DIFFERENTIAL TOPOLOGY

DANIEL S. FREED

What follows are lecture notes from a graduate course given at the University of Texas at Austin in Spring, 2021 and Spring, 2022. The topics follow those in Guilleman-Pollack's *Differential Topology*, but the treatment diverges often. These notes are a bit rough in many places, so use at your own risk!

I have appended homework exercises to these notes.

I welcome feedback of any kind.

Contents

Date: May 6, 2022.

Email: `dafr@math.utexas.edu`.

Lecture 1: Topological manifolds

I posted notes on "multivariable analysis". I will not repeat material in those notes. For example, in the first weeks of the course I use basic notions of affine geometry and of differential calculus from those notes.

The first part of this lecture was motivation for the course, which I won't repeat here. So these notes only cover the last part of the lecture.

(1.1) *Some point set topology.* Let X be a topological space. Recall that X is *Hausdorff* if for all *distinct* $x_1, x_2 \in X$ there exist *disjoint* open sets $U_1, U_2 \subset X$ such that $x_i \in U_i$, $i = 1, 2$. Every metric space is Hausdorff since the distance between distinct points is positive, hence distinct points can be separated by open balls (of radius half the distance). The topological space X is *second countable* if it admits a countable basis. That is, there exists a countable collection $\{U_i\}_{i\in I}$ of open subsets $U_i \subset X$ such that any nonempty open set $U \subset X$ is the union $U = \bigcup_{i\in I'} U_i$ for some subset $I' \subset I$. (The empty set is the union with $I' = \emptyset$, so we can omit 'nonempty' in the previous sentence.) A metric space is not necessarily second countable; it is if it is separable, which means it has a countable dense set.

Definition 1.2. Let X be a topological space.

(i) X is *locally Euclidean* if for all $x \in X$ there exists an open set $U \subset X$ containing x and a homeomorphism of U onto an open subset of a finite dimensional affine space.

(ii) X is a *topological manifold* if it is locally Euclidean, Hausdorff, and second countable.

Remark 1.3.

(i) The *data* of a topological manifold is that of a topological space. The definition specifies three *conditions* on the topology.

(ii) Perhaps 'locally Euclidean' should be 'locally affine', but some terms and notations are ingrained—it would be counterproductive to protest. The affine space in (i) can be chosen to be the standard affine space $\mathbb{A}^n$ of some dimension $n \in \mathbb{Z}^{\geqslant 0}$. The *invariance of domain* theorem shows that the dimension n is well-defined, i.e., it is the same for all choices of local homeomorphism. In other words, there is no local homeomorphism between open subsets of affine spaces of different dimension.

(iii) A topological manifold X is a *regular* topological space. This means that given a point $x \in X$ and a disjoint closed subset $C \subset X$, there exist disjoint open sets $U, V \subset X$ so that $x \in U$ and $C \subset V$; we can separate points and closed sets. This is stronger than the Hausdorff property.

(iv) Urysohn's metrization theorem states that a regular, second countable topological space X in which points are closed sets is metrizable: there exists a metric on X whose underlying topology—set of open sets—agrees with the given topology. In particular, topological manifolds are metrizable.

(v) A subset of a topological manifold is a component if and only if it is a path component.

Definition 1.4. The *dimension* of a topological manifold X is the locally constant function

$$\dim X\colon X \longrightarrow \mathbb{Z}^{\geqslant 0} \tag{1.5}$$

whose value at $x \in X$ is the dimension of an affine space locally homeomorphic to X at x. If $n \in \mathbb{Z}^{\geqslant 0}$, then a *topological manifold of dimension* n, or *topological* n*-manifold*, is one for which (1.5) is the constant function with value n.

(1.6) *Charts.* The local homeomorphisms to affine space on a topological manifold are called coordinate systems or charts.

Definition 1.7. Let X be a topological manifold and A a finite dimensional affine space. An A*-valued chart* on X is a pair (U, ϕ) consisting of an open set $U \subset X$ and a continuous map $\phi\colon U \to A$ which is a homeomorphism onto its image. If $A = \mathbb{A}^n$ is a standard affine space, then we say that (U, ϕ) is a *standard chart*.

A topological manifold admits a covering by charts, i.e., a collection

$$\mathscr{A} = \bigl\{(U_\alpha, \phi_\alpha) \text{ charts}\bigr\}_{\alpha \in \mathcal{A}} \tag{1.8}$$

indexed by some set $\mathcal{A}$ such that $X = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. Here $\phi_\alpha\colon U_\alpha \to A_\alpha$ is a homeomorphism into an affine space A_α. It will sometimes be convenient to take all A_α to be standard affine spaces.

(1.9) *Examples of topological spaces which fail to be topological manifolds.* We first give three examples of topological spaces which are not topological manifolds; each illustrates the failure of precisely one of the three conditions in Definition 1.2.

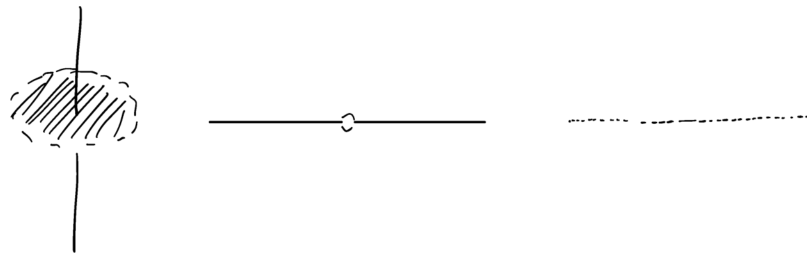


FIGURE 1. Three topological spaces which are not topological manifolds

Example 1.10 (non locally Euclidean)**.** The subspace $X \subset \mathbb{A}^3_{x,y,z}$ defined by

$$X = \{(x^2 + y^2 < 1) \cap z = 0\} \;\cup\; \{x = y = 0\} \tag{1.11}$$

fails to be locally Euclidean at the point $(0,0,0)$.

Example 1.12 (non Hausdorff)**.** The topological space

$$\mathbb{R} \cup_{\mathbb{R}\setminus\{0\}} \mathbb{R} \tag{1.13}$$

obtained by gluing two copies of $\mathbb{R}$ at every point except 0 is locally Euclidean but fails to be Hausdorff: the two copies of 0 cannot be separated.

Example 1.14 (non second countable)**.** The space $X = \mathbb{R}$ with the discrete topology fails to be second countable.

(1.15) *Examples of topological manifolds.* Since countable disjoint unions of topological manifolds are topological manifolds, it suffices to give connected examples.

Example 1.16 (affine space)**.** Any finite dimensional affine space is a topological manifold with its usual topology. (A finite dimensional vector space has a unique topology with respect to which the vector space operations are continuous; see Lecture 4 of the multivariable notes for a closely related theorem.)

Example 1.17 (dimension 1)**.** Any connected topological 1-dimensional manifold is homeomorphic to either S^1 or $\mathbb{A}^1 = \mathbb{R}$. We will prove this for smooth manifolds later in the course.

Example 1.18 (the 2-sphere)**.** Let

$$X = \{(x,y,z) \in \mathbb{A}^3 : x^2 + y^2 + z^2 = 1\}. \tag{1.19}$$

Define a chart (U, ϕ) with $\phi\colon U \to \mathbb{A}^2$ by $U = \{(x,y,z) \in X : x > 0\}$ and

$$\begin{aligned} \phi\colon U &\longrightarrow \mathbb{A}^2 \\ (x,y,z) &\longmapsto (y,z) \end{aligned} \tag{1.20}$$

The 2-sphere X is covered by six such charts; the domains are the open sets where $x > 0$, $x < 0$, $y > 0$, $y < 0$, $z > 0$, and $z < 0$.

Example 1.21 (dimension 2)**.** The classification theorem for surfaces states that there are two infinite families of compact connected topological 2-manifolds, the first family indexed by $\mathbb{Z}^{\geqslant 0}$ and the second by $\mathbb{Z}^{>0}$. The first family starts off with the surfaces $S^2, S^1 \times S^1$, the 2-sphere and the 2-torus. The next in the list is the *connected sum* of two copies of $S^1 \times S^1$, written

$$(S^1 \times S^1)^{\#2} = (S^1 \times S^1) \# (S^1 \times S^1). \tag{1.22}$$

It is formed by removing an open 2-disk from each torus and then gluing the two surfaces remaining along their boundary. For any $g \in \mathbb{Z}^{>0}$ there is a similarly constructed $(S^1 \times S^1)^{\#g}$. (By convention, if $g = 0$ the empty connected sum is the 2-sphere S^2.)

The second family begins with the real projective plane $\mathbb{RP}^2$, which is the projectivization $\mathbb{P}(\mathbb{R}^3)$ of 3-space, the space of lines through the origin of $\mathbb{R}^3$ with a suitable topology. A closely related description is to take S^2 as in (1.19) and let the cyclic group of order 2 act by the antipodal action $(x, y, z) \mapsto (-x, -y, -z)$. Then $\mathbb{RP}^2$ is the quotient space with the quotient topology. For each $g > 0$ we have the surface $(\mathbb{RP}^2)^{\#g}$, and together with the surfaces $(S^1 \times S^1)^{\#g}$ these exhaust the possible compact connected 2-manifolds. The surface $\mathbb{RP}^2 \# \mathbb{RP}^2$ is homeomorphic to the Klein bottle, and there is a homeomorphism $(\mathbb{RP}^2)^{\#3} \approx_{\text{homeo}} \mathbb{RP}^2 \# (S^1 \times S^1)$.

Example 1.23 (empty set)**.** The empty set $\emptyset$ satisfies the conditions of Definition 1.2: it is trivially locally Euclidean, Hausdorff, and second countable ('trivial' meaning 'nothing to check'). It is convenient to regard $\emptyset$ as a manifold of *any* dimension, and even to allow the dimension to be a negative integer.

Lecture 2: Smooth manifolds

I will sometimes use letters like '$M, N, \dots$' for manifolds and other times use '$X, Y, \dots$'.

(2.1) *C^∞ concepts/objects.* In this class I use the word 'smooth' synonymously with 'C^∞', which means infinitely differentiable. A smooth manifold, as defined below, is an abstract space on which one has "C^∞ concepts/objects". These are concepts/objects defined on open subsets of affine space which are invariant under C^∞ diffeomorphisms. (A C^∞ diffeomorphism $\varphi\colon U \to U'$ between open subsets of affine spaces is a bijective map such that both φ and φ^{-1} are C^∞.) They transport to C^∞ concepts/objects on smooth manifolds. These are the concepts/objects of main interest in differential topology. A non-obvious example is the concept of measure zero: a subset $S \subset U$ has measure zero iff its image under a C^∞ diffeomorphism has measure zero.

C^∞-related charts

Recall Definition 1.7 in which charts are defined.

Definition 2.2. Let M be a topological manifold. Let V, W be vector spaces and A, B affine spaces over V, W, respectively. Suppose (U, x) is an A-valued chart and (U', y) is a B-valued chart. We say (U', y) is C^∞*-related* to (U, x) if

$$y \circ x^{-1} \colon x(U \cap U') \longrightarrow B \tag{2.3}$$

is a C^∞ map.

The map (2.3) is called the *overlap* or *transition function*. We illustrate in Figure 2. A smooth manifold is built from open subsets of affine space, glued together by smooth transition functions.

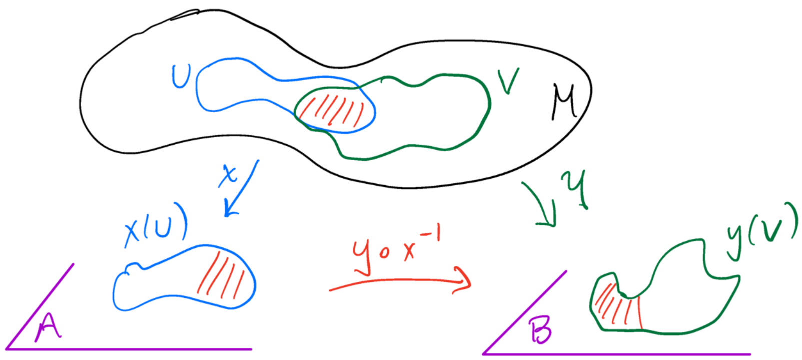


FIGURE 2. The transition function

Example 2.4. Let M be the 2-sphere with charts as defined in Example 1.18. As previously, let U be the chart where $x > 0$, and now let U' be the chart where $y > 0$. Let u, v be the coordinates in U and α, β the coordinates in V. Then the transition function is given by the formulas

$$\begin{aligned} \alpha &= \sqrt{1 - u^2 - v^2} \\ \beta &= v \end{aligned} \tag{2.5}$$

The domain of (2.5) is $\{(u, v) : u > 0 \text{ and } u^2+v^2 < 1\}$; the image is $\{(\alpha, \beta) : \alpha > 0 \text{ and } \alpha^2+\beta^2 < 1\}$.

Proposition 2.6. *There does not exist a covering of S^2 with a single chart.*

Proof. Suppose (S^2, x) is a chart, where $x \colon S^2 \to \mathbb{A}^2_{(x^1,x^2)}$ is a homeomorphism onto the open subset $x(S^2) \subset \mathbb{A}^2$. Then since S^2 is compact so too is $x(S^2)$, hence $x(S^2) \subset \mathbb{A}^2$ is closed and bounded. But then $x(S^2) \subset \mathbb{A}^2$ is open and closed and nonempty. Since $\mathbb{A}^2$ is connected, we conclude $x(S^2) = \mathbb{A}^2$. This contradicts the boundedness of $x(S^2)$. □

Review of calculus

I will rely on you to study the notes on multivariable analysis. The immediately relevant parts are: Lecture 1 (basic definitions), Lecture 2 (bases and coordinates), Lecture 3 (basic definitions), Lecture 5 (shapes and functions to end of lecture), Lecture 6, Lecture 7 (chain rule).

Atlases and differential structures

Remark 2.7. I defined a chart (Definition 1.7) to have values in an abstract affine space. That is convenient in practice—we will experience this for Grassmannian manifolds in the next lecture—and also conceptually: we can separate affine coordinates from the manifestation of the locally affine property of a manifold. But for a *maximal* atlas (Definition 2.9(2) below), we use charts with values in standard affine space, i.e., *standard charts*.

Remark 2.8. For a topological manifold, the collection of pairs (U, x) consisting of an open subset $U \subset M$ and a continuous map $x\colon U \to \mathbb{A}^n$ comprise a *set* $S(M)$, and we can define maximal subsets of $S(M)$. (We can and should let n vary as well.) It is more difficult to control if we replace $\mathbb{A}^n$ by an arbitrary affine space. Nonetheless, in Definition 2.9(2) we allow both general and standard charts, and maximality is formulated by reference to standard charts.

Definition 2.9. Let M be a topological manifold.

(1) An *atlas* on M is a collection $\mathscr{A} = \bigl\{(U_\sigma, x_\alpha)\bigr\}_{\alpha\in\mathcal{A}}$ of charts such that

   (i) $\bigcup\limits_{\alpha\in\mathcal{A}} U_\alpha = M$,

   (ii) for all $\alpha_1, \alpha_2 \in \mathcal{A}$ the charts $(U_{\alpha_1}, x_{\alpha_1})$ and $(U_{\alpha_2}, x_{\alpha_2})$ are C^∞-related.

(2) An atlas $\mathscr{A}$ is a *differential structure* if in addition

   (iii) $\mathscr{A}$ is maximal in the sense that if (U, x) is a standard chart which is C^∞-related to all $(U_\alpha, x_\alpha) \in \mathscr{A}$, then $(U, x) \in \mathscr{A}$.

(3) A *smooth manifold* is a pair $(M, \mathscr{A})$ consisting of a topological manifold M and a differential structure $\mathscr{A}$.

In practice we only need an atlas to define a smooth manifold; we do not need a maximal atlas. This is due to the following completion theorem.

Theorem 2.10. *Let M be a topological manifold and $\mathscr{A} = \{(U_\alpha, x_\alpha)\}_{\alpha\in A}$ an atlas on M. Define*

$$\overline{\mathscr{A}} = \mathscr{A} \;\cup\; \{(U, x) \text{ standard charts on } M : (U, x) \text{ is } C^\infty\text{-related to all charts in } \mathscr{A}\}. \tag{2.11}$$

Then $\overline{\mathscr{A}}$ is a maximal atlas on M, i.e., a differential structure.

Proof. This is a problem on Homework #2, so I won't spoil the fun. □

(2.12) *Useful picture of an atlas.* Let M be a topological manifold equipped with an atlas $\mathscr{A} = \{(U_\alpha, x_\alpha)\}_{\alpha\in A}$. In view of Theorem 2.10 this pair determines a smooth manifold. The atlas $\mathscr{A}$ on M defines a surjective map

$$\coprod_{\alpha\in\mathcal{A}} x_\alpha(U_\alpha) \longrightarrow M \tag{2.13}$$

The domain is a smooth manifold if the indexing set $\mathcal{A}$ is finite or countable. Any *surjective* map $f\colon X \to Y$ (of sets) defines an equivalence relation on the domain X; the map f is a quotient map for the equivalence relation which expresses the codomain Y as the set of equivalence classes in the domain X. Applied to (2.13) we "see" (Figure 3) a manifold as sewn together from opens in affines. The sewing maps are precisely the transition functions depicted in Figure 2.

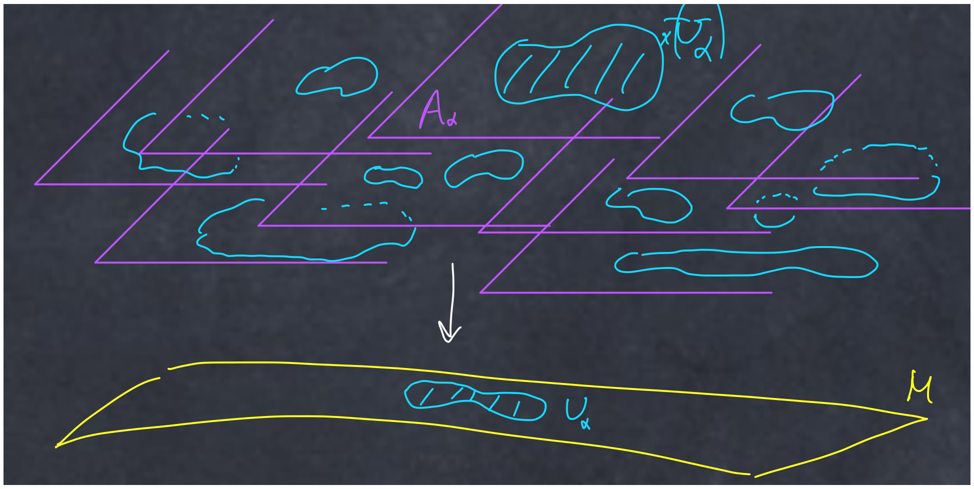


FIGURE 3. A quilt of open subsets of affine spaces

Lecture 3: Examples; smooth functions; tangent space

Examples of smooth manifolds

To specify a smooth manifold M we need to give a topological manifold together with an atlas $\mathscr{A}$. Theorem 2.10 shows that this data determines a smooth manifold.

Example 3.1 (affine space)**.** Let A be a finite dimensional real affine space. Then A is locally Euclidean, Hausdorff, and second countable in its usual topology, so it is a topological manifold. It admits an atlas with a single chart (A, id_A), which defines a smooth manifold structure. This smooth structure is understood when we treat A as a smooth manifold.

