equiv 1$ and $F(1,\cdot)=2f$. Endow the fibers of π with the metric $e^F\cdot g^{(X)}$. The hypotheses of Theorem 2.1 are *not* satisfied—for example, the total volume of the fibers varies with y, which contradicts Lemma 2.2. We sketch a separate argument here to prove the compatibility of the complex structure and the canonical connection, thereby justifying our use of the curvature formula below. (More general versions of Theorem 2.1 are clearly desirable, and will no doubt appear in [BGS]. There is also a version announced in [Bo] which covers the present situation.) For the metric $e^f\cdot g^{(X)}$ on $\mathbb{C}\times X$ a straightforward calculation shows that the connection '$\nabla^{(Z/Y)}$' on the relative tangent bundle evaluates on $\frac{\partial}{\partial\bar y}$ to be the operator $\frac{\partial}{\partial\bar y}$. Equation (1.5) calls for a correction term due to the changing volumes. Since we work with Hermitian metrics here, we choose the 1-form γ to be the (1,0) part of $d\,\mathbf{vol}$ in (1.4). Thus this correction term evaluates trivially on $\frac{\partial}{\partial\bar y}$. Now since the complex structure on X is constant over the family, the bundle $\mathcal{L}_{\text{hol}}$ is holomorphically trivial. Further, we can omit the computation (2.30) by choosing both complexes in (2.25) to be the cohomology (at least over compact regions in $Y=\mathbb{C}$). Finally, (2.31) is still the decomposition of ω into type, since the operator D is constant over the family and so $\overline{\nabla}D = \frac{\partial}{\partial\bar y}D = 0$. For Riemann surfaces the factor $\mu^{(a)}$ in (2.27) is absent, and now (2.31) is trivial. Thus the canonical connection is compatible with the complex structure, as claimed.

By Theorem 1.30 the curvature of the determinant line bundle $\mathcal{L}\to\mathbb{C}$ of $\bar\partial_{K^n}$ is

$$\Omega^{(\mathcal{L})} = 2\pi i\int_X \left[\operatorname{Todd}(\Omega)\operatorname{ch}(-n\Omega)\right]_{(4)}, \tag{4.17}$$

[9] Their formula differs from (4.16) by a factor of 2, as they compare *norms* rather than *metrics*.

where Ω is the relative curvature of π. The relative tangent bundle is a holomorphic line bundle over $\mathbb{C}\times X$, and we easily calculate

$$\Omega = \Omega^{(X)} + \bar\partial\partial F. \tag{4.18}$$

Plug (4.18) into (4.17) to obtain

$$\begin{aligned}\Omega^{(\mathcal{L})} &= \frac{6n(n-1)+1}{24\pi i}\int_X \Big(2\bar\partial\partial F\wedge\Omega^{(X)} + \bar\partial\partial F\wedge\bar\partial\partial F\Big)\\ &= \bar\partial_y\partial_y\left\{\frac{6n(n-1)+1}{24\pi i}\int_X \Big(2F_y\Omega^{(X)} + \partial F_y\wedge\bar\partial F_y\Big)\right\},\end{aligned}$$

where $F_y = F(y,\cdot)$. But $\Omega^{(\mathcal{L})} = \bar\partial_y\partial_y \log\|\cdot\|^2_{\mathcal{L}}(y)$, and $\|\cdot\|_{\mathcal{L}}(y)$ depends only on $|y|$. Equation (4.16) follows immediately.

Equation (4.16) is well-known in the physics literature [P], [Al1], [Fr]. The relationship of this formula to the index theorem was first noticed by Alvarez [Al2]. The application of Proposition 4.15 to the bosonic string has been discussed extensively. For the combination of operators in the string integrand the conformal variation (4.16) vanishes. This is the famous computation which fixes the dimension of the bosonic string to be 26. Geometrically, it allows us to push the metric and connection on the string determinant bundle down to the moduli space. Then another application of Theorem 2.1 and Theorem 1.30 gives the holomorphic factorization of Belavin and Knizhnik [BK] (c.f. [BJ], [Bo], [CCMR], [F1]). Recently Osgood, Phillips, and Sarnak [OPS] used (4.16) as the starting point in a proof of the uniformization theorem for Riemann surfaces.