Remark 3.2. In fact, if $\dim A \neq 4$, then up to diffeomorphism this is the only smooth structure on A. By contrast, $\mathbb{A}^4$ (and therefore any 4-dimensional real affine space) admits infinitely many inequivalent smooth structures. In fact, some come in continuous families, so there are uncountably many. This rather shocking state of affairs was discovered in the early 1980's by Mike Freedman, who combined his own work on the 4-dimensional Poincaré conjecture for topological manifolds with Simon Donaldson's thesis on smooth 4-manifolds. Freedman proved the existence of a single exotic smooth structure on $\mathbb{A}^4$. Soon after, Bob Gompf constructed infinitely many.

Example 3.3 (sphere)**.** We treated the 2-dimensional sphere in Example 1.18 and Example 2.4. Define the n-sphere as the unit sphere in affine space:

$$S^n = \{(x^0, x^1, \dots, x^n) \in \mathbb{A}^{n+1} : (x^0)^2 + \cdots + (x^n)^2 = 1\}. \tag{3.4}$$

Cover S^n with $2(n+1)$ charts: for each $0 \leqslant i \leqslant n$ there is a chart with domain[1] $\{x^i > 0\} \cap S^n$ and a chart with domain $\{x^i < 0\} \cap S^n$. Then as in Example 2.4 you can check that all overlap functions are C^∞.

Remark 3.5. There is an atlas of S^n with 2 charts; the domain of each is the complement of a single point in S^n. The coordinate functions are constructed via stereographic projection.

Example 3.6 (disjoint unions)**.** Let $\mathcal{A}$ be a finite or countable set and $\{M_\alpha\}_{\alpha\in\mathcal{A}}$ a collection of smooth manifolds. Then the disjoint union

$$M = \bigsqcup_{\alpha\in\mathcal{A}} M_\alpha \tag{3.7}$$

has a natural smooth structure: an atlas on M is constructed as the union of atlases on each M_α.

Example 3.8 (Cartesian product)**.** Let $\mathcal{A}$ be a finite set and $\{M_\alpha\}_{\alpha\in\mathcal{A}}$ a collection of smooth manifolds. Then the Cartesian product

$$M = \bigtimes_{\alpha\in\mathcal{A}} M_\alpha \tag{3.9}$$

has a natural smooth structure. Given atlases $\mathscr{A}_\alpha$, $\alpha \in \mathcal{A}$, one obtains an atlas on M by taking Cartesian products of charts. (The Cartesian product of maps $x_\alpha\colon U_\alpha \to A_\alpha$ is a map $\bigtimes_\alpha x_\alpha\colon \bigtimes_\alpha U_\alpha \to \bigtimes_\alpha A_\alpha$.)

So a finite product of spheres, such as the torus $S^1 \times \cdots \times S^1$ (n factors for any $n \in \mathbb{Z}^{>0}$) is a smooth manifold.

Example 3.10 (open subset)**.** Let M be a smooth manifold and $N \subset M$ an open subset. Then N is a topological manifold and it inherits an atlas from an atlas $\mathscr{A}_M$ of M. Namely, for each chart $(U, x) \in \mathscr{A}_M$ we introduce a chart $(U \cap N, x\big|_{U\cap N})$ on N. These have C^∞ overlaps and comprise an atlas $\mathscr{A}_N$ of N.

Remark 3.11. We will soon develop tools for proving that certain closed subsets of smooth manifolds are smooth manifolds.

Example 3.12 (general linear group)**.** As a special case of the preceding, let $M_n\mathbb{R}$ be the vector space of real $n \times n$ matrices. By Example 3.1 it has a natural smooth manifold structure. (A vector space has a canonical affine space structure.) Let $GL_n\mathbb{R} \subset M_n\mathbb{R}$ be the subset of invertible matrices. It is an open subset, since it is the inverse image of the open subset $\mathbb{R}^{\neq 0} \subset \mathbb{R}$ under the continuous determinant map $M_n\mathbb{R} \to \mathbb{R}$. Hence it too is a smooth manifold.

[1]The notation '$\{x^i > 0\}$' is shorthand for the open subset $\{(x^0, \dots, x^n) \in \mathbb{A}^{n+1} : x^i > 0\}$ of affine $(n+1)$-space.

The Grassmannian manifold

We introduce a more abstract manifold, one which does not come to us embedded in affine space. The natural charts take values in affine spaces which are not standard affine space, and indeed they vary from chart to chart.

(3.13) *Definition of the Grassmannian.* Let V be a real vector space of dimension n. (The same construction works for complex or quaternionic vector spaces.) Fix $k \in \{0, \dots, n\}$. Define the *Grassmannian* as the set

$$\operatorname{Gr}_k(V) = \bigl\{W \subset V \text{ subspaces of dimension } k\bigr\}. \tag{3.14}$$

For $k = 1$ the Grassmannian is called the *projectivization* of V, and this *projective space* is denoted

$$\mathbb{P}V = \operatorname{Gr}_1(V). \tag{3.15}$$

So far $\operatorname{Gr}_k(V)$ is a set. We simultaneously construct a topology and an atlas.

(3.16) *Charts and an atlas.* For each $X \in \operatorname{Gr}_{n-k}(V)$—that is, for each subspace $X \subset V$ of dimension $(n-k)$—define

$$\begin{aligned} V_X &= \operatorname{Hom}(V/X, X) \\ A_X &= \bigl\{W \in \operatorname{Gr}_k(V) : W \cap X = 0\bigr\}. \end{aligned} \tag{3.17}$$

We define on A_X the structure of an affine space over the vector space V_X. Namely, any $W \in A_X$ is a linear complement to X. Equivalently, $V = W \oplus X$. Or, in another formulation, the restriction of the quotient map $V \to V/X$ to W is an isomorphism $\theta_W \colon W \to V/X$. Then given $T \in V_X$, define $W + T \in A_X$ to be the graph of the linear map $T \circ \theta_W \colon W \to X$. This graph is a subspace of $W \oplus X = V$ of dimension k, and it intersects X in the zero vector. The reader can easily check that this defines a simply transitive action of V_X on A_X. Choose a finite set $\{X_i\}_{i \in \{1,\dots,N\}} \subset \operatorname{Gr}_{n-k}(V)$ so that $A_{X_1} \cup \dots \cup A_{X_N} = \operatorname{Gr}_k(V)$. (For example, choose a basis $e_1, \dots, e_n$ of V and take the spans of all cardinality $(n-k)$ subsets of the basis.) The surjective map

$$\bigsqcup_{i=1}^{N} A_{X_i} \longrightarrow \operatorname{Gr}_k(V) \tag{3.18}$$

induces the quotient topology on $\operatorname{Gr}_k(V)$. Then for all $X \in \operatorname{Gr}_{n-k}(V)$ the pair $(A_X, \operatorname{id}_{A_X})$ is a chart on $\operatorname{Gr}_k(V)$. We claim that the overlap functions are smooth. Fix $X, Y \in \operatorname{Gr}_{n-k}(V)$, and choose $W_0 \in A_X \cap A_Y$. Using W_0 as a basepoint we identify the affine space A_X with the vector space $\operatorname{Hom}(W_0, X)$ and similarly identify A_Y with $\operatorname{Hom}(W_0, Y)$. Let $\Phi \colon W_0 \oplus X \to W_0 \oplus Y$ be the linear map transported from id_V under the identifications $V \cong W_0 \oplus X \cong W_0 \oplus Y$. Then the transition function, defined on a subset $U \subset \mathrm{Hom}(W_0, X)$ is

$$\begin{array}{rlllllll} U & \longrightarrow & \mathrm{Hom}(W_0, W_0 \oplus X) & \longrightarrow & \mathrm{Hom}(W_0, W_0 \oplus Y) & \longrightarrow & \mathrm{Hom}(W_0, Y) \\ T & \longmapsto & \mathrm{id}_{W_0} \oplus T & \longmapsto & \Phi \circ (\mathrm{id}_{W_0} \oplus T) & \longmapsto & \pi \circ \Phi \circ (\mathrm{id}_{W_0} \oplus T) \end{array} \tag{3.19}$$

where $\pi\colon W_0 \oplus Y \to Y$ is projection. The map (3.19) is a composition of linear maps, hence is smooth. Therefore,

$$\mathscr{A} = \{(A_X, \mathrm{id}_{A_X})\}_{X \in \mathrm{Gr}_{n-k}(V)} \tag{3.20}$$

is an atlas on $\mathrm{Gr}_k(V)$.

(3.21) *A cellular decomposition.* The Grassmannian has a rich geometric structure. We illustrate one aspect—a decomposition into "cells", which in this case are affine spaces—for the projective space $\mathbb{P}V = \mathrm{Gr}_1(V)$ of an n-dimensional vector space. Any $X_1 \in \mathrm{Gr}_{n-1}(V)$ determines a partition

$$\mathbb{P}V = A^{(n-1)}_{X_1} \amalg \mathbb{P}X_1. \tag{3.22}$$

The superscript is the dimension of the affine space. Now choose $X_2 \in \mathrm{Gr}_{n-2}(X_1)$ to obtain

$$\mathbb{P}V = A^{(n-1)}_{X_1} \amalg A^{(n-2)}_{X_2} \amalg \mathbb{P}X_2. \tag{3.23}$$

Continuing we partition the projective space $\mathbb{P}V$ into a disjoint union of affine spaces of dimensions $0, 1, \dots, n-1$. Note that $A^{(n-1)}_{X_1}$ is dense, so projective space is a compactification of affine space.

Smooth maps

Any C^∞ concept on open subsets of affine space transports to smooth manifolds using an atlas. One need only define/check in a single chart since the charts in an atlas are C^∞ related. We apply this principle to define a smooth function between smooth manifolds.

Definition 3.24. Let M, N be smooth manifolds and $f\colon M \to N$. Fix $p \in M$. Then f *is smooth at* p if there exists a chart (U_α, x_α) about p and a chart (V_β, y_β) about $f(p)$ so that the composite function $y_\beta \circ f \circ x_\alpha^{-1}$ is C^∞ at $x_\alpha(p) \in x(U)$.

If the charts take values in affine spaces A, B, respectively, then

$$y_\beta \circ f \circ x_\alpha^{-1}\colon x_\alpha\big(U_\alpha \cap f^{-1}(V_\beta)\big) \longrightarrow B, \tag{3.25}$$

which is a function from an open subset of the affine space A to the affine space B, and its smoothness is defined using standard multivariable calculus. It suffices to check smoothness in one pair of charts, since the answer is the same no matter which pair is chosen, as in the following.

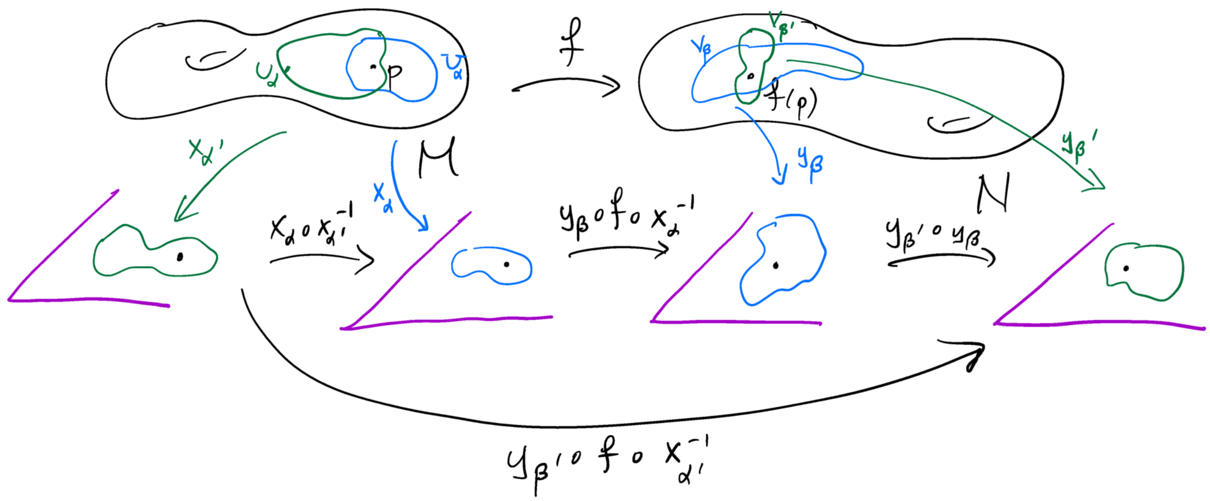


FIGURE 4. Checking smoothness of f at p in different charts

Lemma 3.26. *If in Definition 3.24 we choose different charts* $(U_{\alpha'}, x_{\alpha'})$ *and* $(V_{\beta'}, y_{\beta'})$*, then the local representation of* f *with respect to these charts is smooth at* $x_{\alpha'}(p)$ *iff the local representation* (3.25) *is smooth at* $x_\alpha(p)$*.*

See Figure 4 for an illustration.

Proof. Observe

$$y_{\beta'} \circ f \circ x_{\alpha'}^{-1} = (y_{\beta'} \circ y_\beta^{-1}) \circ (y_\beta \circ f \circ x_\alpha^{-1}) \circ (x_\alpha \circ x_{\alpha'}^{-1}), \tag{3.27}$$

and the overlap functions $y_{\beta'} \circ y_\beta^{-1}$ and $x_\alpha \circ x_{\alpha'}^{-1}$ are smooth. Now apply the chain rule. □

Example 3.28. Consider the antipodal map $f\colon S^2 \to S^2$. Regarding $S^2 \subset \mathbb{A}^3_{x,y,z}$ as usual, f is the restriction of the automorphism $(x,y,z) \mapsto (-x,-y,-z)$ of $\mathbb{A}^3$. Let $p = (1/\sqrt{2}, 1/\sqrt{2}, 0)$ and choose the charts $U_\alpha = \{x > 0\}$, $V_\beta = \{y < 0\}$ of the type in Example 3.3. Use coordinates u, v and u', v' in the charts, which are then given by the maps

$$x_\alpha : \begin{cases} u = y \\ v = z \end{cases} \qquad\qquad y_\beta : \begin{cases} u' = x \\ v' = z \end{cases} \tag{3.29}$$

So the local expression $y_\beta \circ f \circ x_\alpha^{-1}$ is

$$\begin{aligned} u' &= -\sqrt{1-u^2-v^2} \\ v' &= -v \end{aligned} \tag{3.30}$$

which is a smooth function.

The proper notion of isomorphism for smooth manifolds is the following.

Definition 3.31. Let M, N be smooth manifolds and $f\colon M \to N$ a smooth map. Then f is a *diffeomorphism* if f is bijective and f^{-1} is smooth.

Example 3.32. Consider two different differential structures on $\mathbb{R}$, each defined by an atlas with a single chart with domain $\mathbb{R}$. Let $\mathscr{A}_1 = \{(\mathbb{R}, \mathrm{id}_{\mathbb{R}})\}$ and $\mathscr{A}_2 = \{(\mathbb{R}, \phi)\}$, where $\phi(x) = x^3$. Then the map $x \mapsto x^{1/3}$ is a diffeomorphism $(\mathbb{R}, \mathscr{A}_1) \to (\mathbb{R}, \mathscr{A}_2)$. Observe that the charts $(\mathbb{R}, \mathrm{id}_{\mathbb{R}})$ and $(\mathbb{R}, \phi)$ are not C^∞-compatible, so cannot appear in the same differential structure.

Tangent space

Let V be a vector space and let A be an affine space over V. Then the tangent space to A at any $p \in A$ is the vector space V. A smooth manifold M is a pasting of open subsets of affine space. We linearize the pasting to define the tangent space at a point $p \in M$.

(3.33) *Direct product of vector spaces.* Let $\{V_\alpha\}_{\alpha\in\mathcal{A}}$ be a collection of vector spaces indexed by a set $\mathcal{A}$. Then the *direct product* vector space $\prod\limits_{\alpha\in\mathcal{A}} V_\alpha$ is the Cartesian product of the sets V_α with componentwise addition and scalar multiplication; the zero vector is the componentwise zero vector. This gives $\prod\limits_{\alpha\in\mathcal{A}} V_\alpha$ the structure of a vector space. An element of the direct product is denoted

$$\xi = \{\xi_\alpha\} \in \prod_{\alpha\in\mathcal{A}} V_\alpha; \tag{3.34}$$

the α-component of ξ is ξ_α. The vector sum is defined by the formula

$$(\xi + \eta)_\alpha = \xi_\alpha + \eta_\alpha, \tag{3.35}$$

which may also be written

$$\{\xi_\alpha\} + \{\eta_\alpha\} = \{\xi_\alpha + \eta_\alpha\}. \tag{3.36}$$

The zero vector is $\{0_\alpha\}$, where 0_α is the zero vector in V_α.

(3.37) *Tangent space to a smooth manifold.* Now let M be a smooth manifold with atlas $\mathscr{A} = \{(U_\alpha, x_\alpha)\}_{\alpha\in\mathcal{A}}$. The coordinate function $x_\alpha\colon U_\alpha \to A_\alpha$ has codomain an affine space A_α with a vector space V_α of translations. For $p \in M$ let $\mathcal{A}_p \subset \mathcal{A}$ be the set of indices $\alpha \in \mathcal{A}$ such that $p \in U_\alpha$.

Definition 3.38. The *tangent space* T_pM is the subspace of the direct product $\prod\limits_{\alpha\in\mathcal{A}_p} V_\alpha$ consisting of vectors $\xi = \{\xi_\alpha\}$ such that

$$\xi_{\alpha''} = d(x_{\alpha''} \circ x_{\alpha'}^{-1})_{x_{\alpha'}(p)}(\xi_{\alpha'}) \tag{3.39}$$

for all $\alpha', \alpha'' \in \mathcal{A}_p$.

Since (3.39) is a linear equation, it indeed defines a linear subspace of the direct product.

Lemma 3.40. *For each $\alpha \in \mathcal{A}_p$, projection onto the α-component is an isomorphism*

$$j_\alpha \colon T_pM \longrightarrow V_\alpha. \tag{3.41}$$

Proof. Fix $\alpha \in \mathcal{A}_p$. We first prove that j_α is injective. Suppose $\xi = \{\xi_{\alpha'}\}_{\alpha' \in \mathcal{A}_p} \in T_pM$ and assume $\xi_\alpha = j_\alpha(\xi) = 0_\alpha$. Then from (3.39) we deduce that $\xi_{\alpha'} = 0_{\alpha'}$ for all $\alpha' \in \mathcal{A}_p$. Hence $\xi = \{0_{\alpha'}\}_{\alpha' \in \mathcal{A}_p}$ is the zero vector.