Determinant line bundles play a role in arithmetic. The Hermitian norm was introduced by Faltings [Fa] in that context, based on ideas of Arakelov [Ar]. They study line bundles L over a Riemann surface X, but restrict to a class of *admissible metrics*. Let $\mu\colon X\to \mathrm{Pic}^0(X)$ be the Abel map; it is defined up to translations. The polarization of the Jacobian $\mathrm{Pic}^0(X)$ (coming from the intersection pairing) determines a translation invariant (1,1) form, whose pullback to X we denote $\Omega^{(X)}$. It is normalized by $\frac{i}{2\pi}\int_X \Omega^{(X)} = 2-2g$, where g is the genus of X. A metric on a holomorphic line bundle $L\to X$ is admissible if its curvature $\Omega^{(L)}$ is proportional to $\Omega^{(X)}$. Admissible metrics always exist and are unique up to a constant. Applied to the tangent bundle we obtain a metric on X whose curvature is $\Omega^{(X)}$. Fix a point $Q\in X$ and consider the line bundle $\mathcal{O}(Q)$. It has a canonical section s whose divisor is Q, and we normalize the admissible metric on $\mathcal{O}(Q)$ by requiring

$$\int_X \big(\log|s|^2\big)\,\Omega^{(X)} = 0. \tag{4.19}$$

Let $\lambda_X(L)$ denote the determinant line bundle for the $\bar\partial$ complex coupled to L. Then the section s gives rise to a canonical isomorphism

$$\hat s\colon \lambda_X\big(L\otimes\mathcal{O}(Q)\big)\cong\lambda_X(L)\otimes\big(L\otimes\mathcal{O}(Q)\big)[Q], \tag{4.20}$$

where the last term is the fiber of $L\otimes\mathcal{O}(Q)$ at Q. Faltings characterizes his metric on the determinant by requiring that $\hat s$ be an isometry (c.f. [Ar,§4]). He than proves [Fa,Theorem 1] that this property, together with two other axioms, characterizes the metric up to a scale factor. The issue of whether the Faltings metric and the Quillen metric coincide was settled by Bost.

Proposition 4.21 [Bo]**.** *Let $L \to X$ be a holomorphic bundle over a Riemann surface of genus $g > 2$, and suppose both X and L are endowed with admissible metrics.*[10] *Then the Faltings metric on $\lambda_X(L)$ agrees with the Quillen metric, up to a constant depending only on g.*

We omit the proof, but remark that it is similar to an argument of Donaldson [D,§2]. In both cases a variational formula governing the restriction of the determinant to a divisor is derived using Theorem 1.30 together with Theorem 2.1. The universal family of Riemann Surfaces with Arakelov metrics satisfies the hypothesis of Theorem 2.1, since the (1,1)-form τ can be induced from the universal family of abelian varieties via the Abel map.

As for elliptic curves, we consider the family of Dirac operators twisted by a flat holomorphic line bundle. Fix a spin structure on X, and let $\mathcal{S}_g$ be the moduli space of Riemann surfaces with this spin structure.[11] Over $\mathcal{S}_g$ lies the family of Jacobians $q\colon Y \to \mathcal{S}_g$ with typical fiber $\mathrm{Pic}^0(X)$. The spin structure determines a Θ-divisor on Y, consisting of bundles $L \in \mathrm{Pic}^0(X)$ with $H^0(X, K^{1/2} \otimes L) \neq 0$. Let $p\colon Z \to \mathcal{S}_g$ be the universal family of Riemann surfaces with the fixed spin structure. We use the Arakelov metrics along the fibers. Then points of p^*Y are equivalence classes of such surfaces together with a basepoint and flat bundle. Hence there is a Poincaré line bundle $P \to p^*p^*Y$, trivial over the diagonal. (The typical fiber of p^*p^*Y is $\mathrm{Pic}^0(X) \times X \times X$; the first X is the basepoint which defines the Poincaré bundle over $\mathrm{Pic}^0(X) \times X_{(2)}$.) The family of Dirac operators D_P is defined on the fibers of $\pi\colon p^*p^*Y \to p^*Y$. Then $\det D_P$ is a section of $\mathcal{L} \to p^*Y$. The Dirac operator is blind to the basepoint, which we now discard. Finally, then, $\det D_P$ is a section of $\mathcal{L} \to Y$.

Recall[12] that to a point Z in the Siegel upper half-plane $\mathcal{H}_g$ (which consists of symmetric $g \times g$ matrices Z with $\mathrm{Im}\, Z > 0$) corresponds a principally polarized abelian variety

$$A_Z = \mathbb{C}^g/(\mathbb{Z}^g + Z \cdot \mathbb{Z}^g).$$

The ϑ-function, defined on $\mathcal{H}_g \times \mathbb{C}^g$ by

$$\vartheta(Z, z) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i n \cdot Z \cdot n + 2\pi i n \cdot z), \tag{4.22}$$

is a section of a line bundle $\mathcal{O}(\Theta)_Z$ over A_Z. Its divisor Θ is symmetric. A metric on $\mathcal{O}(\Theta)_Z$ is determined up to a constant by requiring its curvature to be translation invariant. Fix the constant by specifying the norm square of the canonical section (integrated over A_Z). Identify A differentiably with the square torus. Then $Sp(2g;\mathbb{Z})$ acts by diffeomorphisms on A. Let Γ denote the subgroup which fixes the spin structure. The theta group Γ also acts on $\mathcal{H}_g$, and the quotient $\mathcal{A}_g = \mathcal{H}_g/\Gamma$ parametrizes principally polarized abelian varieties with a fixed spin structure. The abelian varieties fit together into a family $\mathcal{Y} \to \mathcal{A}_g$. The metaplectic double cover $\tilde{\Gamma}$ of Γ acts on the theta line bundle (c.f. [JM]), and we obtain a holomorphic line bundle $\mathcal{O}(\Theta) \to \mathcal{Y}$. The ϑ-function (4.22) is a section of $\mathcal{O}(\Theta)$ and its divisor Θ sits on $\mathcal{Y}$.

[10] The restriction to $g > 2$ can probably be circumvented.

[11] Technically, we should work on a finite cover of $\mathcal{S}_g$, due to nontrivial automorphisms of special surfaces. The fiber of $\mathcal{S}_g \to \mathcal{M}_g$ can be identified with the orbit of the fixed spin structure under $\mathrm{Diff}\, X$.

[12] The reader should compare this discussion with [Fa,§6].

Returning to the Riemann surface X there is a diagram

$$\begin{array}{ccc} Y & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{S}_g & \longrightarrow & \mathcal{A}_g \end{array}$$

The bottom arrow is the map which attaches the period matrix to a Riemann surface, and the lift to Y sends $\mathrm{Pic}^0(X)$ into the corresponding abelian variety. We choose the spin structure on the abelian varieties so that the Θ-divisors correspond. By pullback we have a line bundle $\mathcal{O}(\Theta) \to Y$ with its ϑ-function. By definition, the Dirac operator D_P vanishes precisely on the Θ-divisor. The line bundles $\mathcal{L}$ and $\mathcal{O}(\Theta)^{-1}$ are isomorphic, and an isomorphism is unique up to a constant.[13] However, this isomorphism is not an isometry. Faltings denotes the ratio of the metrics on the zero section of q by $\exp(\delta(X)/8)$. Over the zero section we compare $\det D$ with the determinant $\det \bar{\partial}$ of the $\bar{\partial}$ operator, which is a section of a line bundle $\mathcal{K}$. The curvature formula (1.30) shows that $\mathcal{L}^{\otimes 2}$ and $\mathcal{K}^{-1}$ have the same curvature. hence their metrics agree up to a constant. Equating the norms of the determinants, Bost and Nelson [BN] prove

$$|\det D| = C_g \left(\frac{\det' \triangle}{\det(\operatorname{Im} Z) \cdot \operatorname{Area}(X)} \right)^{-1/4} |\vartheta(0, Z)|, \tag{4.23}$$

where Z is the period matrix of X (relative to some basis of $H_1(X)$), and $\triangle$ is the Laplacian on X. Notice that Equation (4.23) was derived here for the Arakelov metrics, though extensions to other metrics are possible.

Equation (4.23) corresponds to "bosonization" in the physics literature. Alvarez-Gaumé, Bost, Moore, Nelson, and Vafa [ABMNV] use these ideas to obtain other bosonization formulæ by coupling Dirac to a power of the canonical bundle (c.f. [Bo]). Their bosonization program goes beyond the determinant. They build a Lagrangian for the bosonic theory, but we do not pursue those ideas here. Also, Bost [Bo] uses (4.23) to determine Falting's constant $\delta(X)$.