Next, we prove that j_α is surjective. Suppose $\xi_\alpha \in V_\alpha$ is given. For each $\alpha' \in \mathcal{A}_p$, define $\xi_{\alpha'} \in V_{\alpha'}$ using (3.39):

$$\xi_{\alpha'} = d(x_{\alpha'} \circ x_\alpha^{-1})_{x_\alpha(p)}(\xi_\alpha) \tag{3.42}$$

We claim that $\xi = \{\xi_{\alpha'}\}_{\alpha' \in \mathcal{A}_p}$ lies in $T_pM \subset \prod\limits_{\alpha' \in \mathcal{A}_p} V_{\alpha'}$. Namely, if $\alpha', \alpha'' \in \mathcal{A}_p$, then

$$\begin{aligned} \xi_{\alpha''} &= d(x_{\alpha''} \circ x_\alpha^{-1})_{x_\alpha(p)}(\xi_\alpha) \\ &= d(x_{\alpha''} \circ x_{\alpha'}^{-1})_{x_{\alpha'}(p)} \circ d(x_{\alpha'} \circ x_\alpha^{-1})_{x_\alpha(p)}(\xi_\alpha) \\ &= d(x_{\alpha''} \circ x_{\alpha'}^{-1})_{x_{\alpha'}(p)}(\xi_{\alpha'}), \end{aligned} \tag{3.43}$$

where we use the chain rule to pass to the final equality. Finally,

$$j_\alpha(\xi) = \xi_\alpha = d(x_\alpha \circ x_\alpha^{-1})_{x_\alpha(p)}(\xi_\alpha) = \xi_\alpha. \tag{3.44}$$

□

(3.45) *Local framing in a standard chart.* Fix $\alpha \in \mathcal{A}_p$, let n be the dimension of M at p, and suppose the chart (U_α, x_α) takes values in standard affine space $\mathbb{A}^n$ with vector space of translations $\mathbb{R}^n$. Let $e_1, e_2, \dots, e_n$ denote the standard basis of $\mathbb{R}^n$. Define the basis

$$\left.\frac{\partial}{\partial x_\alpha^1}\right|_p, \left.\frac{\partial}{\partial x_\alpha^2}\right|_p, \dots, \left.\frac{\partial}{\partial x_\alpha^n}\right|_p \tag{3.46}$$

of T_pM by the formula

$$j_\alpha\left(\left.\frac{\partial}{\partial x_\alpha^i}\right|_p\right) = e_i, \qquad i = 1, 2, \dots, n. \tag{3.47}$$

We often omit '$\big|_p$' from the notation.

(3.48) *The differential of a smooth map.* Now let $f\colon M \to N$ be a smooth map. Define for each $p \in M$ the differential

$$df_p\colon T_pM \longrightarrow T_{f(p)}N, \tag{3.49}$$

a linear map from the tangent space to M at p to the tangent space to N at $f(p)$, as follows. Use the notation as above on M, and let $\mathscr{B} = \{(V_\beta, y_\beta)\}_{\beta\in\mathcal{B}}$ denote an atlas on N, and for a point $q \in N$ let $\mathcal{B}_q \subset \mathcal{B}$ the subset of indices β for which $q \in V_\beta$. Then the differential is defined by

$$df_p\bigl(\{\xi_\alpha\}_{\alpha\in\mathcal{A}_p}\bigr) = \{\eta_\beta\}_{\beta\in\mathcal{B}_{f(p)}}, \tag{3.50}$$

where $\{\xi_\alpha\}_{\alpha\in\mathcal{A}_p} \in T_pM$ and we for any $\alpha \in \mathcal{A}_p$ we set

$$\eta_\beta = d(y_\beta \circ f \circ x_\alpha^{-1})_{x_\alpha(p)}(\xi_\alpha). \tag{3.51}$$

Lemma 3.52. *The value of η_β in (3.51) is independent of $\alpha \in \mathcal{A}_p$, and $\{\eta_\beta\}_{\beta\in\mathcal{B}_{f(p)}} \in T_{f(p)}N$.*

Proof. If $\alpha, \alpha' \in \mathcal{A}_p$, then

$$\begin{aligned} d(y_\beta \circ f \circ x_{\alpha'}^{-1})_{x_{\alpha'}(p)}(\xi_{\alpha'}) &= d(y_\beta \circ f \circ x_{\alpha'}^{-1})_{x_{\alpha'}(p)} \circ d(x_{\alpha'} \circ x_\alpha^{-1})(x_\alpha) \\ &= d(y_\beta \circ f \circ x_\alpha^{-1})_{x_\alpha(p)}(\xi_\alpha). \end{aligned} \tag{3.53}$$

Choose $\alpha \in \mathcal{A}_p$. For $\beta', \beta'' \in \mathcal{B}_{f(p)}$ we have

$$\begin{aligned} \eta_{\beta''} &= d(y_{\beta''} \circ f \circ x_\alpha^{-1})_{x_\alpha(p)}(\xi_\alpha) \\ &= d(y_{\beta''} \circ y_{\beta'}^{-1}) \circ d(y_{\beta'} \circ f \circ x_\alpha^{-1})_{x_\alpha(p)}(\xi_\alpha) \\ &= d(y_{\beta''} \circ y_{\beta'}^{-1})(\eta_{\beta'}), \end{aligned} \tag{3.54}$$

so from Definition 3.38 we conclude that $\{\xi_\beta\} \in T_{f(p)}N$. ☐

We summarize the definition of the differential in the *commutative diagram*[2]

$$\begin{array}{ccc} T_pM & \xrightarrow{\;df_p\;} & T_{f(p)}N \\ {\scriptstyle j_\alpha}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle j_\beta} \\ V_\alpha & \xrightarrow{\;d(y_\beta\circ f\circ x_\alpha^{-1})\;} & W_\beta \end{array} \tag{3.55}$$

Here V_α is the vector space of translations in the codomain of the chart (U_α, x_α) on M, and W_β is the vector space of translations in the codomain of the chart (V_β, y_β) on N.[3]

[2]A commutative diagram is one in which compositions of arrows with the same initial and final nodes agree.

[3]There is an awful notational conflict between the 'V' in the charts on N and the 'V' which is the vector space of translations underlying the codomains of the charts on M. I hope to find a palatable alternative.

(3.56) *Frames (bases) in a chart.* Apply the forgoing to the coordinate functions. Thus suppose (U, x) is a chart on M with values in standard affine space. Then $x = (x^1, \dots, x^n)\colon U \to \mathbb{A}^n$ is a smooth function defined on the open set U, and we compute its differential. (The domain of x is a manifold: recall that any open subset U of a manifold M inherits a manifold structure.) For the codomain $\mathbb{A}^n$ we use the identity chart, and for the domain U we use the chart (U, x). Then in this chart the map $f = x$ is represented by the composition $f \circ x^{-1}$, which is the identity map on $x(U) \subset \mathbb{A}^n$. Its differential is the identity map of $\mathbb{R}^n$. Writing this in terms of the component functions of x, note that the differential dx^i of the i^{th} component function is a linear map $T_pU = T_pM \to \mathbb{R}$, so an element of the dual space T_p^*M to T_pM. Then using the natural basis (3.46), the fact that the differential is the identity map is the equation

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases} \tag{3.57}$$

In other words, the differentials $dx^1, \dots, dx^n$ are the dual basis to the basis $\partial/\partial x^1, \dots, \partial/\partial x^n$.

Remark 3.58. For a standard chart one should first define the basis $dx^1, \dots, dx^n$ of the *cotangent space* T_p^*M using the definition of the differential. In fact, functions $x^1, \dots, x^n\colon U \to \mathbb{R}$ defined on an open subset $U \subset M$ of a smooth manifold M are the coordinate functions of a standard chart if together they define a homoeomorphism onto their image in $\mathbb{A}^n$ and the differentials $dx^1_p, \dots, dx^n_p$ form a basis of T_p^*M for all $p \in U$. With $dx^1, \dots, dx^n$ in hand, define $\partial/\partial x^1, \dots, \partial/\partial x^n$ to be the dual basis of the tangent space T_pM.

Lecture 4: More on tangent vectors and differentials

The material in the notes on Multivariable Analysis, specifically pp. 14–15, 32–35, 66–67 are relevant to this lecture.

In this lecture we begin with a review of the linear algebra of duality. Linear algebra is an important foundation for this course, so it is worth spending time reviewing/learning as we go along. We then give a few other views of tangent vectors: as motion germs and as derivations. In the first we map open intervals into a smooth manifold M; in the second we map in the other direction (infinitesimally). These dual ways of probing a smooth manifold—by maps in and by maps out—are useful, often in juxtaposition. We conclude by illustrating computational technique, with which you should develop some facility to the point you don't have to rethink the definitions each time.

Linear algebra preliminaries

(4.1) *The dual space.* Let V, W be finite dimensional real vector spaces. The space of linear maps

$$\mathrm{Hom}(V,W) = \{T\colon V \to W :\ T \text{ is linear}\} \tag{4.2}$$

is a vector space: $(T_1 + T_2)(\xi) = T_1\xi + T_2\xi$ for all $T_1, T_2 \in \mathrm{Hom}(V,W)$ and $\xi \in V$. The (abstract) *dual space* is a special case:

$$V^* = \mathrm{Hom}(V, \mathbb{R}). \tag{4.3}$$

Remark 4.4. Suppose V, V' are finite dimensional real vector spaces and

$$B\colon V \times V' \longrightarrow \mathbb{R} \tag{4.5}$$

is a bilinear map. We say B is nondegenerate if the linear maps

$$\begin{aligned} V &\longrightarrow (V')^* \\ \xi &\longmapsto \big(\xi' \mapsto B(\xi,\xi')\big) \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} V' &\longrightarrow V^* \\ \xi' &\longmapsto \big(\xi \mapsto B(\xi,\xi')\big) \end{aligned} \tag{4.7}$$

are isomorphisms. Then B exhibits a duality between V and V'.

(4.8) *Dual bases.* Suppose $\dim V = n$ and $e_1, \dots, e_n$ is a basis of V. The *dual basis* $e^1, \dots, e^n$ of V^* is defined by

$$e^i(e_j) = \delta^i_j = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases} \tag{4.9}$$

On each side of the equality the indices i, j do not repeat, so there is no sum. The compact notation in (4.9) represents n^2 equations, one for each choice of a pair $i, j \in \{1, \dots, n\}$. In the context of Remark 4.4, the dual basis $e'_1, \dots, e'_n$ of V' is characterized by

$$B(e_i, e'_j) = \delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases} \tag{4.10}$$

For the standard vector space $\mathbb{R}^n$ a vector $\xi = (\xi^1, \dots, \xi^n)$ is represented by a column vector. The dual space $(\mathbb{R}^n)^*$ consists of linear functionals $\omega = (\omega_1, \dots, \omega_n)$ which are represented as row vectors. (Recall that a collection of real numbers A^i_j with one superscript and one subscript are organized into a matrix in which the upper index is the row number and the lower index the column number.) The pairing $\omega(\xi) \in \mathbb{R}$ is computed as the product of a row vector and a column vector:

$$\omega(\xi) = \omega_i \xi^i = \begin{pmatrix} \omega_1 & \cdots & \omega_n \end{pmatrix} \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^n \end{pmatrix}. \tag{4.11}$$

(4.12) *The dual of a linear map.* Let W be another vector space and $T\colon V \to W$ a linear map. There is an induced *dual* or *pullback* or *transpose* linear map $T^*\colon W \to V$ defined by

$$T^*(w^*)(v) = w^*(Tv), \qquad v \in V, \quad w^* \in W^*. \tag{4.13}$$

Each side of (4.13) is a real number.

Tangent vectors as motion germs

(4.14) *Motion germs in affine space.* Let A be an affine space over a vector space V. A *local motion* or *local parametrized curve* in A is, for $\delta > 0$ a smooth function $\hat\gamma\colon (-\delta, \delta) \to A$. Its *initial position* is the point $\hat p = \hat\gamma(0) \in A$ and its *initial velocity* is the vector

$$\hat\xi = \hat\gamma'(0) = \lim_{h\to 0} \frac{\hat\gamma(h) - \hat p}{h} \quad \in V. \tag{4.15}$$

Define two local motions in A to be equivalent if their initial positions and initial velocities agree. An equivalence class is a *motion germ* with well-defined position and velocity.[4]

Remark 4.16. There is a distinguished *affine motion* $t \mapsto \hat p + t\hat\xi$ in each equivalence class.

Let $U \subset A$ be an open set, B is an affine space, and $f\colon U \to B$ a smooth function. Then if $\hat p \in U$, $\hat\xi \in V$, and $\hat\gamma\colon (-\delta, \delta) \to U$ represents a motion germ with position $\hat p$ and velocity $\hat\xi$,

$$\hat\xi f(\hat p) = df_{\hat p}(\hat\xi) = \left.\frac{d}{dt}\right|_{t=0} f\bigl(\hat\gamma(t)\bigr) \tag{4.17}$$

by the chain rule.

(4.18) *Motion germs on a smooth manifold.* Let M be a smooth manifold. Define a local motion $\gamma\colon (-\delta, \delta) \to M$ and its initial position $p = \gamma(0)$ as in affine space. To define an equivalence relation on local motions with initial position p, choose a chart (U, x) about p with values in an affine space A, and transport the equivalence relation on the corresponding A-valued local motions $\hat\gamma = x \circ \gamma$. Smoothness of overlap functions shows, via the chain rule, that the equivalence relation is independent of the chart; see Lemma 3.52 and its proof for a similar assertion. The velocity of a motion germ represented by γ is an element of T_pM; it is the transport of the velocity of $\hat\gamma = x \circ \gamma$ using the inverse of the isomorphism (3.41). Again one can check that it is independent of the chart. The tangent vector represented by a motion germ is

$$\xi = \dot\gamma(0) = \left.\frac{d}{dt}\right|_{t=0} \gamma(t) = d\gamma_0\Bigl(\frac{d}{dt}\Bigr) \quad \in T_pM. \tag{4.19}$$

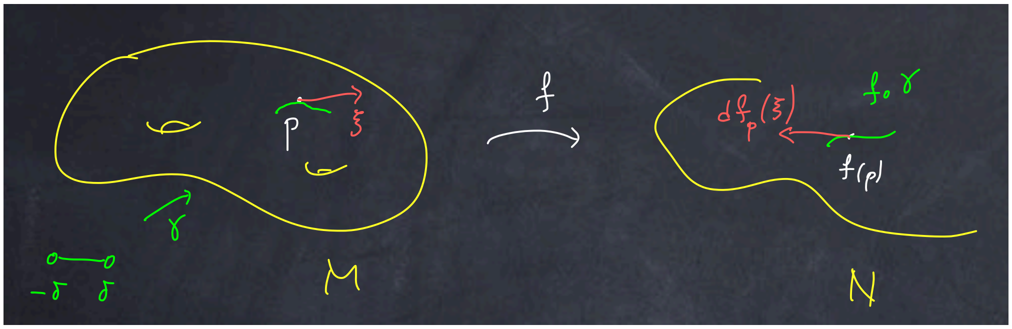


FIGURE 5. The differential as a map of local motions

Remark 4.20. The differential $d\gamma_0$ is defined in (**3.48**), so (4.19) does not depend on the statements about an equivalence relation on local motions in M.

If $f\colon M \to N$ is a smooth map, then one can compute its differential on tangent vectors as the induced map on motion germs; see Figure 5. Namely, given $p \in M$ and $\xi \in T_pM$, find a local motion $\gamma\colon (-\delta, \delta) \to M$ with initial position p and initial velocity ξ. Then $df_p(\xi)$ is represented by the motion germ $f \circ \gamma\colon (-\delta, \delta) \to N$. The chain role follows easily using motion germs; see Figure 6.

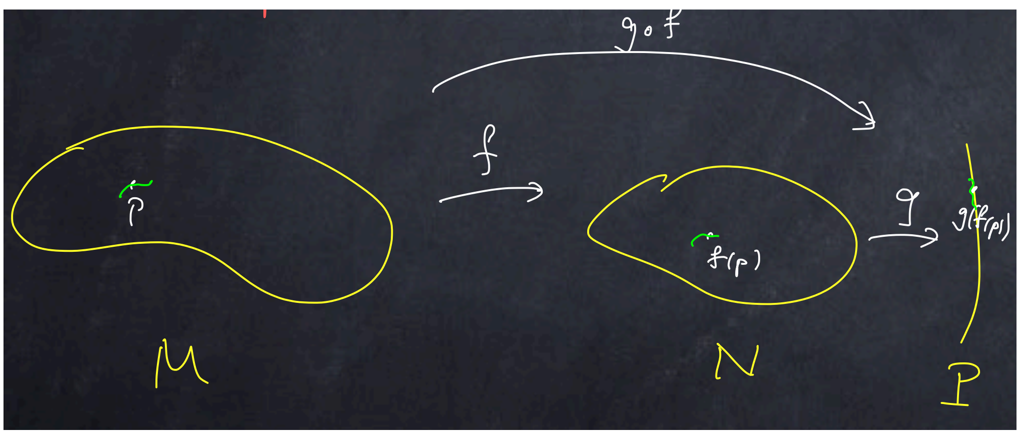


FIGURE 6. The chain rule as composition of motion germs

The cotangent space and the ring of C^∞ functions

Let M be a smooth manifold and $p \in M$.

Definition 4.21. The *cotangent space* to M at p is $T_p^*M = (T_pM)^*$, the dual to the tangent space.

If $f\colon M \to \mathbb{R}$ is a smooth function, then its differential at p is a linear map $df_p\colon T_pM \to \mathbb{R}$. In other words, the differential $df_p \in T_p^*M$ is an element of the cotangent space.

Remark 4.22. It is worth contemplating df, which is a function $p \mapsto df_p$ from M to. . . where?

[4]More properly it is a C^1 *motion germ*. There are higher order C^k and C^∞ motion germs.

Let $C^\infty(M)$ denote the ring of smooth real-valued functions on M. Addition and multiplication are inherited from the ring structure of $\mathbb{R}$. We have not yet proved that $C^\infty(M)$ contains more than locally constant functions; we will do so when we study partitions of unity. Then we will prove that about each point $p \in M$ there exist functions $x^1, \dots, x^n \in C^\infty(M)$ whose differentials at p form a basis $dx^1_p, \dots, dx^n_p$ of the cotangent space T^*_pM. This leads to a definition of a smooth structure on a topological manifold in terms of the ring of smooth functions. We remark that from this point of view the cotangent space is more fundamental and the tangent space is defined to be the dual space to the cotangent space. That world order is reflected in practice as we do computations, as illustrated in the next section.

Tangent vectors as derivations

The most basic derivative operation, out of which the differential and higher derivatives are built, is the directional derivative. Here we interpret the directional derivative on a smooth manifold as a derivation on smooth function.

Definition 4.23. Let A be an algebra over $\mathbb{R}$. A ($\mathbb{R}$-valued) *derivation* on A is a linear map

$$\mathcal{D}\colon A \longrightarrow \mathbb{R} \tag{4.24}$$

which satisfies

$$\mathcal{D}(a_1a_2) = \mathcal{D}(a_1)a_2 + a_1\mathcal{D}(a_2), \qquad a_1, a_2 \in A. \tag{4.25}$$

Equation (4.25) is an abstract formulation of the Leibniz rule. A derivation is a general first derivative operator. One can generalize this definition to maps between A-modules, a form in which we encounter it later when we study differential forms.

Let M be a smooth manifold, and suppose $\xi \in T_pM$ is a tangent vector at some point $p \in M$. Define

$$\begin{aligned} \mathcal{D}_\xi\colon C^\infty(M) &\longrightarrow \mathbb{R} \\ f &\longmapsto \xi f(p) \end{aligned} \tag{4.26}$$

where $\xi f(p) = df_p(\xi)$ is the directional derivative of f in the direction ξ at p. Then $\mathcal{D}_\xi$ is a derivation.

Remark 4.27. Conversely, every derivation $\mathcal{D}\colon C^\infty(M) \to \mathbb{R}$ is of the form $\mathcal{D} = \mathcal{D}_\xi$ for some $p \in M$ and $\xi \in T_pM$, though we do not yet have enough tools to prove this statement.

Local computations in coordinate charts

(4.28) *The moving frame induced by a chart.* Let (U, ϕ) be a standard chart on a smooth manifold M, so $\phi\colon U \to \mathbb{A}^n$ for some open set $U \subset M$ and a positive integer n. Fix $p \in U$. Denote the coordinate functions as $x^i\colon U \to \mathbb{R}$.

Lemma 4.29. *The differentials $dx^1_p, \dots, dx^n_p$ form a basis of T^*_pM.*

Proof. It suffices to check in any chart at p, so we check in the chart (U, ϕ). But in that chart the function x^i is the standard affine coordinate functions with differential $\begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \end{pmatrix} \in (\mathbb{R}^n)^*$. These form the standard basis. ☐

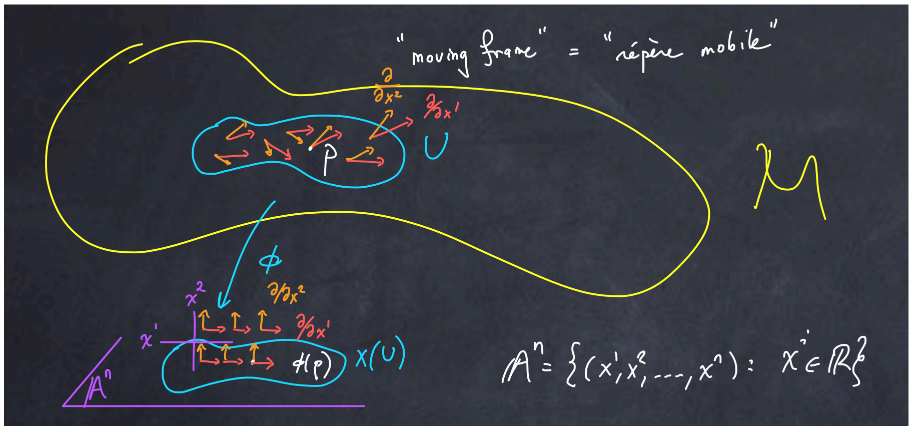


FIGURE 7. The moving frame of a local coordinate system

Introduce the dual basis

$$\left.\frac{\partial}{\partial x^1}\right|_p, \;\cdots\;, \left.\frac{\partial}{\partial x^n}\right|_p \tag{4.30}$$

of T_pM. As p varies each $\partial/\partial x^i$ is a *vector field* on U. This basis of vector fields, or vector fields of bases, is called a *moving frame* or *repère mobile* by Elie Cartan. (The word 'frame' is a synonym for 'basis'.) It is illustrated in Figure 7.

Remark 4.31. A general moving frame assigns a basis $b_p = (\xi_1\big|_p, \dots, \xi_n\big|_p)$ of T_pM to each $p \in U$ in an open subset $U \subset M$, and the assignment must be smooth; it need not be induced from a local coordinate system. To make sense of 'smooth' in the last sentence, we view $p \mapsto b_p$ as a map from U to... where? The 'where' should be a smooth manifold.

(4.32) *Computational technique.* We present a relatively simple computation to illustrate these ideas in practice.

Example 4.33 (sample computation)**.** We compute the Gauss map of a torus M embedded in Euclidean 3-space $\mathbb{E}^3$. It is a map $f\colon M \to S^2$ which takes each point of M to a unit normal vector to M at that point. The computation is illustrated in Figure 8. Let x, y, z be the standard affine coordinates on $\mathbb{E}^3$. Fix positive real numbers $R > r$. Define M as the surface obtained by revolving a circle of radius r about the z-axis, assuming its center to be at distance R from the axis. Then M is the image of the map $S^1 \times S^1 \to \mathbb{E}^3$ defined by

$$\begin{aligned} x &= (R + r\cos\theta)\cos\phi \\ y &= (R + r\cos\theta)\sin\phi \\ z &= r\sin\theta \end{aligned} \tag{4.34}$$

Restrict $0 < \theta, \phi < 2\pi$ to obtain a homeomorphism onto the image U, which is the complement of the union of two circles in M, and invert to obtain a chart with coordinate functions θ, ϕ.

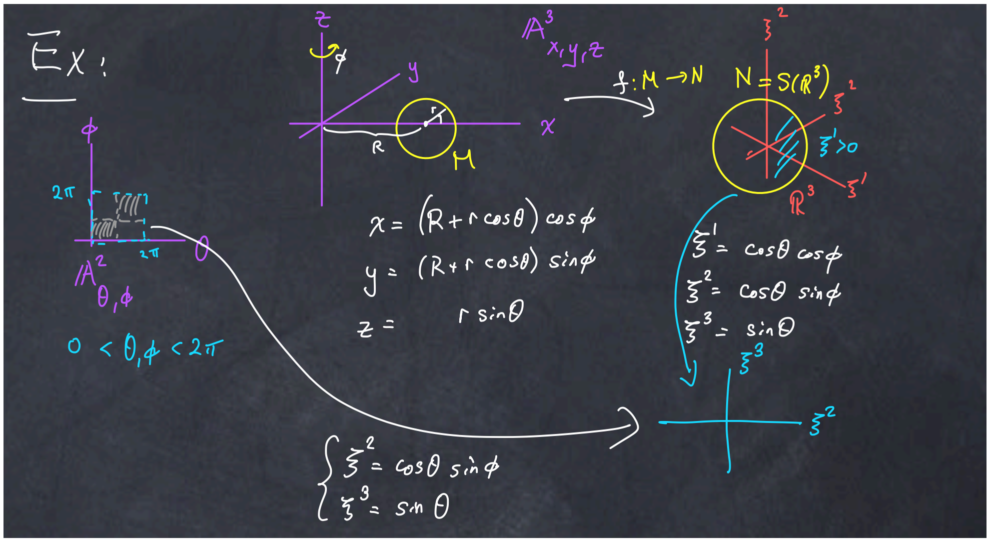


FIGURE 8. Local computation of the Gauss map of a torus in Euclidean 3-space

The sphere S^2 is the unit sphere in the vector space $\mathbb{R}^3$, which we take to have standard coordinates ξ^1, ξ^2, ξ^3. In these coordinates the Gauss map $S^1 \times S^1 \to S^2$ is

$$\begin{aligned} \xi^1 &= \cos\theta\cos\phi \\ \xi^2 &= \cos\theta\sin\phi \\ \xi^3 &= \sin\theta \end{aligned} \tag{4.35}$$

as can be deduced by differentiating (4.34) with respect to r. (The motion with parameter r has initial velocity the unit normal to M.) Now take a chart on S^2 to be the open subset where $\xi^1 > 0$ and use ξ^2, ξ^3 as coordinate functions on that chart. Then the local representation of the Gauss map from coordinates θ, ϕ to coordinates ξ^2, ξ^3 is

$$\begin{aligned} \xi^2 &= \cos\theta\sin\phi \\ \xi^3 &= \sin\theta \end{aligned} \tag{4.36}$$

The dual (4.13) to the differential $df_p\colon T_pM \to T_{f(p)}S^2$ is what we compute at any point $p \in U$ by differentiating the equations (4.36) which define f:

$$\begin{aligned} d\xi^2 &= -\sin\phi\sin\theta\,d\theta \;+\; \cos\phi\cos\theta\,d\phi \\ d\xi^3 &= \qquad\quad \cos\theta\,d\theta \end{aligned} \tag{4.37}$$

These can be transposed to compute the map on tangent vectors, i.e., the differential:

$$\begin{aligned} \frac{\partial}{\partial\theta} &\longmapsto -\sin\phi\sin\theta\frac{\partial}{\partial\xi^2} \;+\; \cos\phi\cos\theta\frac{\partial}{\partial\xi^3} \\ \frac{\partial}{\partial\phi} &\longmapsto \qquad\quad \cos\theta\frac{\partial}{\partial\xi^2} \end{aligned} \tag{4.38}$$

You can derive (4.38) by evaluating (4.37) on $\partial/\partial\theta$ and $\partial/\partial\phi$. This is the differential at the point on M with coordinates (θ, ϕ).

Lecture 5: The inverse function theorem

Extensive notes for this lecture are in the text on Multivariable Analysis, especially pp. 16–17, 24–26, 30, 58–63*, 65. Please be sure to think about the examples in the notes as well as the general theorems.

Lecture 6: Normal forms for maximal rank maps

We begin in this lecture with the maximal rank condition in linear algebra. We prove that the space of maximal rank linear maps is open in the vector space of linear maps. This fits the slogan "invertibility is an open condition"; see Example 3.12. A maximal rank linear map has a simple normal form. We then go on to the case of a map between smooth manifolds, where the maximal rank condition makes sense on the differential. The circle of ideas around the inverse function theorem (Lecture 5) is used to pass from this first-order infinitesimal condition to a local normal form. We end by defining some global conditions on smooth maps.

Maximal rank linear maps

The ground field is arbitrary for the linear algebra of this section. In our application to smooth manifolds, we use the field of real numbers.

Definition 6.1. Let V, W be finite dimensional vector spaces and $T\colon V \to W$ a linear map.

(i) The *rank* of T is the dimension of its image:

$$\operatorname{rank} T = \dim T(V) \leqslant \min(\dim V, \dim W). \tag{6.2}$$

(ii) T has maximal rank if there is equality in (6.2).

A maximal rank map is injective/bijective/surjective if $\dim V \leqslant / = / \geqslant \dim W$, respectively.

Lemma 6.3. *Let V, W be finite dimensional vector spaces.*

(1) *The space of maximal rank linear maps* $\operatorname{MaxRank}(V, W) \subset \operatorname{Hom}(V, W)$ *is open.*

(2) *If $T\colon V \to W$ has maximal rank, then there exist bases $e_1, \dots, e_m$ of V and $f_1, \dots, f_n$ of W such that*

$$T(e_j) = f_j, \qquad j = 1, \dots, m, \quad \text{if } \dim V \leqslant \dim W, \tag{6.4}$$

and

$$T(e_j) = \begin{cases} f_j, & j = 1 \dots, n; \\ 0, & j = n+1, \dots, m, \end{cases} \qquad \text{if } \dim V \geqslant \dim W. \tag{6.5}$$

Proof. We already gave the argument for (1) in case $\dim V = \dim W$: then the subset of isomorphisms $\mathrm{Iso}(V,W) \subset \mathrm{Hom}(V,W)$ is open.

If $\dim V < \dim W$ and $T_0\colon V \to W$ has maximal rank, choose $W_0 \subset W$ complementary to $T_0(V)$. Equivalently, $W = T_0(V) \oplus W_0$. Note that $\dim V = \dim W/W_0$. Let $\pi\colon W \to W/W_0$ be projection onto the quotient. Define

$$\begin{aligned} p\colon \mathrm{Hom}(V,W) &\longrightarrow \mathrm{Hom}(V,W/W_0) \\ T &\longmapsto \pi \circ T \end{aligned} \tag{6.6}$$

Then $p^{-1}\bigl(\mathrm{Iso}(V,W/W_0)\bigr) \subset \mathrm{MaxRank}(V,W) \subset \mathrm{Hom}(V,W)$ is an open subset containing T_0, since $\mathrm{Iso}(V,W/W_0) \subset \mathrm{Hom}(V,W/W_0)$ is open, and this proves that $\mathrm{MaxRank}(V,W)$ is open.

The argument for $\dim V > \dim W$ is similar. Given $T_0\colon V \to W$ of maximal rank, choose $V_0 \subset V$ so that $T\big|_{V_0}\colon V_0 \to W$ is an isomorphism. Let $\iota\colon V_0 \to V$ be the inclusion. Define

$$\begin{aligned} r\colon \mathrm{Hom}(V,W) &\longrightarrow \mathrm{Hom}(V_0,W) \\ T &\longmapsto T \circ \iota \end{aligned} \tag{6.7}$$

to be restriction to V_0. Then $r^{-1}\bigl(\mathrm{Iso}(V_0,W)\bigr) \subset \mathrm{MaxRank}(V,W) \subset \mathrm{Hom}(V,W)$ is open and contains T_0, which proves $\mathrm{MaxRank}(V,W) \subset \mathrm{Hom}(V,W)$ is open.

For (2), if $\dim V \leqslant \dim W$ choose an arbitrary basis $e_1, \dots, e_m$ of V, define $f_j = T(e_j)$, $j = 1, \dots, m$, and fill out the linearly independent set $f_1, \dots, f_m$ to a basis of W. Similarly, if $\dim V \geqslant \dim W$, choose an arbitrary basis $f_1, \dots, f_n$ of W, use surjectivity to find vectors $e_1, \dots, e_n$ in V with $T(e_j) = f_j$, $j = 1 \dots, n$, and choose $e_{n+1}, \dots, e_m$ to be a basis of $\ker T$. ☐

The maximal rank condition for smooth maps of manifolds

We introduce special terminology for the maximal rank condition.

Definition 6.8. Let M, N be smooth manifolds and $f\colon M \to N$ a smooth map. Fix $p \in M$ and set $q = f(p)$. The differential of f at p is a linear map $df_p\colon T_pM \to T_qN$.

(i) If df_p is injective, then f is an *immersion* at p.
(ii) If df_p is surjective, then f is a *submersion* at p. We also say f is *regular* at p, or p is a *regular point* of f.
(iii) If df_p is not surjective, then p is a *critical point* of f.
(iv) If all $p \in f^{-1}(q)$ are regular points, then q is a *regular value* of f.
(v) If there exists $p \in f^{-1}(q)$ a critical point, then q is a *critical value* of f.

Be cognizant that regular and critical *points* lie in the *domain* and regular and critical *values* lie in the *codomain*. Repeat: points/domain, values/codomain. Points/domain, values/codomain.

Remark 6.9. If $q \in N \backslash f(M)$ is not in the image of f, then q is a regular value, since the condition in (iv) is trivially satisfied.

A fundamental result, Sard's Theorem, asserts that the set of critical values of any smooth function has measure zero in the codomain. (Part of that circle of ideas is defining measure zero.) As a corollary, the set of regular values is dense, and in particular is nonempty. On the other hand, the set of critical values can be empty, as for the identity map $\mathrm{id}_M\colon M \to M$ on any M. We discuss Sard's theorem a few lectures hence.

Recall Definition 3.31 of a diffeomorphism between smooth manifolds.

Remark 6.10.

(1) If f is a diffeomorphism, differentiate the equality $f^{-1} \circ f = \mathrm{id}_M$ at $p \in M$ to find

$$(df^{-1})_{f(p)} = (df_p)^{-1}, \tag{6.11}$$

as we already observed when proving the inverse function theorem.

(2) Compositions of diffeomorphisms are diffeomorphisms.

(3) If $U \subset M$ is open and $x\colon U \to \mathbb{A}^n$ is a smooth map to standard affine space, then (U, x) is a (standard) chart on M iff x is a diffeomorphism onto its image $x(U) \subset \mathbb{A}^n$. (The chart exists in the differential structure, i.e., a maximal atlas.)

The following is a corollary of the inverse function theorem in affine space; it is the inverse function theorem for smooth manifolds.

Theorem 6.12. *Let M, N be smooth manifolds, $f\colon M \to N$ a smooth map, and suppose $df_p\colon T_pM \to T_{f(p)}N$ is an isomorphism for some $p \in M$. Then there exist open subsets $U \subset M$ containing p and $V \subset N$ containing $f(p)$ such that $f\big|_U\colon U \to V$ is a diffeomorphism.*

The converse statement follows from Remark 6.10(i). The conclusion of Theorem 6.12 can be summarized in the assertion: f is a *local diffeomorphism* at p.

Proof. Choose an A-valued chart $(\widetilde{U}, x)$ about p and a B-valued chart $(\widetilde{V}, y)$ about $f(p)$, for some affine spaces A, B. Apply the inverse function theorem in affine space to

$$y \circ f \circ x^{-1}\colon x\big(\widetilde{U} \cap f^{-1}(\widetilde{V})\big) \longrightarrow B \tag{6.13}$$

using the fact that $d(y \circ f \circ x^{-1})_{x(p)} = dy_{f(p)} \circ df_p \circ (dx^{-1})_{x(p)}$ is an isomorphism. Thus on a subset of its domain, which transports by x^{-1} to a subset $U \subset \widetilde{U}$, the map (6.13) is a diffeomorphism onto its image. The theorem follows. $\square$

Now we turn to special coordinate systems and local forms.

Proposition 6.14. *Let M be a smooth manifold, $p \in M$ a point, $n = \dim_p M$ the dimension of M at p, and $U \subset M$ an open set containing p.*

(1) *Let $x^1, \dots, x^n\colon U \to \mathbb{R}$ be smooth functions whose differentials $dx^1_p, \dots, dx^n_p$ at p are a basis of T^*_pM. Then there exists an open subset $U' \subset U$ such that $(U'; x^1, \dots, x^n)$ is a standard chart.*

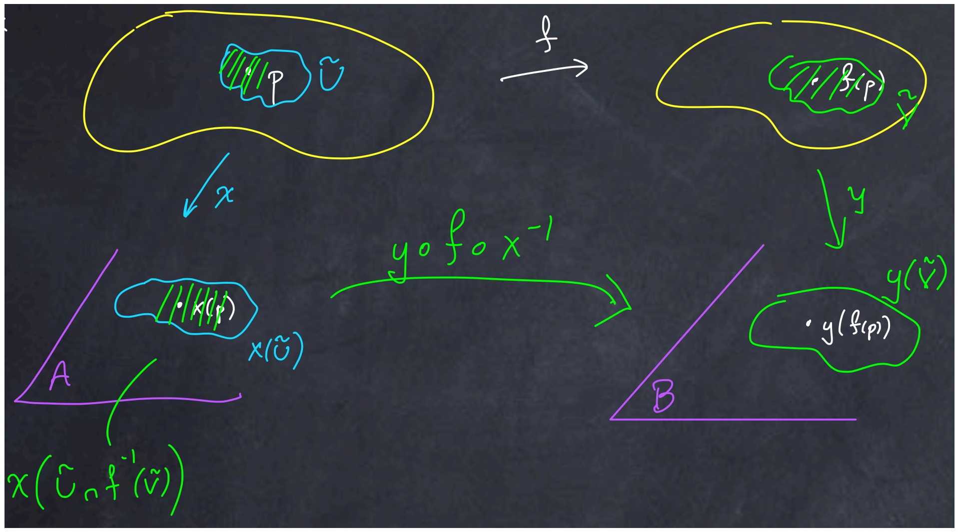


FIGURE 9. Transporting the inverse function theorem from affine space to manifolds

(2) *Let $x^1,\dots,x^k\colon U\to\mathbb{R}$, $k<n$, be smooth functions whose differentials $dx^1_p,\dots,dx^k_p$ at p are linearly independent in T^*_pM. Then there exists an open subset $U'\subset U$ and functions $x^{k+1},\dots,x^n\colon U'\to\mathbb{R}$ such that $(U';x^1,\dots,x^n)$ is a standard chart.*

(3) *Let $x^1,\dots,x^\ell\colon U\to\mathbb{R}$, $\ell>n$, be smooth functions whose differentials $dx^1_p,\dots,dx^\ell_p$ at p span T^*_pM. Then there exists an open subset $U'\subset U$ and a subset $\{i_1,\dots,i_n\}\subset\{1,\dots,\ell\}$ such that $(U';x^{i_1},\dots,x^{i_n})$ is a standard chart.*

Proof. Assertion (1) is a direct consequence of Theorem 6.12 and Remark 6.10(3).