Finally, consider bundles $L \in \mathrm{Pic}^0(X)$ of finite order N. Suppose φ is a diffeomorphism of X preserving both the spin structure and the topological class $[L] \in H^1(X; \mathbb{Z}/N\mathbb{Z})$. Form the 3-manifold P by gluing the ends of $X \times [0,1]$ via φ; then P fibers over S^1. Furthermore, a lift $\tilde{\varphi}$ acts on spinors, so determines a spin structure along the fibers. Also, L extends to a flat bundle $L \to P$. Let $\phi(\tilde{\varphi})$ denote the ratio of the holonomies of the determinant line bundles $\mathcal{L}_L$ and $\mathcal{L}$ for the Dirac operator coupled to L and the ordinary Dirac operator, respectively. This is the phase in a certain transformation of ϑ-functions.

Proposition 4.24 [FV],[LMW]**.** *Fix a canonical basis a_j, b_j of $H_1(X)$ and let $A \in Sp(2g; \mathbb{Z})$ be the matrix of the induced action by φ on homology. Let $u_j, v_j \in \mathbb{Q}/\mathbb{Z}$ be the value of $[L]$ on a_j, b_j, and choose lifts $u_j, v_j \in \mathbb{Q}$. Suppose*

$$(1 - A^*) \begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} s_j \\ t_j \end{pmatrix}$$

for integers s_j, t_j. Then $\phi_L(\tilde{\varphi})$ depends only on A and is given by

$$\phi_L(A) = \exp \pi i \left\{ q(\begin{pmatrix} \bar{s} \\ t \end{pmatrix}) + \sum_j (s_j v_j - t_j u_j) \right\}, \tag{4.25}$$

[13] This uses a compactness principle on moduli space—any holomorphic function is constant—which is valid for genus $g > 2$.

where q is the quadratic form on $H_1(X;\mathbb{Z}/2\mathbb{Z}) = H^1(X;\mathbb{Z}/2\mathbb{Z})$ determined by the spin structure, and $\bar{s},\bar{t}$ are the reductions of s,t modulo 2.

As before, the phase $\phi_L(A)$ is a $(2N)^{\text{th}}$ root of unity. Proposition 4.24 was proved in [FV,Corollary 7.45] for the case $[L] \equiv 0 \pmod 2$. The argument there also proves (4.25) up to a sign in the general case. The approach of Lee, Miller, and Weintraub [LMW], based on [BM], relates this sign to the quadratic form q, and so proves the proposition. A direct argument in K-theory would be nice here; the quadratic form should appear as in (4.2). (The bundle L, together with its trivialization, determines an element of $K^{-1}(P;\mathbb{Q}/\mathbb{Z})$. Then $\phi_L(A)$ is the image of this element under the direct image $K^{-1}(P;\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ determined by the spin structure. One needs to compute this direct image in terms of the matrix A.) Also, an explicit comparison of (4.25) with the transformation law for the ϑ-function may prove interesting. Proposition 4.24 reduces to Proposition 4.12 for elliptic curves with the odd spin structure.

For $N = 2$ the bundle L is real, and $\phi_L(A)$ expresses the ratio of holonomies for different spin structures. That this is of order 4 was proved in [AMV], [F1,Appendix], [LMW], and [W2] by four distinct methods! (Notice that in this case L is a real 2-plane bundle, and the Pfaffian of Dirac is defined. Then the ratio of the holonomies of the Pfaffians for different spin structures is of order 8.) Lee, Miller, and Weintraub [LMW] identify $\phi_L(A)$ as a difference of *Rohlin invariants*.