For (2), choose a standard chart (V,y) about p, and suppose $y\colon V\to\mathbb{R}^n$ takes values in the vector space $\mathbb{R}^n$ and $y(p)=0$. We may also assume $V\subset U$. Then the ordered k-tuple $(x^1\circ y^{-1},\dots,x^k\circ y^{-1})$ defines a map $g\colon y(V)\to\mathbb{R}^k$ whose differential at 0 is surjective. By Lemma 6.3 we can find a linear automorphism $S\colon\mathbb{R}^n\to\mathbb{R}^n$ so that $dg_0\circ S\colon\mathbb{R}^n\to\mathbb{R}^k$ is projection onto the first k components. Let $x^{k+1},\dots x^n\colon V\to\mathbb{R}$ be the last $(n-k)$ coordinates of the chart $(V,S^{-1}\circ y)$. Then the n functions $x^1,\dots,x^n\colon V\to\mathbb{R}$ satisfy the hypothesis of (1), as we can check in the chart $(V,S^{-1}\circ y)$. Now apply the conclusion of (1) to prove (2).

For (3), choose a subset $\{i_1,\dots,i_n\}\subset\{1,\dots,\ell\}$ such that $dx^{i_1}_p,\dots,dx^{i_n}_p$ is a basis of T^*_pM and then apply (1). □

We apply Proposition 6.14 to prove the analog of the normal form theorem Lemma 6.3(2) on smooth manifolds.

Theorem 6.15. *Let $f\colon M\to N$ be a map of smooth manifolds. Fix $p\in M$ and suppose $\dim_p M=m$ and $\dim_{f(p)}N=n$. Assume $df_p\colon T_pM\to T_{f(p)}N$ has maximal rank. Then there exist standard charts (U,x) about p and (V,y) about $f(p)$ such that $y\circ f\circ x^{-1}$ takes the form*

$$y^i=x^i,\qquad i=1,\dots,n,\quad \text{if } m\geqslant n, \tag{6.16}$$

and

$$(6.17) \qquad y^i = \begin{cases} x^i, & i = 1\dots,m; \\ 0, & i = m+1,\dots,n, \end{cases} \qquad \text{if } m \leqslant n.$$

Proof. Choose an arbitrary standard chart $(V; y^1,\dots,y^n)$ on N about $f(p)$. If $m \geqslant n$, consider the n functions $x^1 := y^1 \circ f,\dots,x^n := y^n \circ f$, defined on the neighborhood $f^{-1}(V)$ of p in M. Their differentials at p are linearly independent, so by Proposition 6.14(2) they can be completed to a standard chart with domain $U \subset f^{-1}(V)$. In this way we obtain the normal form (6.16). In case $m \leqslant n$, the differentials of these n functions at p span T_p^*M, so by Proposition 6.14(3) we can choose m of them which form a basis. Renumbering so these are the first m, we obtain the normal form (6.17). □

Global properties of smooth maps; submanifolds

We now turn to global properties.

Definition 6.18. Let $f\colon M \to N$ be a smooth map of smooth manifolds. Then f is an *embedding* if it is an injective immersion which is a homeomorphism onto its image.

There are three conditions on f in the definition: the local condition that f be an immersion, i.e., its differential df_p is injective for all $p \in M$; the global condition that f be injective; and the global condition that the inverse $f^{-1}\colon f(M) \to M$ of f be continuous. Here $f(M) \subset N$ has the subspace topology. In Figure 10 we depict three immersions $f\colon \mathbb{R} \longrightarrow \mathbb{A}^2$, illustrating these two global properties. Regard the function f as a motion in $\mathbb{A}^2$; then the immersion condition states that the velocity is nonzero at every time.

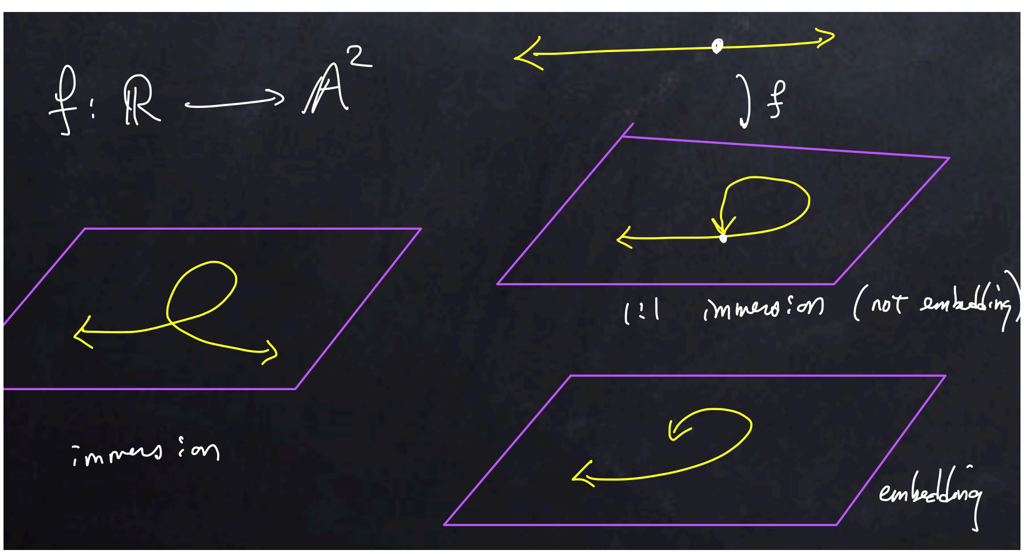


FIGURE 10. Types of map with injective differential

Just as a smooth manifold is defined by a local normal form—locally a smooth manifold is diffeomorphic to affine space—so too is a *submanifold* defined by a local normal form: locally a submanifold is diffeomorphic to an affine subspace of an affine space.

Definition 6.19. Let M be a smooth manifold and $Q \subset N$ a subset. Then Q is a *submanifold* of N if for all $q \in Q$ there exists a chart (V, y) about q with values in an affine space A together with an affine subspace $A' \subset A$ such that

$$y(Q \cap V) = A' \cap y(V). \tag{6.20}$$

The submanifold chart is *standard* if the chart (V, y) is standard and there exists $\ell \in \{0, \dots, n\}$ such that

$$y(Q \cap V) = \{(y^1, \dots, y^n) \in \mathbb{A}^n : y^{\ell+1} = \dots = y^n = 0\} \cap y(V). \tag{6.21}$$

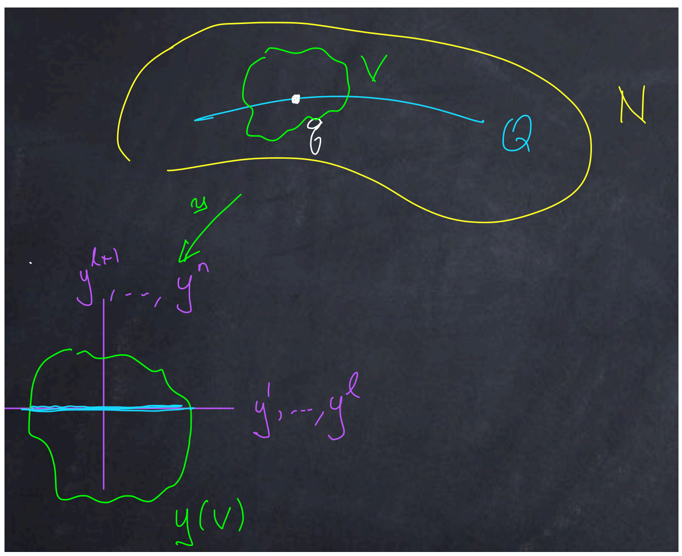


FIGURE 11. A submanifold chart

We illustrate a standard submanifold chart in Figure 11. The integer ℓ is the *codimension* of Q in N at the point q, which is written $\operatorname{codim}_q(Q \subset N)$.

Remark 6.22.

(1) A submanifold is a manifold in its own right. Namely, if $\mathscr{A} = \{(V_\alpha, y_\alpha)\}_{\alpha \in \mathcal{A}}$ is a covering of Q by submanifold charts, then $\mathscr{A} = \{(V_\alpha \cap Q; y^1_\alpha, \dots, y^\ell_\alpha)\}$ is an atlas of Q. (This assumes constant codimension ℓ and standard submanifold charts; the reader needs merely to change the notation to accommodate general submanifold charts and varying codimension.)

(2) A submanifold may be open, it may be closed, or it may be neither. Let $N = \mathbb{A}^2$. Then an open disk in N is an open submanifold, an affine line in N is a closed submanifold, and an open interval in an affine line is a submanifold which is neither open nor closed.

(3) A submanifold $Q \subset N$ is always a locally closed subset. That is, for every $q \in Q$ there exists an open neighborhood $V \subset N$ of q such that $Q \cap V \subset V$ is closed. (To prove this, let V be the domain of a submanifold chart.)

Example 6.23 (skew line on a torus)**.** Consider the 2-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$. Fix $(x_0, y_0) \in \mathbb{R}^2$ so that $x_0 \neq 0$ and y_0/x_0 is irrational. Then the map

$$\begin{aligned} f\colon \mathbb{R} &\longrightarrow \mathbb{R}^2/\mathbb{Z}^2 \\ t &\longmapsto (tx_0, ty_0) \pmod{\mathbb{Z}^2} \end{aligned} \tag{6.24}$$

is an injective immersion. The image $f(\mathbb{R}) \subset \mathbb{R}^2/\mathbb{Z}^2$ is dense in the torus. About every $t \in \mathbb{R}$ we can choose special charts as in Theorem 6.15 so that (6.16) is satisfied, but the chart on the torus is not a submanifold chart since (6.21) is not satisfied.

Lecture 7: Submanifolds, embeddings, and regular values; a counting invariant

Recall that there are three basic ways to associate a "shape" to a function $f\colon M \to N$. We can take the image $f(M) \subset N$; the preimage $f^{-1}(q) \subset M$ of a point of N, or more generally the preimage $f^{-1}(Q) \subset M$ of a subset of N; and the graph $\Gamma(f) \subset M \times N$ of f. If M, N are smooth manifolds and f a smooth function, then the graph $\Gamma(f)$ is always a submanifold of $M \times N$, and it is diffeomorphic to the domain M. The first theorem in this lecture gives a sufficient condition on f for its image $f(M)$ to be a submanifold of the codomain N, namely that f be an embedding; then $f(M)$ is diffeomorphic to M. The second theorem gives a sufficient condition on $q \in N$ for the inverse image $f^{-1}(q)$ to be a submanifold of the domain M, namely that q be a regular value. In both cases the condition is not necessary: a constant map has image a submanifold, and the inverse image of a critical value can "accidentally" be a manifold. We will soon study the condition—*transversality*—for the inverse image $f^{-1}(Q) \subset M$ of a submanifold $Q \subset N$ to be a submanifold.

In the last part of the lecture we construct our first topological invariant and use it to prove the fundamental theorem of algebra.

Embeddings and submanifolds

Theorem 7.1. *Let $f\colon M \to N$ be an embedding. Then $f(M) \subset N$ is a submanifold.*

Proof. Fix $q \in Q$; we must construct a submanifold chart about q. Let $p \in M$ be the unique point so that $f(p) = q$. Since f is immersive, by Theorem 6.15 there exist charts (U, x) about p and (V, y) about q so that $y \circ f \circ x^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^m, 0, \dots, 0)$; see Figure 12. We claim that there exists an open subset $V' \subset V$ so that the restricted chart (V', y) is a submanifold chart. If (6.21) fails, then there exists a sequence $\{p_k\}_{k=1}^{\infty} \subset M \backslash U$ so that $\lim_{k\to\infty} y^j\big(f(p_k)\big) = 0$ for $j = m+1, \dots, n$. Hence the sequence $\{f(p_k)\} \subset V$ converges to a point of $f(U)$, and since f is a homeomorphism onto its image we conclude that $\{p_k\} \subset M\backslash U$ converges to a point of U, which is absurd since $M\backslash U$ is a closed subset of M. ∎

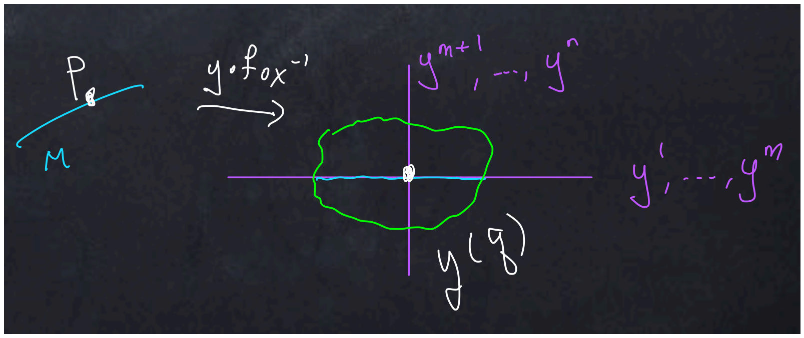


FIGURE 12. Constructing a submanifold chart

Regular values and submanifolds

We first introduce some terminology which will recur throughout the course.

Definition 7.2. A sequence

$$V \xrightarrow{T} W \xrightarrow{S} X \tag{7.3}$$

of linear maps of vector spaces is *exact* if $S \circ T = 0$ and $\ker S = T(V)$ as subspaces of W. A *long exact sequence*

$$\cdots \longrightarrow V^i \longrightarrow V^{i+1} \longrightarrow V^{i+2} \longrightarrow \dots \tag{7.4}$$

is a sequence of linear maps in which every two consecutive maps forms an exact sequence. A *short exact sequence* is a long exact sequence of the form

$$0 \longrightarrow V' \xrightarrow{T} V \xrightarrow{S} V'' \longrightarrow 0. \tag{7.5}$$

In (7.5) the linear map $T\colon V' \to V$ is injective with cokernel (isomorphic to) V'', and the linear map $S\colon V \to V''$ is surjective with kernel (isomorphic to) V'. Furthermore, if V', V, V'' are finite dimensional, then

$$\dim V = \dim V'' + \dim V'. \tag{7.6}$$

Definition 7.7. Let $P \subset M$ be a submanifold and $p \in P$.

(1) The quotient space T_pM/T_pP is the *normal* (*space*) to P at p.
(2) The *codimension* of P in M at p is

$$\operatorname{codim}_p(P \subset M) = \dim_p M - \dim_p P = \dim\bigl(T_pM/T_pP\bigr). \tag{7.8}$$

We sometimes use the notation 'ν_p' for the normal space at p. There is a short exact sequence

$$0 \longrightarrow T_pP \longrightarrow T_pM \longrightarrow \nu_p \longrightarrow 0. \tag{7.9}$$

Theorem 7.10. *Let $f\colon M \to N$ be a smooth map of smooth manifolds and $q \in N$ a regular value. Then $P := f^{-1}(q) \subset M$ is a submanifold of codimension equal to $\dim_q N$. Furthermore, if $p \in P$,*

$$T_pP = \ker\big(df_p\colon T_pM \longrightarrow T_pN\big). \tag{7.11}$$

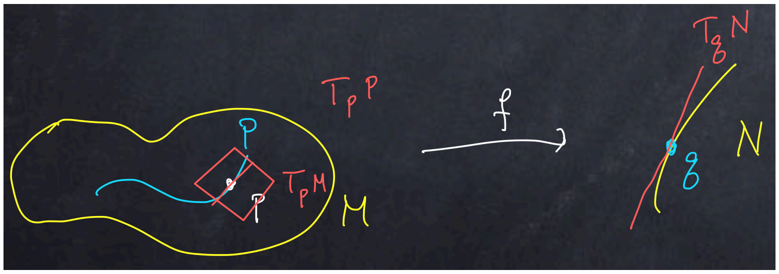


FIGURE 13. The linearization of f cuts out the tangent space to the submanifold cut out by f

We express (7.11) as the short exact sequence

$$0 \longrightarrow T_pP \longrightarrow T_pM \xrightarrow{df_p} T_qN \longrightarrow 0, \tag{7.12}$$

illustrated in Figure 13. In general, the codimension of a submanifold $P \subset M$ is a locally constant function $\operatorname{codim}\colon P \to \mathbb{Z}^{\geqslant 0}$. Theorem 7.10 asserts that if P is cut out by a single function, then codim is a constant function.

Remark 7.13. Not every submanifold is cut out by a single function. Compare (7.9) and (7.12) to conclude that in the situation of Theorem 7.10, the differential df_p identifies each normal space ν_p with the fixed vector space T_qN. It is not true that the normal spaces to every submanifold admit such a smoothly varying identification.[5]

Proof of Theorem 7.10. Fix $p \in P$ and choose charts (U, x) of M about p and (V, y) of N about q as in Theorem 6.15, so that $y \circ f \circ x^{-1}(x^1, \dots, x^k, \dots, x^m) = (x^1, \dots, x^k)$ and $x^i(p) = 0$, $i = 1, \dots, m$. Then (U, x) is a submanifold chart: $x(P \cap U) = \{(x^1, \dots, x^m) \in x(U) : x^1 = \cdots = x^k = 0\}$. The codimension is k, and the exact sequence (7.12) is immediate in these charts. $\square$

Example 7.14 (the 2-sphere redux)**.** The 2-sphere $S^2 \subset \mathbb{A}^3_{x,y,z}$ is cut out by the single function

$$\begin{aligned} f\colon \mathbb{A}^3 &\longrightarrow \mathbb{R} \\ (x, y, z) &\longmapsto x^2 + y^2 + z^2 \end{aligned} \tag{7.15}$$

Namely, $S^2 = f^{-1}(1)$. To verify that $1 \in \mathbb{R}$ is a regular value of f, it suffices to observe that the differential $df = 2x\,dx + 2y\,dy + 2z\,dz$ does not vanish at any point of $f^{-1}(1)$. Note that $df_{(0,0,0)} = 0$, so $(0, 0, 0) \in \mathbb{A}^3$ is a critical point, $0 \in \mathbb{R}$ is a critical value, and yet $f^{-1}(0) \subset \mathbb{A}^3$ is a submanifold (though not of the expected codimension $\dim \mathbb{R}$).

[5]We will soon study fiber bundles, and express this as the trivializability of the normal bundle to a submanifold cut out by a global function.

Example 7.16 (the orthogonal group)**.** Recall that $M_n\mathbb{R}$ is the n^2-dimensional vector space of $n \times n$ matrices. Let ${}^tA$ denote the transpose of the matrix A. The orthogonal group $O_n \subset M_n\mathbb{R}$ is defined by the single condition $A\,{}^tA = I$, where I is the identity matrix. To re-express this condition as the inverse image of a *regular* value of a function, we must note that the matrix $S = A\,{}^tA$ is symmetric: ${}^tS = S$. Let $S_n\mathbb{R} \subset M_n\mathbb{R}$ denote the vector subspace of symmetric matrices. Define

$$\begin{aligned} f\colon M_n\mathbb{R} &\longrightarrow S_n\mathbb{R} \\ A &\longmapsto A\,{}^tA \end{aligned} \tag{7.17}$$

Then $O_n = f^{-1}(I)$. To prove that $O_n \subset M_n\mathbb{R}$ is a submanifold, we show that I is a regular value of f and apply Theorem 7.10. For any $A, \dot{A} \in M_n\mathbb{R}$ we compute

$$df_A(\dot{A}) = A\,{}^t\dot{A} + \dot{A}\,{}^tA. \tag{7.18}$$

For $A \in O_n$ and $S \in S_n\mathbb{R}$ we must prove that the equation

$$A\,{}^t\dot{A} + \dot{A}\,{}^tA = S \tag{7.19}$$

has a solution $\dot{A} \in M_n\mathbb{R}$, which it does: $\dot{A} = \frac{1}{2}SA$.

The orthogonal group O_n is an example of a Lie group.

Definition 7.20. Let G be a set endowed with both a group structure and a smooth manifold structure. Suppose these structures are compatible in the sense that multiplication $G \times G \to G$ and inversion $G \to G$ are both smooth maps. Then G is a *Lie group.*

To verify that multiplication on O_n is smooth, we first observe that matrix multiplication $M_n\mathbb{R} \times M_n\mathbb{R} \to M_n\mathbb{R}$ is a polynomial map, hence is smooth. Since $O_n \subset M_n\mathbb{R}$ is a submanifold, so too is $O_n \times O_n \subset M_n\mathbb{R} \times M_n\mathbb{R}$, and hence the restriction of multiplication to a map

$$O_n \times O_n \longrightarrow M_n\mathbb{R} \tag{7.21}$$

is smooth. Furthermore, (7.21) factors through a map with codomain O_n. The smoothness of the factored map follows from a general result, as does the smoothness of inversion on O_n.

Proposition 7.22. *Suppose $f\colon M \to N$ is a smooth map of smooth manifolds, $M' \subset M$ and $N' \subset N$ are submanifolds, and the restriction of f to M' factors through a map $f'\colon M' \to N'$. Then f' is smooth.*

Proof. Let $p' \in M'$ and choose submanifold charts (U, x) about p' and (V, y) about $f(p')$ such that $f(U) \subset V$. If $m' = \dim_{p'} M'$, $m = \dim_{p'} M$, $n' = \dim_{f(p')} N'$, and $n = \dim_{f(p)} N$, and if the smooth functions $y^i = y^i(x^1, \dots, x^m)$, $i = 1, \dots, n$, are the expression of $y \circ f \circ x^{-1}$, then the smooth functions $y^i = y^i(x^1, \dots, x^{m'}, 0, \dots, 0)$, $i = 1, \dots, n'$, are the expression of $y' \circ f' \circ (x')^{-1}$ in the charts on M', N' induced from the corresponding charts on M, N. □

Proposition 7.23. *The orthogonal group O_n is a Lie group.*

A counting invariant; the fundamental theorem of algebra

We use the inverse function theorem to construct our first topological invariant. It illustrates a main theme of the class: we use calculus—local control from the infinitesimal hypothesis of maximal rank—to set up global invariants.

Theorem 7.24. *Let M be a compact smooth manifold, N a smooth manifold with* $\dim N = \dim N$, *and $f\colon M \to N$ a smooth function. Set $N_{\mathrm{reg}} \subset N$ the subset of regular values. Then the function*

$$\begin{aligned} \#\colon N_{\mathrm{reg}} &\longrightarrow \mathbb{Z}^{\geqslant 0} \\ q &\longmapsto \#f^{-1}(q) \end{aligned} \tag{7.25}$$

is well-defined and locally constant.

The conclusion is that for any regular value $q \in N$ the subset $f^{-1}(q) \subset M$ is finite and its cardinality is a locally constant function of the regular value.

Proof. Fix $q \in N_{\mathrm{reg}}$. Theorem 7.10 implies that $f^{-1}(q)$ is a 0-dimensional submanifold of M, so a finite or countable set of isolated points, and since $f^{-1}(q) \subset M$ is closed and M is compact it follows that $f^{-1}(q)$ is a finite set. Hence (7.25) is well-defined. Suppose $\#f^{-1}(q) = N$ and write $f^{-1}(q) = \{p_1, \dots, p_N\}$. The Inverse Function Theorem 6.12 implies that f is a local diffeomorphism at each p_i; choose $U_i \subset M$ open so that $f\colon U_i \to f(U_i)$ is a diffeomorphism. Set

$$V = \bigcap_{i=1}^{N} f(U_i) \ \setminus \ f\bigl(M \setminus \bigcup_{i=1}^{N} U_i\bigr). \tag{7.26}$$

Then $V \subset N_{\mathrm{reg}}$ is open, $q \in V$, and $\#\big|_V$ is constant. ∎

In the next lecture we prove that N_{reg} is nonempty; in fact, it is a dense subset of N. In general we cannot control its connectivity. But for polynomial functions we have good control.

Lemma 7.27. *Let $Q\colon \mathbb{C} \to \mathbb{C}$ be a polynomial of degree d. Then Q has at most d roots.*

Proof. Argue by induction: if $Q(z_0) = 0$, then $Q(z) = (z - z_0)Q_1(z)$ for a polynomial Q_1 of degree $d - 1$. ∎

Theorem 7.28 (fundamental theorem of algebra)**.** *Any polynomial $P\colon \mathbb{C} \to \mathbb{C}$ has a root.*

The one-point compactification of $\mathbb{C}$ is the complex projective line $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$; see (3.23). It is defined as the projectivization $\mathbb{P}(\mathbb{C}^2)$ of the standard 2-dimensional complex vector space, which we take to have components z, w. Thus a point of $\mathbb{CP}^1$ is an equivalence class of ordered pairs $[z, w]$ with at least one of z, w nonzero and $[z, w] \sim [\lambda z, \lambda w]$ for all $\lambda \neq 0$. The complex line $\mathbb{C}$ embeds in $\mathbb{CP}^1$ as $z \mapsto [z, 1]$; the point ∞ is $[1, 0]$.

Proof. Define a smooth map $f\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ by

$$\begin{aligned} f\bigl([z,1]\bigr) &= [P(z), 1] \\ f\bigl([1,0]\bigr) &= [1,0] \end{aligned} \tag{7.29}$$

The critical points of f outside $\infty = [1,0]$ are the roots of P', hence by Lemma 7.27 there are finitely many. Therefore, if $N = \mathbb{CP}^1$ is the codomain of f, then $N_{\text{reg}} \subset N$ is the complement of a finite set, so is connected. It follows that the locally constant function (7.25) is constant. It is clearly nonzero. If $0 \in N_{\text{reg}}$, then $f^{-1}(0) \subset \mathbb{C} \subset \mathbb{CP}^1$ is nonempty, so P has a root. If $0 \in N$ is a critical value, then in particular it is a value of f, so again P has a root. □

Lecture 8: Measure zero, Sard's theorem, introduction to fiber bundles

The main topic of this lecture, Sard's theorem, is proved in an appendix in Guillemin-Pollack, and I will not duplicate the proof here. That appendix also lays out the theory of measure zero on a smooth manifold, which I will expand upon in these notes. In the last part of the lecture we switch gears and single out another important class of maps: *fiber bundles*.

Sard's Theorem

Recall from Theorem 7.10 that the inverse image of a regular value of a smooth map of smooth manifolds is a submanifold. This method of producing manifolds gains its power from a companion theorem which asserts that regular values are plentiful. This companion theorem has a long history, which includes Morse (1939), Sard (1942), Brown (1935), Dubrovickii (1953), and Thom (1954). The main result goes simply by the name of Sard.

Theorem 8.1. *Let X, Y be C^∞ manifolds and $f\colon X \to Y$ a C^∞ map. Denote by $C \subset X$ the subset of critical points of f. Then $f(C) \subset Y$ has measure zero.*

Recall that $f(C) \subset Y$ is the subset of critical values. Its complement $Y \backslash F(C)$ is the set of regular values. The proof implies a stronger result for C^k maps, where k is sufficiently large depending on the dimensions of X and Y.

Corollary 8.2. *The subset $Y \backslash f(C)$ of regular values is dense.*

This follows from the fact that sets of measure zero have nonempty interior. We often apply a weaker result: the set of regular values is nonempty. Another measure zero fact, that a finite or countable union of sets of measure zero has measure zero, implies the next result.

Corollary 8.3. *Let $\{X_i\}_{i \in I}$ be a collection of C^∞ manifolds, where I is finite or countable. Let Y be a C^∞ manifold and $f_i\colon X_i \to Y$, $i \in I$, a C^∞ map. Then the set of simultaneous regular values of f_i is a dense subset of Y.*

Consider now the special case in which the domain has smaller dimension than the codomain. Every point of the domain is critical, since the differential cannot be surjective.

Corollary 8.4. *Suppose X, Y are C^∞ manifolds with $\dim X < \dim Y$ and $f\colon X \to Y$ is a C^∞ map. Then $f(X) \subset Y$ has measure zero.*

As we will see, the proof of Corollary 8.4 can be given in a more elementary fashion independently of Sard's Theorem 8.1. As a particular application we prove the following.

Corollary 8.5. *Any smooth map $f\colon S^n \to S^m$ is homotopically trivial if $n < m$.*

Proof. By Corollary 8.4 there exists a point $q \in S^m$ not in the image of f, so f factors through a map $f'\colon S^n \to S^m \backslash \{q\}$. Stereographic projection is a diffeomorphism $\varphi\colon S^m\backslash\{q\} \xrightarrow{\approx} \mathbb{R}^m$. Define the family of homotheties

$$\begin{aligned} h_t\colon \mathbb{R}^m &\longrightarrow \mathbb{R}^m \\ \xi &\longmapsto (1-t)\xi \end{aligned} \tag{8.6}$$

Let $\iota\colon \mathbb{R}^m \hookrightarrow S^m$ denote the inclusion. Then $\iota \circ h_t \circ \varphi \circ f'\colon S^n \to \mathbb{R}^m$ is a null homotopy of f' ☐

Measure zero in affine space

We define the measure, or volume, of some standard subsets of $\mathbb{A}^n$ and use them to define when an arbitrary subset $E \subset \mathbb{A}^n$ has measure zero.

Definition 8.7.

(i) A *standard (n-dimensional) box* defined by real numbers $a^1, \dots, a^n, b^1, \dots, b^n$ with $a^i < b^i$, $i = 1, \dots, n$, is the set

$$S = S(a^1, b^1; \dots; a^n, b^n) = \{(x^1, \dots, x^n) \in \mathbb{A}^n : a^i < x^i < b^i \text{ for all } i = 1, \dots, n\}. \tag{8.8}$$

If $\lambda = b^i - a^i$ is independent of i, then we call S a *standard cube* of side length λ.

(ii) The *(n-dimensional) volume* of the standard box (8.8) is

$$\mu(S) = \prod_{i=1}^{n} (b^i - a^i). \tag{8.9}$$

(iii) A set $E \subset \mathbb{A}^n$ has *(n-dimensional) measure zero* if for all $\epsilon > 0$ there exists a covering $\{S_i\}_{i\in I}$ of E with I finite or countable such that $\sum_{i\in I} \mu(S_i) < \epsilon$.

The definition (iii) of measure zero does depend on the ambient dimension: a nonempty open interval in $\mathbb{A}^1$ does not have 1-dimensional measure zero, but if we regard $\mathbb{A}^1 \subset \mathbb{A}^n$ for $n > 1$, then it has n-dimensional measure zero. We use 'measure zero' if the dimension is clear from context.

We prove some basic properties of sets of measure zero.

Proposition 8.10.

(1) *Let $E \subset \mathbb{A}^n$ be a set of measure zero and $E' \subset E$ a subset. Then E' has measure zero.*

(2) *Let $\{E_i\}_{i\in I}$ be a finite or countable collection of measure zero subsets of $\mathbb{A}^n$. Then $\bigcup_{i\in I} E_i$ has measure zero.*

(3) *The affine subspace $\mathbb{A}^m \subset \mathbb{A}^n$ has n-dimensional measure zero if $m < n$.*

(4) *Let $U \subset \mathbb{A}^n$ be an open subset, $E \subset U$ a set of measure zero, and $f\colon U \to \mathbb{A}^n$ a C^1 map. Then $f(E) \subset \mathbb{A}^n$ has measure zero.*
(5) *A standard box does not have measure zero.*
(6) *If $F \subset \mathbb{A}^n$ has nonempty interior, then F does not have measure zero.*
(7) *Let $E \subset \mathbb{A}^n$ be a closed subset. Suppose that for all $c \in \mathbb{R}$ the set $E \cap \bigl(\{c\} \times \mathbb{A}^{n-1}\bigr) \subset \mathbb{A}^{n-1}$ has $(n-1)$-dimensional measure zero. Then E has n-dimensional measure zero.*

A special case of (4) is that the image of a set of measure zero under a C^∞ diffeomorphism has measure zero. In other words, measure zero is a C^∞ concept.

Proof. Assertion (1) is immediate since a cover of E by standard boxes of total volume $< \epsilon$ is *a fortiori* such a cover of E'.

For (2), write $I = \{1, 2, \dots, N\}$ or $I = \mathbb{Z}^{\geqslant 0}$. Then given $\epsilon > 0$, for each $i \in I$ let $\{S_{i,j}\}_{j \in J_i}$ be a cover of E_i by at most countably many standard boxes of total volume $< \epsilon/2^i$. Then $\{S_i, j\}_{i\in I, j \in J_i}$ is an at most countable cover of $\bigcup_{i\in I} E_i$ by standard boxes of volume $< \epsilon$.

For (3), let $\{p_i\}_{i\in\mathbb{Z}^{\geqslant 0}}$ be a countable dense subset of $\mathbb{A}^m$, for example the set of points with rational coordinates. Given $\epsilon > 0$, for each $i \in \mathbb{Z}^{\geqslant 0}$ let S_i be the standard box in $\mathbb{A}^n$ with center p_i and side lengths $d_1 = \cdots = d_m = 1$, and $d_{m+1} = \cdots = d_n = (\epsilon/2^i)^{1/(n-m)}$. Then $\{S_i\}_{i\in\mathbb{Z}^{\geqslant 0}}$ covers $\mathbb{A}^m \subset \mathbb{A}^n$ and has total n-dimensional volume ϵ.

For (4), fix $p \in E$ and let $B_p \subset U$ be a ball whose closure lies in U. We prove that $f(E \cap B_p)$ has measure zero. Since df is continuous, we can choose $C > 0$ so that $\|df\| \leqslant C$ on the compact set $\overline{B_p}$. It follows that

$$d\bigl(f(x), f(x')\bigr) \leqslant C\, d(x, x'), \qquad x, x' \in \overline{B_p}. \tag{8.11}$$

Hence if $S \subset \overline{B_p}$ is a standard cube of side length λ, then $f(S)$ is contained in a standard cube of side length $C\sqrt{n}\,\lambda$. Given $\epsilon > 0$, cover $E \cap B_p$ by at most countably many standard cubes of total volume $< \epsilon/(C\sqrt{n})^n$. It follows that $f(E \cap B_p)$ is covered by at most countably many standard cubes of total volume $< \epsilon$. This proves that $f(E \cap B_p)$ has measure zero. Since $f(E)$ is covered by countably many sets of this form, (4) is proved.

The proof of (5) is a beautiful argument which Guillemin-Pollack credit to von Neumann. It is based on the following estimate. Let T be a standard box. Denote by $I(T)$ the number of points $(x^1, \dots, x^n) \in T$ such that $x^j \in \mathbb{Z}$ for all $j = 1, \dots, n$. Let the side lengths of T be $d_1, \dots, d_n$. Assume for the moment that each $d_j > 1$. Then

$$\prod_{j=1}^n (d_j - 1) \;<\; I(T) \;<\; \prod_{j=1}^n (d_j + 1). \tag{8.12}$$

Now to the proof of (5). Let S be a standard box of side lengths $d_1, \dots, d_n$, and let $\{S_i\}_{i\in I}$ be an at most countable covering of $\overline{S}$ by standard boxes. Then we claim

$$\sum_{i\in I} \mu(S_i) \geqslant \mu(S). \tag{8.13}$$

It follows immediately that $\overline{S}$, hence also S, cannot have measure zero. To prove (8.13), observe that since $\overline{S}$ is compact, we can and do choose a finite subcover of $\{S_i\}_{i\in I}$, whose elements we denote $S_1, \dots, S_N$. For $i \in \{1, \dots, N\}$, denote the side lengths of S_i by $d_1(i), \dots, d_n(i)$. Apply (8.12) to deduce[6]

$$\prod_{j=1}^{n}(d_j - 1) \;<\; I(S) \;\leqslant\; \sum_{i=1}^{N} I(S_i) \;<\; \sum_{i=1}^{N}\prod_{j=1}^{n}\bigl(d_j(i) + 1\bigr). \tag{8.14}$$

For any $c > 1$ the standard boxes $cS_1, \dots, cS_N$ cover $\overline{cS}$, and we can apply (8.14) to conclude

$$\prod_{j=1}^{n}(c\,d_j - 1) \;<\; \sum_{i=1}^{N}\prod_{j=1}^{n}\bigl(c\,d_j(i) + 1\bigr) \tag{8.15}$$

which, after dividing by c^n, is

$$\prod_{j=1}^{n}(d_j - \frac{1}{c}) \;<\; \sum_{i=1}^{N}\prod_{j=1}^{n}\bigl(d_j(i) + \frac{1}{c}\bigr) \tag{8.16}$$

Now take $c \to \infty$ to deduce the first inequality in

$$\mu(S) \;=\; \prod_{j=1}^{n} d_j \;\leqslant\; \sum_{i=1}^{N}\prod_{j=1}^{n} d_j(i) \;=\; \sum_{i=1}^{N}\mu(S_i) \;\leqslant\; \sum_{i\in I}\mu(S_i), \tag{8.17}$$

which is (8.13).

Assertion (6) now follows quickly: if $F \subset \mathbb{A}^n$ has nonempty interior, it properly contains a standard box, which by (5) does not have measure zero. Now apply (1).

It[7] suffices to assume in (7) that E is compact, since any closed $E \subset \mathbb{A}^n$ is the (countable) union of the compact sets $E \cap \overline{B_n(x)}$, $n \in \mathbb{Z}^{>0}$, where $B_n(x)$ is the ball of radius n about a fixed point $x \in \mathbb{A}^n$. Then $E \subset [a,b] \times \mathbb{A}^{n-1}$ for some finite closed interval $[a,b] \subset \mathbb{R}$. For any subset $J \in [a,b]$, let

$$E_J = E \cap \bigl(J \times \mathbb{A}^{n-1}\bigr). \tag{8.18}$$

Let $\epsilon > 0$ be given. For each $c \in [a,b]$ let $S_1(c), \dots, S_{N_c}(c)$ be a covering of the compact set $E_c \subset \mathbb{A}^{n-1}$ by $(n-1)$-dimensional standard boxes of total volume $< \epsilon/2(b-a)$. By compactness of E there is an open interval $J(c) \subset [a,b]$ about c so that $\{J(c) \times S_i(c)\}_{i=1}^{N_c}$ covers $E_{J(c)}$. The intervals $\{J(c)\}_{c\in[a,b]}$ cover $[a,b]$, so there exists a finite subcover. It is not difficult then to find a finite collection of open intervals $J_1, \dots, J_M$ which cover $[a,b]$; each $J_j \subset J(c_j)$ for some c_j such that $J(c_j)$ is in the finite subcover; and the total length of $J_1, \dots, J_M$ is $< 2(b-a)$. The n-dimensional standard boxes $\bigl\{J_j \times S_i(c_j)\bigr\}_{j=1,\;i=1}^{M\quad N_c}$ cover E and have total n-dimensional volume $< \epsilon$. □

[6] If $d_j < 1$ for some j, then (8.12) does not apply. However, for sufficiently large c we have (8.15) and this suffices.