§5 Anomalies and Orientation

A vector space V is oriented by choosing a basis (nonzero element) of its determinant $\det V$. A family of vector spaces, say a *real* vector bundle $V \to M$, is oriented in ordinary cohomology by a trivialization of its determinant bundle $\det V$. There is a single topological obstruction to orientability—the first Stiefel-Whitney class $w_1(V) \in H^1(M;\mathbb{Z}/2\mathbb{Z})$. In extraordinary cohomology there are extraordinary orientations [ABS], [Ad]. Suppose $w_1(V) = 0$. Then V is orientable in KO-theory if and only if $w_2(V) \in H^2(M;\mathbb{Z}/2\mathbb{Z})$ vanishes, while for K-theory it is the integral Stiefel-Whitney class $W_3(V) \in H^3(M;\mathbb{Z})$ which obstructs orientability. Recently, new topological constraints have entered string theory through work of Ed Witten [W2], [W3], [F1], [FV]. Roughly speaking, V is "oriented" if its first Pontrjagin class $p_1(V) \in H^4(M;\mathbb{Z})$ is zero.[14] (This ignores a factor of two, as we explain below.) In string theory this vanishing is a sufficient condition for modular invariance, i.e., for the absence of anomalies in the theory. Topologists are familiar with this condition in a different theory—they call it $MO\langle 8\rangle$. (See [Da], [Pe], [St], for example. The 7-connected cover $BO\langle 8\rangle$ of BO classifies certain bundles, and topologists study the bordism theory represented by the associated Thom spectrum $MO\langle 8\rangle$.) The link [W3] between string theory and the burgeoning "elliptic cohomology" [La] suggests that the new orientability condition will enter the definitive version of this most extraordinary cohomology theory. One should compare the situation here with Atiyah's interpretation [A3] of Witten's assertion [W4], [W2] that the free loop space $\mathfrak{L}M$ is orientable if and only if M is spin. Ultimately, that too involves real K-theory. Killingback [K] takes a different (and perhaps more fundamental) approach to this new orientation condition, which is related to the Hamiltonian viewpoint of anomalies [Seg].

The anomaly in physics has a topological interpretation due to Atiyah and Singer [AS2]—it is the obstruction to trivializing the determinant line bundle $\mathcal{L}$ of a family of Dirac operators. (This topological obstruction is the Chern class $c_1(\mathcal{L})$.) Let X be the two dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$, and $\mathfrak{L}^2M = \mathrm{Map}(X,M)$ the double free loop space of M. Fix a spin structure on X. There is a natural evaluation map

$$e\colon \mathfrak{L}^2M \times X \longrightarrow M, \tag{5.1}$$

by which we pull back the real vector bundle $V \to M$. Let $\pi^X : \mathfrak{L}^2M \times X \to \mathfrak{L}^2M$ be projection onto the second factor, and choose differential geometric data as in (1.1). This determines a family of Dirac operators (coupled to e^*V) along the fibers of π^X, and by Theorem 3.1 its Pfaffian is a section of a complex line bundle $\mathcal{L}^{1/2} \to \mathfrak{L}^2M$. Proposition 3.15 computes the Chern class of $\mathcal{L}^{1/2}$:

$$c_1(\mathcal{L}^{1/2}) = \mathrm{Pfaff}\left(\pi^X_! e^![V]\right). \tag{5.2}$$

Here $[V] \in KO(M)$ is the KO class, $e^!$ is the pullback in KO-theory, $\pi_!\colon KO(\mathfrak{L}^2M \times X) \to KO^{-2}(\mathfrak{L}^2M)$ is the direct image determined by the spin structure on X, and Pfaff is the map (3.13). Of course, we will denote the determinant line bundle, which is the square of $\mathcal{L}^{1/2}$, by $\mathcal{L}$.

An important refinement occurs by noting that $\mathrm{Diff}\, X$ acts on $\mathfrak{L}^2M \times X$ preserving the evaluation map (5.1). We restrict our attention to the group of components $\pi_0 \mathrm{Diff}\, X = SL(2;\mathbb{Z})$. Introduce the

[14] The setup of heterotic string theory is somewhat more complicated than the situation we consider here.

upper half-plane H with its $SL(2;\mathbb{Z})$ action. (This essentially amounts to working with equivariant KO-theory.) Let $\Gamma \subseteq SL(2;\mathbb{Z})$ be the subgroup preserving the given spin structure on X. Then a double cover $\tilde{\Gamma}$ lifts the Γ action to spinors, and we obtain a diagram

$$\begin{array}{ccc} e^*V & & V \\ \big\downarrow & & \big\downarrow \\ Z = H \times \mathfrak{L}^2 M \times X \;/\; \Gamma & \xrightarrow{\;e\;} & M \\ \big\downarrow{\scriptstyle \pi^X} & & \\ Y = H \times \mathfrak{L}^2 M \;/\; \Gamma & & \end{array} \tag{5.3}$$

with a KO orientation along the fibers of π (the symbol of the Dirac family). Equation (5.2) applies to this setup, and so expresses $c_1(\mathcal{L}^{1/2})$ in KO-theoretic terms. Over the space H/Γ we also have the Pfaffian line bundle $\mathcal{L}_0^{1/2}$ of the ordinary Dirac operator (not coupled to any extrinsic data). This is the bundle discussed in §4. Recall from Proposition 4.5 that the Pfaffian of the Dirac operator is a theta function divided by the Dedekind η-function. The line bundle $\mathcal{L}_0^{1/2}$ lifts to a bundle over Y.