[7] Guillemin-Pollack, Appendix A, has more detail for the proof in this paragraph.

Measure zero on smooth manifolds

Definition 8.19. Let Y be a smooth manifold. A subset $E \subset Y$ has *measure zero* if for all $\mathbb{A}^n$-valued charts $(V, y) \subset Y$, the set $y(E \cap Y) \subset \mathbb{A}^n$ has measure zero.

The choice of n may vary on components of Y. Definition 8.19 would be impractical if we had to check all charts in a maximal atlas, but Proposition 8.10(4) guarantees the following.

Proposition 8.20. *A subset $E \subset Y$ has measure zero if the condition of Definition 8.19 holds for a set of charts of Y which cover E.*

The following implies in particular that the complement of a set of measure zero on a smooth manifold is nonempty.

Proposition 8.21. *Let $E \subset Y$ have measure zero. Then $Y \backslash E$ is dense.*

Proof. $(\overline{Y \backslash E})^c \subset E$ is open and has measure zero (Proposition 8.10(1)), so by Proposition 8.10(6) must be empty. Therefore, $\overline{Y \backslash E} = Y$ as claimed. □

Now we are in a position to prove a special version of Sard's Theorem.

Proof of Corollary 8.4. By second countability it suffices to check locally on X. Let $n = \dim X$ and $m = \dim Y$. With respect to local charts we represent f by a smooth map $g\colon U \to \mathbb{A}^m$, where $U \subset \mathbb{A}^n$ is open. Define

$$
\begin{aligned}
G\colon U \times \mathbb{A}^{m-n} &\longrightarrow \mathbb{A}^m \\
(x, y) &\longmapsto g(x)
\end{aligned}
\tag{8.22}
$$

Then since $U \times \{0\} \subset \mathbb{A}^n \times \{0\} \subset \mathbb{A}^n \times \mathbb{A}^{m-n} = \mathbb{A}^m$ has m-dimensional measure zero (Proposition 8.10(3)), it follows from Proposition 8.10(4) that $G\big(U \times \{0\}\big) = g(U)$ has m-dimensional measure zero. □

Proof of Sard's Theorem

I defer to the appendix in Guillemin-Pollack.

Introduction to fiber bundles

(8.23) *Special types of maps: review.* Let X, Y be smooth manifolds and suppose $f\colon X \to Y$ is a smooth map. In previous lectures we introduced conditions on f that define special classes of maps. The infinitesimal condition that the rank of the differential be maximal, which we can impose at a point of the domain X or at all points of X, leads to a local normal form for f (Theorem 6.15). We also imposed global conditions. For example, an injective immersion which is a homeomorphism onto its image is an embedding, and the image of an embedding is a submanifold (Theorem 7.1). A submersion is a map whose differential is surjective everywhere, and in that case the fibers (inverse images of points) are submanifolds (Theorem 7.10). Now we introduce a special type of submersion in which the diffeomorphism type of the fibers does not jump. The key notion of *local triviality* was flagged by Steenrod in his classic 1951 text *The Topology of Fibre Bundles*, though it had already been around for more than a decade at that point.

Definition 8.24. Let $\pi\colon E \to M$ be a smooth map of smooth manifolds. We say π is a *fiber bundle* if there exists $\{(U_\alpha, F_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ such that $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M, each F_α is a smooth manifold, and φ_α is a diffeomorphism $\varphi\colon U_\alpha \times F_\alpha \to \pi^{-1}(U)$ such that the diagram

$$\begin{array}{ccc} U_\alpha \times F_\alpha & \xrightarrow{\varphi_\alpha} & \pi^{-1}(U_\alpha) \\ & {\scriptstyle \mathrm{pr}_1}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & U_\alpha & \end{array} \tag{8.25}$$

commutes. The domain E of π is called the *total space* and the codomain M is called the *base* of the fiber bundle π.

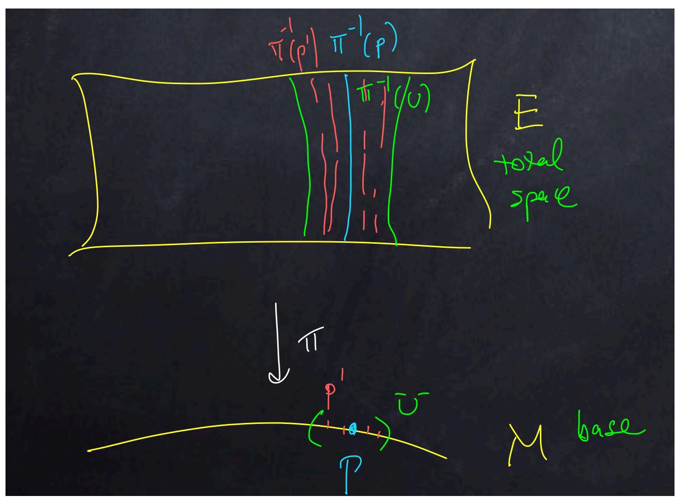


FIGURE 14. A fiber bundle with local trivialization about p

Here pr_1 is projection onto the first factor, followed by the inclusion $U \hookrightarrow M$. The *condition* that pairs (U, φ) exist about every point $p \in M$ is called *local triviality*. It implies that π is a submersion, hence the fibers are submanifolds.

Remark 8.26. We emphasize that $\{(U_\alpha, F_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ is *not* data. We do not want to carry it around! Its existence is a *condition* on the map π.

Remark 8.27. Some authors require that π be surjective. We allow the fibers $\pi^{-1}(p)$ to be empty. A fiber bundle is sometimes labeled by the total space, which is a smooth manifold; however, the fiber bundle *is* the function.

Remark 8.28. There is also a notion of a fiber bundle with fixed fiber F; it is the case in which $F_\alpha = F$ for all $\alpha \in A$.

Remark 8.29. Local triviality (8.25) provides a diffeomorphism $\pi^{-1}(p) \xrightarrow{\approx} F_\alpha$ of the fiber over any $p \in U_\alpha$ with the fixed manifold F_α. If we, heuristically, rewrite π as a map from M to sets, then local triviality expresses the local constancy of this map.

Example 8.30 (trivial fiber bundle)**.** Let M, F be smooth manifolds. Then projection $\mathrm{pr}_1\colon M \times F \to M$ is a fiber bundle. To verify the condition of local triviality, for any $p \in M$ we can choose $U = M$ and $\varphi = \mathrm{id}$. This is the *trivial* fiber bundle over M with fiber F. Any fiber bundle is locally isomorphic to a trivial fiber bundle. (We introduce the appropriate notion of isomorphism in the next lecture.)

Example 8.31 (a nontrivial example)**.** The map

$$\begin{aligned} \pi\colon O_n &\longrightarrow S^{n-1} \\ A &\longmapsto A\xi_0 \end{aligned} \tag{8.32}$$

is a fiber bundle, where, say, $\xi_0 = (1, 0, \dots, 0)$. It is nontrivial (meaning not isomorphic to a trivial bundle) if $n \geqslant 3$. We will not prove that statement now.

In the next lecture we give more examples and focus on the cases of interest in this course: the tangent bundle to a smooth manifold and the normal bundle to a submanifold.

(8.33) *Perspective.* Any map (of sets) $\pi\colon E \to M$ induces a partition of the domain into its fibers. Fiber bundles induce "regular" partitions in that the fibers are locally diffeomorphic to each other. In this way fiber bundles provide useful decompositions of smooth manifolds, and that can be a great tool to study global properties. In a different direction, fiber bundles can encode the geometry of the base manifold. The tangent bundle is an example, and there are many associated bundles that also come into play. Since the total space is a smooth manifold in its own right, we can apply the tools of manifold theory to it and often learn about the base in the process.

Lecture 9: Fiber bundles and vector bundles

Recollection of fiber bundles

We resume our discussion of fiber bundles. Recall that a fiber bundle $\pi\colon E \to M$ is a special kind of map between smooth manifolds. It satisfies the *local triviality* condition specified in Definition 8.24. Note here that locality is in the codomain, which is called the *base* of the fiber bundle. In that version we ask that the fibers be "locally constant" functions of the base: nearby fibers are diffeomorphic in a smooth way, which is the content of the map φ in (8.25). A closely related notion fixes a model manifold for the fiber.

Definition 9.1. Let $\pi\colon E \to M$ be a smooth map of smooth manifolds, and let F be a (nonempty) smooth manifold. We say π is a *fiber bundle with fiber* F if for all $p \in M$ there exists an open neighborhood $U \subset M$ about p and a diffeomorphism $\varphi\colon U \times F \to \pi^{-1}(U)$ such that the diagram

(9.2)
$$\begin{array}{ccc} U \times F & \xrightarrow{\;\varphi\;} & \pi^{-1}(U) \\ & {\scriptstyle \mathrm{pr}_1}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & U & \end{array}$$

commutes.

(9.3) *Notation.* If $\pi\colon E \to M$ is a fiber bundle, then for $p \in M$ we set $E_p = \pi^{-1}(p)$ to be the fiber over the point p in the base.

(9.4) *Maps of fiber bundles.* The parametrized version of a smooth map of manifolds is a map of fiber bundles. Let $\pi'\colon E' \to M$ and $\pi\colon E \to M$ be fiber bundles over the same base. Then a map of fiber bundles is a smooth map $\varphi\colon E' \to E$ which fits into the commutative diagram

(9.5)
$$\begin{array}{ccc} E' & \xrightarrow{\;\varphi\;} & E \\ & {\scriptstyle \pi'}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & M & \end{array}$$

The single smooth map φ encodes a smooth family of smooth maps $\varphi_p\colon E'_p \to E_p$ parametrized by $p \in M$; indeed 'smoothness' of the family means smoothness of the single map φ. (Consider the special case in which $M = \mathbb{R}_x$, $E' = \mathbb{R}_x \times \mathbb{R}_y$, and $E = \mathbb{R}_x \times \mathbb{R}_z$. The function φ is given by a function $z = z(x,y)$ of two variables, and we are saying that smoothness jointly in the two variables x, y is what we mean by a smoothly varying family of functions of y parametrized by x.) When the fiber bundles π', π have extra structure—such as bundles of affine spaces, vector spaces, Lie groups, etc.—then we may require that each φ_p preserve that structure.

Examples of fiber bundles

Example 9.6 (covering spaces)**.** A (smooth) covering space $\pi\colon E \to M$ is a fiber bundle. To see this, recall the definition: every point $p \in M$ has an open neighborhood $U \subset M$ which is evenly covered, i.e., there is a discrete set S (which can depend on Uy) and a diffeomorphism

(9.7)
$$\varphi\colon U \times S \longrightarrow \pi^{-1}(U)$$

which commutes with projection to U. This is precisely the local trivialization condition. So a fiber bundle with discrete fibers is a covering space.

Example 9.8 (affine lines in a plane)**.** Let V be a 2-dimensional real vector space, and let A be an affine space over V. Let E be the 2-dimensional manifold of affine lines in A. Each affine line determines a line in V—a 1-dimensional subspace—namely its tangent line. The assignment of a tangent line is a smooth map

$$\pi\colon E \longrightarrow \mathbb{P}V. \tag{9.9}$$

We claim that π is a fiber bundle. Fix $K \in \mathbb{P}V$ and $p \in A$. We produce a local trivialization of π on $U = \mathbb{P}V \backslash \{K\}$. First, observe that p determines a section $s_p\colon \mathbb{P}V \to E$ of (9.9) which assigns to each $L \in \mathbb{P}V$ the unique affine line through p with tangent line L. As depicted in Figure 15, define

$$\begin{aligned} \varphi\colon U \times K &\longrightarrow \pi^{-1}(U) \\ L\ ,\ \xi &\longmapsto s_p(L) + \xi \end{aligned} \tag{9.10}$$

It is easily checked that φ is a diffeomorphism which commutes with projection to U, i.e., φ is a local trivialization.

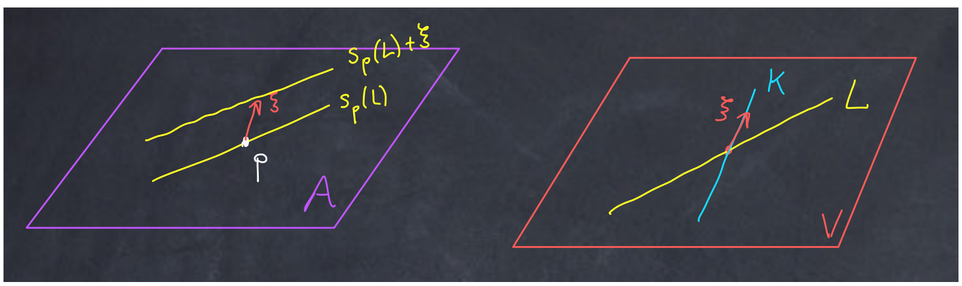


FIGURE 15. Affine lines, their tangent lines, and a local trivialization of (9.9)

Remark 9.11. The fibers of (9.9) have more structure: they are affine spaces. More precisely, $\pi^{-1}(L)$ is affine over the quotient vector space V/L. (If the affine line $\ell \subset A$ has tangent line $L \subset V$, and $\xi \in V$, then the affine line $\ell + \xi$ only depends on $\xi \pmod L$ since translation by vectors in L preserves ℓ.) In fact, there is a vector bundle $Q \to \mathbb{P}V$ whose fiber at $L \in \mathbb{P}V$ is the one-dimensional vector space V/L, and (9.9) is a bundle of affine lines over $Q \to \mathbb{P}V$, a parametrized version of a single affine space over a single vector space. This illustrates the idea that fiber bundles can have more structure, in which case we require that local trivializations preserve that structure. Note that in this case the local trivialization (9.10) *is* an affine map on each fiber.

Example 9.12 (a surjective submersion which is not a fiber bundle)**.** Work in the affine space $\mathbb{A}^3$ with coordinates (x, y, z). Define

$$E = \bigl\{(x,y,z) \in \mathbb{A}^3 : y^2 + z^2 = 1\bigr\} \setminus \bigl\{(0,0,+1), (0,0,-1)\bigr\}. \tag{9.13}$$

This is a cylinder with two points n, s deleted. Let P denote the space of affine planes in $\mathbb{A}^3$ which contain the z-axis; then P is diffeomorphic to $\mathbb{RP}^1$, the space of lines through the origin in the x, y-plane. Define

$$\pi\colon E \longrightarrow P \tag{9.14}$$

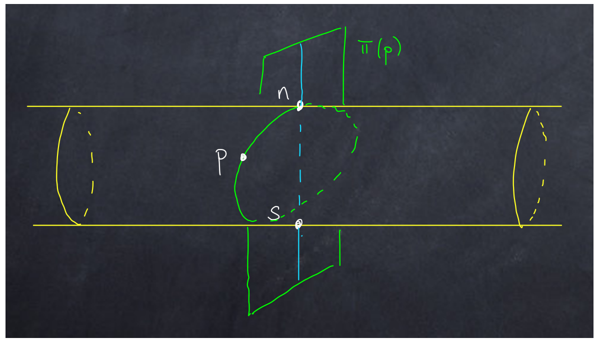


FIGURE 16. A surjective submersion which is not a fiber bundle

the map which takes $p \in E$ to the plane containing the distinct non-collinear points n, s, p, as depicted in Figure 16. Then π is surjective and a submersion. (Proof of the latter: a motion germ in P is represented by a curve Π_t of planes through the z-axis. Intersect with the affine line $x = 1$, $z = 0$ to lift to a motion p_t in E such that $\pi(p_t) = \Pi_t$.) However, π is not a fiber bundle. The typical fiber of π is an ellipse minus the points n, s, whereas the fiber over the x, z-plane $\Pi_{x,z}$ is the union of two affine lines minus n, s. The former has two components while the latter has four components. Hence the special fiber over $\Pi_{x,z}$ is not diffeomorphic to the other fibers. Therefore, π cannot be locally trivial over $\Pi_{x,z}$.

Vector bundles

(9.15) *Fiber product.* As a preliminary, we introduce the fiber product of fiber bundles. It is the parametrized version of the Cartesian product of manifolds. Let M be a smooth manifold and $\pi_i \colon E_i \to M$, $i = 1, 2$, fiber bundles over M. Define

$$E_1 \times_M E_2 = \big\{(e_1, e_2) \in E_1 \times E_2 : \pi_1(e_1) = \pi_2(e_2)\big\}. \tag{9.16}$$

Then $E_1 \times_M E_2 \subset E_1 \times E_2$ is a submanifold. This is easily proved once we introduce transversality, though it is not difficult to do directly. The diagram

$$\begin{array}{ccc} E_1 \times_M E_2 & \xrightarrow{\mathrm{pr}_2} & E_2 \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ E_1 & \xrightarrow{\pi_1} & M \end{array} \tag{9.17}$$

commutes, and the composition

$$\pi \colon E_1 \times_M E_2 \longrightarrow M. \tag{9.18}$$

from northwest to southeast is a fiber bundle: local trivializations φ_1, φ_2 of E_1, E_2 over open neighborhoods U_1, U_2 of a point $p \in M$ combine to a local trivialization of (9.18) over $U_1 \cap U_2$. The construction generalizes to the fiber product of a finite set of fiber bundles over a common base.

(9.19) *Vector space.* Recall the definition of a vector space. It consists of data $(V, 0, +, \times)$ where V is a set; $0 \in V$ is a distinguished element, the zero vector; $+\colon V \times V \to V$ is called vector addition; and $\times\colon \mathbb{R} \times V \to V$ is called scalar multiplication. There are many axioms which tell that $(V, 0, +)$ is an abelian group, scalar multiplication distributes over vector addition, etc.

(9.20) *Vector bundle.* Just as the fiber product **(9.15)** is a parametrized version of Cartesian product, a vector bundle is a parametrized version of a vector space.

Definition 9.21. A *vector bundle* $(\pi, 0, +, \times)$ consists of a fiber bundle $\pi\colon E \to M$; a section $0\colon M \to E$ of π, called the *zero section*; a smooth map $+\colon E \times_M E \to E$ such that

(9.22)
$$\begin{array}{ccc} E \times_M E & \xrightarrow{\;+\;} & E \\ & \searrow \quad \swarrow{\scriptstyle \pi} & \\ & M & \end{array}$$

commutes; and a smooth map $\times\colon \mathbb{R} \times E \to E$ such that

(9.23)
$$\begin{array}{ccc} \mathbb{R} \times E & \xrightarrow{\;\times\;} & E \\ & \searrow \quad \swarrow{\scriptstyle \pi} & \\ & M & \end{array}$$

commutes. We require the vector space axioms and also that local trivializations for π be linear maps on fibers.