Our work in this section concerns the passage from (5.2) to characteristic classes in cohomology. Over the rationals this is standard, but over the integers there are not usually good general formulæ. In low dimensions, however, the situation is quite tractable. Let $\lambda \in H^4(B\,\mathrm{Spin};\mathbb{Z})$ denote the generator such that 2λ lifts $p_1 \in H^4(BSO;\mathbb{Z})$. For a real vector bundle V with $w_1(V) = w_2(V) = 0$ the characteristic class $\lambda(V)$ is well-defined. Our proof of the following proposition is in the spirit of index theory, though surely other (homotopy-theoretic) arguments can be given instead.

Proposition 5.4. *In the situation described above,*

$$c_1(\mathcal{L}) - c_1(\mathcal{L}_0^{(\dim V)}) = \pi_*^X e^* p_1(V), \tag{5.5}$$

where π_* is integration along the fiber. If $w_2(V) = 0$, then

$$c_1(\mathcal{L}^{1/2}) - c_1(\mathcal{L}_0^{(\dim V)/2}) = \pi_*^X e^* \lambda(V). \tag{5.6}$$

Proof. A cohomology class is determined by its periods on integral homology classes and $\mathbb{Q}/\mathbb{Z}$ homology classes. (It can also be specified by these periods [MS,§2].) In two dimensions we can represent these classes by manifolds. Suppose $\Sigma \to Y$ is a closed Riemann surface mapping to Y, and let $\pi^X : Q \to \Sigma$ be the pullback of π^X in (5.3). Here Q is a closed 4-manifold. It inherits a spin structure along the fibers and a bundle V by pullback. Let $\pi_!^\Sigma : KO^{-2}(\Sigma) \to KO^{-4}(\text{point}) \cong \mathbb{Z}$ denote the direct image for some fixed spin structure on Σ. We claim that the diagram

$$\begin{array}{ccccc} KO^{-2}(\Sigma) & & \xrightarrow{\;\pi_!^\Sigma\;} & & \mathbb{Z} \\ & {\scriptstyle \mathrm{Pfaff}^{\otimes 2}} & & & \\ & & H^2(\Sigma;\mathbb{Z}) & & {\scriptstyle 2} \\ & {\scriptstyle \det} & & {\scriptstyle [\Sigma]} & \\ K^{-2}(\Sigma) & & \xrightarrow{\;\pi_!^\Sigma\;} & & \mathbb{Z} \end{array} \tag{5.7}$$

commutes. The left triangle is (3.14), and the bottom triangle commutes by the standard formula for the direct image in K-theory. The outer rectangle

$$\begin{array}{ccc} KO^{-2}(\Sigma) & \xrightarrow{\pi_!^\Sigma} & KO^{-4}(\text{point}) \\ \big\downarrow & & \big\downarrow \\ K^{-2}(\Sigma) & \xrightarrow{\pi_!^\Sigma} & K^{-4}(\text{point}) \end{array} \tag{5.8}$$

commutes since the spin structure defines both the KO orientation of the top arrow and the K orientation of the bottom arrow. The right arrow is multiplication by 2 on the generators (c.f. the discussion of Rohlin's theorem in §3). Hence the upper right triangle in (5.7) also commutes, from which

$$\operatorname{Pfaff}(x)[\Sigma] = \pi_!^\Sigma(x) \tag{5.9}$$

for any $x \in KO^{-2}(\Sigma)$. Applied to $x = \pi_!^X e^!([V] - \dim V)$ using (5.2) we obtain

$$\{c_1(\mathcal{L}^{1/2}) - c_1(\mathcal{L}_0^{(\dim V)/2})\}[\Sigma] = \pi_!^\Sigma \pi_!^X e^!([V] - \dim V). \tag{5.10}$$

But the transitivity of direct images [AS3] implies

$$\{c_1(\mathcal{L}^{1/2}) - c_1(\mathcal{L}_0^{(\dim V)/2})\}[\Sigma] = \pi_!^Q e^!([V] - \dim V), \tag{5.11}$$

where $\pi_!^Q \colon KO(Q) \to KO^{-4}(\text{point}) \cong \mathbb{Z}$ is the direct image on Q. Now $\pi_!^Q(y) = \frac{1}{2}\hat{A}(Q)\operatorname{ch}(y)[Q]$ for $y \in KO(Q)$ (see [AS4]). Hence

$$\{c_1(\mathcal{L}^{1/2}) - c_1(\mathcal{L}_0^{(\dim V)/2})\}[\Sigma] = e^*\lambda(V)[Q] = \pi_*^X e^*\lambda(V)[\Sigma], \tag{5.12}$$

which proves (5.5) and (5.6) over $\mathbb{Q}$.

Following Sullivan [Su], [MS] we use $\mathbb{Z}/k$-manifolds to represent $\mathbb{Z}/k\mathbb{Z}$ cycles.[15] Let $\overline{\Sigma} \to Y$ be a $\mathbb{Z}/k$-Riemann surface in Y. So now Σ is a smooth surface whose boundary consists of k disjoint circles, which we glue together to obtain $\overline{\Sigma}$. From this description it follows that a $\mathbb{Z}/k$-manifold carries a fundamental class in $\mathbb{Z}/k\mathbb{Z}$ homology. Now the pullback of π^X in (5.3) is $\pi^X \colon \overline{Q} \to \overline{\Sigma}$ for some four dimensional $\mathbb{Z}/k$-manifold $\overline{Q}$. Here $\overline{Q}$ is obtained from a manifold Q with boundary by gluing together the k diffeomorphic boundary components. (See Figure 2.) Pull back the spin structure along the fibers and the KO class $[V]$. Let P be the *Bockstein* of $\overline{Q}$; it is diffeomorphic to each boundary component of Q and fibers over the circle (which is diffeomorphic to each boundary component of Σ).

[15] Our arguments here follow [F2]. See also [F1,§3], [FM].

FIGURE 2

A spin structure on a $\mathbb{Z}/k$-manifold determines a fundamental class in KO theory. The resulting direct image has no direct cohomological formula in general, but in two dimensions we claim (compare (5.9))

$$\operatorname{Pfaff}(x)[\overline{\Sigma}] = \pi_!^{\overline{\Sigma}}(x) \tag{5.13}$$

for $x \in KO^{-2}(\overline{\Sigma})$. To prove (5.13) write the Bockstein of $\overline{\Sigma}$ (which is a circle) as the boundary of some spin manifold R. Then $\Sigma - kR$ is closed, and x extends to a class $\tilde{x} \in KO^{-2}(\Sigma - kR)$. By (5.9) we obtain the desired result:

$$\pi_!^{\overline{\Sigma}}(x) \equiv \pi_!^{\Sigma - kR}(\tilde{x}) \equiv \operatorname{Pfaff}(\tilde{x})[\Sigma - kR] \equiv \operatorname{Pfaff}(x)[\overline{\Sigma}] \pmod{k}. \tag{5.14}$$

The direct images obey a multiplicative law for fiberings of closed manifolds over $\mathbb{Z}/k$-manifolds. Applied to $\pi^X : \overline{Q} \to \overline{\Sigma}$, imitating (5.10)–(5.11),

$$\{c_1(\mathcal{L}^{1/2}) - c_1(\mathcal{L}_0^{(\dim V)/2})\}[\overline{\Sigma}] = \pi_!^{\overline{Q}} e^!([V] - \dim V). \tag{5.15}$$

To evaluate the right hand side of (5.15) we repeat the argument of (5.14), now using (5.12). But for this we must write the Bockstein P of $\overline{Q}$ as a spin boundary $P = \partial S$ and extend the bundle V over S. In case $w_2(V) = 0$ we can lift V to a spin bundle. Then the relevant bordism group $\Omega_3^{\text{spin}}(B\,\text{Spin})$ vanishes, by the Atiyah-Hirzebruch spectral sequence, whence the bounding manifold S exists. Hence by (5.12),

$$\pi_!^{\overline{Q}} e^!([V] - \dim V) \equiv \pi_!^{Q-kS} e^!([V] - \dim V) \equiv \pi_*^X e^* \lambda(V)[\Sigma - kR] \equiv \pi_*^X e^* \lambda(V)[\overline{\Sigma}] \pmod{k}, \tag{5.16}$$

which completes the proof of (5.6).

The assertion in (5.5) is obtained by applying (5.6) to the bundle $V \oplus V$.

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