The vector space axioms tell that each fiber E_p, $p \in M$, of $\pi\colon E \to M$ is a vector space. Thus π exhibits a family of vector spaces parametrized by M. The last condition—that local trivializations be linear on fibers—imposes that this be a locally trivial family of vector spaces. Explicitly, about each $p \in M$ there exists an open neighborhood $U \subset M$ and a diffeomorphism φ in the diagram

(9.24)
$$\begin{array}{ccc} U \times E_p & \xrightarrow{\;\varphi\;} & \pi^{-1}(U) \\ & {\scriptstyle \mathrm{pr}_1}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & U & \end{array}$$

such that $\varphi\big|_{p' \times E_p}\colon E_p \to E_{p'}$ is a linear isomorphism for all $p' \in U$.

Analogous to Definition 9.1 is the definition of a vector bundle with fiber a fixed vector space. We leave the reader to formulate it.

Constructions of vector bundles

(9.25) *Transition functions.* Let V be a finite dimensional real vector space and suppose $\pi\colon E \to M$ is a vector bundle with fiber V. Let $U_1, U_2 \subset M$ be open sets equipped with local trivializations

$$\begin{array}{ccc} U_i \times V & \xrightarrow{\varphi_i} & \pi^{-1}(U_i) \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & U_i & \end{array} \tag{9.26}$$

Over the intersection $U_1 \cap U_2$ the ratio of the trivializations is the *transition function*

$$\begin{aligned} g_{21}\colon U_1 \cap U_2 &\longrightarrow \qquad\qquad \mathrm{Aut}(V) \\ p \quad &\longmapsto \bigl(\xi \mapsto (\mathrm{pr}_2 \circ \varphi_2^{-1} \circ \varphi_1)(p,\xi)\bigr) \end{aligned} \tag{9.27}$$

which compares them. Order matters: g_{21} is the transition from trivialization 1 to trivialization 2. The transition functions satisfy

$$g_{12} \circ g_{21} = \mathrm{id}_V \qquad \text{on } U_1 \cap U_2, \tag{9.28}$$

$$g_{32} \circ g_{21} = g_{31} \qquad \text{on } U_1 \cap U_2 \cap U_3. \tag{9.29}$$

Equation (9.29) is called the *cocycle condition*. It can be written

$$g_{23} \circ g_{13}^{-1} \circ g_{12} = \mathrm{id}_V, \tag{9.30}$$

whose three constituents on the left hand side are obtained by striking out a single digit from 123 and alternating the "signs" as we go across: $\not{1}23 \circ 1\not{2}3^{-1} \circ 12\not{3}$.

(9.31) *Patching construction.* One method to construct vector bundles, or to construct new vector bundles from old ones, is to patch local trivial bundles using transition functions.

Theorem 9.32. *Let M be a smooth manifold and $\{U_\alpha\}_{\alpha\in A}$ an open cover. Let V be a finite dimensional real vector space, and suppose given smooth functions*

$$g_{\alpha_2\alpha_1}\colon U_{\alpha_1} \cap U_{\alpha_2} \longrightarrow \mathrm{Aut}(V), \qquad \alpha_1, \alpha_2 \in A, \tag{9.33}$$

such that for all $\alpha_1, \alpha_2, \alpha_3 \in A$,

$$g_{\alpha_1\alpha_2} \circ g_{\alpha_2\alpha_1} = \mathrm{id}_V \qquad \text{on } U_{\alpha_1} \cap U_{\alpha_2}, \tag{9.34}$$

$$g_{\alpha_3\alpha_2} \circ g_{\alpha_2\alpha_1} = g_{\alpha_3\alpha_1} \qquad \text{on } U_{\alpha_1} \cap U_{\alpha_2} \cap U_{\alpha_3}. \tag{9.35}$$

Then there is a canonical vector bundle $\pi\colon E \to M$ equipped with local trivializations over the sets U_α such that the transition functions equal (9.33).

I will not spell out the precise meaning of 'canonical'; heuristically, it means there are no choices made beyond the given data.

Proof. Define an equivalence relation $\sim$ on the disjoint union $\bigsqcup_{\alpha\in A} U_\alpha \times V$ as

$$(p_1,\xi_1)\sim(p_2,\xi_2) \qquad \text{iff} \qquad \begin{aligned} p_1 &= p_2 \\ \xi_2 &= g_{\alpha_2\alpha_1}(p)(\xi_1) \end{aligned} \tag{9.36}$$

where $p_i \in U_{\alpha_i}$ and $\xi_i \in V$, $i = 1, 2$. Define

$$E = \bigsqcup_{\alpha\in A} U_\alpha \times V \;/\!\sim, \tag{9.37}$$

the set of equivalence classes. Endow E with the quotient topology of the product topology. For each $\alpha \in A$ the composition

$$U_\alpha \times V \longrightarrow \bigsqcup_{\alpha\in A} U_\alpha \times V \longrightarrow E \tag{9.38}$$

is a homeomorphism onto its image, and the images cover E. It follows that E is locally Euclidean. The projections $U_\alpha \times V \to U_\alpha$ stitch to a surjective continuous map

$$\pi\colon E \longrightarrow M. \tag{9.39}$$

Since M is second countable and the fibers of π are homeomorphic to the second countable space V, it follows that E is second countable. Also, M is Hausdorff, so if two points in E project to distinct points in M they can be separated by open sets. The fibers of E are homeomorphic to V, so are Hausdorff. It follows that E is Hausdorff. Therefore, E is a topological manifold.

Choose an atlas of M such that the domain U of each chart (U,x) is contained in some U_α. Assume the charts are standard and that M is n-dimensional. The composition

$$\pi^{-1}(U) \longrightarrow U \times V \xrightarrow{\;x\times\mathrm{id}\;} \mathbb{A}^n \times V \tag{9.40}$$

is a homeomorphism onto its image, so is a chart on E, and these charts have C^∞ overlaps. (The first map in (9.40) is the inverse of the restriction of (9.38) to $U \times V$, inverted on its image.) This endows E with the structure of a smooth manifold, and the map (9.39) is smooth.

The vector space structure on V—the zero vector, vector addition, and scalar multiplication—induce a vector space structure on (9.39) since the equivalence relation (9.36) is defined by linear maps. The vector space axioms hold on each fiber of π since they hold on V. The diffeomorphisms (9.38) are local trivializations which are linear on each fiber. ☐

Example 9.41. Theorem 9.32 applies also to *complex* vector bundles. (A complex vector bundle is defined by replacing '$\mathbb{R}$' in Definition 9.21 with '$\mathbb{C}$'.) Write points of $\mathbb{CP}^1$ as equivalence classes $[z^0, z^1]$ of pairs of complex numbers, not both zero, under the equivalence $[z^0, z^1] = [\lambda z^0, \lambda z^1]$ for $\lambda \in \mathbb{C}^{\neq 0}$. Cover $\mathbb{CP}^1$ by two (affine) open sets

$$\begin{aligned} U_1 &= \{[z,1] : z \in \mathbb{C}\} \\ U_2 &= \{[1,w] : w \in \mathbb{C}\}, \end{aligned} \tag{9.42}$$

and for $k \in \mathbb{Z}$ define the transition function

$$\begin{aligned} g_{21}^{(k)} \colon U_1 \cap U_2 &\longrightarrow \operatorname{Aut}(\mathbb{C}) = \mathbb{C}^{\times} \\ [z,1] &\longmapsto \quad z^k \end{aligned} \tag{9.43}$$

where $\mathbb{C}^{\times} = \mathbb{C}^{\neq}$. Note that $z \neq 0$ if $[z,1] \in U_1 \cap U_2$. The preceding construction gives a complex line bundle $L^{(k)} \to \mathbb{CP}^1$ for each k.

(9.44) *Another construction.* In practice, we often encounter a variation of Theorem 9.32 in which we have a family of vector spaces parametrized by a smooth manifold and we want to construct from it a vector bundle.

Theorem 9.45. *Let M be a smooth manifold and*

$$\pi \colon E = \bigsqcup_{p \in M} E_p \longrightarrow M \tag{9.46}$$

a set of vector spaces parametrized by M. Suppose given an open cover $\{U_\alpha\}_{\alpha \in A}$ together with a family of vector spaces $\{V_\alpha\}_{\alpha \in A}$ and isomorphisms φ_α, $\alpha \in A$, such that the diagram

$$\begin{array}{ccc} U_\alpha \times V_\alpha & \xrightarrow{\ \varphi_\alpha\ } & \bigsqcup_{p \in U_\alpha} E_p \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & U_\alpha & \end{array} \tag{9.47}$$

commutes and $\varphi_\alpha\big|_{\{p\} \times V_\alpha}$ is a linear isomorphism $V_\alpha \to E_p$ for all $p \in U_\alpha$. There are induced transition functions

$$g_{\alpha_2 \alpha_1} \colon U_{\alpha_1} \cap U_{\alpha_2} \longrightarrow \operatorname{Iso}(V_{\alpha_1}, V_{\alpha_2}) \tag{9.48}$$

defined by composition $\varphi_{\alpha_2}^{-1}$ and φ_{α_1}. Assume they satisfy the cocycle conditions

$$g_{\alpha_1 \alpha_2} \circ g_{\alpha_2 \alpha_1} = \mathrm{id}_{V_{\alpha_1}} \qquad \text{on } U_{\alpha_1} \cap U_{\alpha_2}, \tag{9.49}$$

$$g_{\alpha_3 \alpha_2} \circ g_{\alpha_2 \alpha_1} = g_{\alpha_3 \alpha_1} \qquad \text{on } U_{\alpha_1} \cap U_{\alpha_2} \cap U_{\alpha_3}. \tag{9.50}$$

Then there is a canonical manifold structure on E so that (9.46) is a vector bundle and the φ_α are smooth local trivializations.

The proof is similar to that of Theorem 9.32. Use the disjoint union of the φ_α to topologize E and make charts on E as before.

(9.51) *Dual bundle.* As an application of Theorem 9.45 we construct the dual bundle to a vector bundle. It is the parametrized version of the passage from a vector space to its linear dual. As a preliminary, recall that if $T\colon V \to W$ is a linear map of vector spaces, then there is a dual map $T^*\colon W^* \to V^*$ defined by

$$\langle T^*w^*, v\rangle = \langle w^*, Tv\rangle, \qquad v \in V, \quad w^* \in W^*, \tag{9.52}$$

where the pairing on the left hand side is between V^* and V, and the pairing on the right hand side is between W^* and W. If T is invertible, then so is T^*. For V finite dimensional, there is an isomorphism of Lie groups

$$\begin{aligned} \operatorname{Aut}(V) &\longrightarrow \operatorname{Aut}(V^*) \\ T &\longmapsto (T^*)^{-1} \end{aligned} \tag{9.53}$$

Let $\pi\colon E \to M$ be a vector bundle. Choose an open cover $\{U_\alpha\}_{\alpha\in A}$ of M together with, for each $\alpha \in A$, a local trivialization

$$\begin{array}{ccc} U_\alpha \times V_\alpha & \xrightarrow{\varphi_\alpha} & \pi^{-1}(U_\alpha) \\ & {\scriptstyle \mathrm{pr}_1}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & U_\alpha & \end{array} \tag{9.54}$$

over each U_α, $\alpha \in A$. (Such open covers exist by the definition of a vector bundle.) Define the disjoint union $E^* = \bigsqcup_{p\in M} E_p^*$, and for $\alpha \in A$ set

$$\begin{aligned} \widetilde{\varphi}_\alpha\colon U_\alpha \times V_\alpha^* &\longrightarrow \bigsqcup_{p\in U_\alpha} E_p^* \\ (p \;\;,\;\; \xi^*) &\longmapsto \bigl(\varphi_\alpha(p)^*\bigr)^{-1}(\xi^*) \end{aligned} \tag{9.55}$$

where $\varphi_\alpha(p) \in \operatorname{Iso}(V_\alpha, E_p)$ and we apply its inverse dual to obtain an isomorphism $V_\alpha^* \to E_p^*$. Since (9.53) is a homomorphism, the cocycle conditions for the transition functions derived from the φ_α imply the cocycle conditions (9.49), (9.50) for the transition functions derived from the $\widetilde{\varphi}$. Then Theorem 9.45 applies to construct a smooth vector bundle structure on $E^* \to M$.

Remark 9.56. A similar procedure allows us to carry over functorial maps in linear algebras to vector bundles. For example, if $E' \to M$ and $E \to M$ are vector bundles, then there is a vector bundle $\operatorname{Hom}(E', E) \to M$ whose fiber at $p \in M$ is $\operatorname{Hom}(E'_p, E_p)$.

Remark 9.57. A more powerful approach is to form the *frame bundle* of $\pi\colon E \to M$, a fiber bundle whose fiber at $p \in M$ is the manifold of bases of E_p. From the frame bundle we can directly form the dual bundle as well as many other associated vector bundles and associated fiber bundles. (For example, there is an associated fiber bundle $\mathbb{P}E \to M$ whose fiber at $p \in M$ is the projective space $\mathbb{P}(E_p)$.)

(9.58) *Quotient bundle.* Let M be a smooth manifold and $\pi'\colon E'\to M$, $\pi\colon E\to M$ smooth vector bundles. Suppose $\iota\colon M\to \operatorname{Hom}(E',E)$ is a section of $\operatorname{Hom}(E',E)\to M$ which is injective on each fiber. Then we can replace E'_p with $\iota(E'_p)\subset E_p$ and so assume that π' is a *subbundle* of π. Define the family of vector spaces

$$E''=\bigsqcup_{p\in M}E_p/E'_p\longrightarrow M \tag{9.59}$$

Then (9.59) is a vector bundle in a natural way. Namely, choose an open cover $\{U_\alpha\}_{\alpha\in A}$ of M together with a family $\{V_\alpha\}_{\alpha\in A}$ of vector spaces, a family $\{V'_\alpha\}_{\alpha\in A}$ of subspaces, and local trivializations $\varphi_\alpha\colon U_\alpha\times V_\alpha\to\pi^{-1}(U_\alpha)$ of π which restrict to local trivializations $\varphi'_\alpha\colon U_\alpha\times V'_\alpha\to(\pi')^{-1}(U_\alpha)$ of π'. Define

$$\begin{aligned}\varphi''_\alpha\colon U_\alpha\times V_\alpha/V'_\alpha&\longrightarrow\bigsqcup_{p\in U_\alpha}E_p/E'_p\\ p\ ,\ [\xi]&\longmapsto[\varphi_\alpha(\xi)]\end{aligned} \tag{9.60}$$

Now apply Theorem 9.45.

Tangent, cotangent, and normal bundles

Let M be a smooth manifold. Recall the construction of the tangent space T_pM from Lecture 3.

(9.61) *Tangent bundle.* Define

$$\pi\colon TM=\bigsqcup_{p\in M}T_pM\longrightarrow M \tag{9.62}$$

Let $\{(U_\alpha,x_\alpha)\}_{\alpha\in\mathcal{A}}$ be an atlas with $x_\alpha\colon U_\alpha\to A_\alpha$, where A_α is an affine space over a vector space V_α. Recall that for each $p\in U_\alpha$ the chart gives an isomorphism $T_pM\to V_\alpha$. Use its inverse to construct a set isomorphism

$$\varphi_\alpha\colon U_\alpha\times V_\alpha\longrightarrow\bigsqcup_{p\in M}T_pM \tag{9.63}$$

Now apply Theorem 9.45 to produce the tangent bundle π as a smooth vector bundle.

Definition 9.64. A *vector field* is a section of the tangent bundle $\pi\colon TM\to M$.

(9.65) *Charts on TM.* Suppose $(U; x^1, \dots, x^n)$ is a standard chart on M. We use it to construct a standard chart $(\pi^{-1}U; \pi^*x^1, \dots, \pi^*x^n, \dot{x}^1, \dots, \dot{x}^n)$, where $\pi^*x^i = x^i \circ \pi$ is the *pullback* function on $\pi^{-1}U$. The functions $\dot{x}^i \colon \pi^{-1}U \to \mathbb{R}$ are determined by

$$\xi = \dot{x}^i(\xi)\, \frac{\partial}{\partial x^i}\bigg|_{\pi(\xi)}, \qquad \xi \in \pi^{-1}U. \tag{9.66}$$

The overlap functions with a chart induced from a second chart $(U; y^1, \dots, y^n)$ of M with the same domain are computed by:

$$\xi = \dot{x}^i \frac{\partial}{\partial x^i} = \dot{x}^i \frac{\partial y^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha} = \dot{y}^\alpha \frac{\partial}{\partial y^\alpha}, \tag{9.67}$$

which leads to the overlap functions

$$\begin{aligned} y^\alpha &= y^\alpha(x^1, \dots, x^n) \\ \dot{y}^\alpha &= \dot{x}^i \frac{\partial y^\alpha}{\partial x^i}, \end{aligned} \tag{9.68}$$

where $y^\alpha(x^1, \dots, x^n)$ are the overlap functions on M.

(9.69) *Cotangent bundle.* Apply **(9.51)** to construct the cotangent bundle

$$T^*M \longrightarrow M \tag{9.70}$$

as the dual vector bundle to the tangent bundle.

Definition 9.71. A *1-form* is a section of the cotangent bundle $T^*M \to M$.

Observe that if $f \colon M \to \mathbb{R}$ is a smooth real-valued function on M, then its differential df is a 1-form on M.

(9.72) *Normal bundle.* Suppose M is a smooth manifold and $N \subset M$ is a smooth submanifold. Then for all $p \in N$, the tangent space to N is a subspace $T_pN \subset T_pM$ of the tangent space to M. This leads to an inclusion of vector bundles $TN \subset TM\big|_N$ over N. Now apply **(9.58)** to construct the *normal bundle* $\nu \to N$ as the quotient bundle. There is a short exact sequence of vector bundles

$$0 \longrightarrow TN \longrightarrow TM\big|_N \longrightarrow \nu \longrightarrow 0 \tag{9.73}$$

over N. It is a family of short exact sequences of vector spaces parametrized by N. The fiber at $p \in N$ is the quotient space

$$\nu_p = T_pM/T_pN. \tag{9.74}$$

Lecture 10: Partitions of unity

Let M be a smooth manifold. Often we encounter a situation in which we have in hand a geometric object locally on M and we want to patch together to a global object. As a concrete example, consider the tangent bundle $\pi\colon TM \to M$. We seek a smoothly varying inner product on the fibers of π. A chart (U, x) on M identifies T_pM, $p \in U$, with a fixed vector space V. So we can choose an inner product on V and use the chart to transport it to a smoothly varying family of inner products on T_pM, $p \in U$. We can do so on each chart in an atlas, and so cover M by open sets on which we have the desired inner products. What a partition of unity enables us to do is splice these together into a single global inner product (called a *Riemannian metric*). A partition of unity gives a weighted average a geometric quantity, such as an inner product, and the technique applies as long as the space of those quantities is *convex*, as is the space of positive definite inner products on a real vector space.

Preliminary: some point-set topology

In lecture I proved that a locally compact,[8] Hausdorff, second countable topological space is paracompact. Here I'll simplify a bit and replace 'locally compact' by 'locally Euclidean', i.e., prove the theorem for topological manifolds.

(10.1) *Exhaustion of a manifold by compact subsets.* An *exhaustion* of a space is a nested sequence of subsets whose union is the entire space. The following theorem proves that a topological manifold admits an exhaustion by compact subsets. To give a bit of space between them, each compact subset is conveniently expressed as the closure of an open set. Of course, if the manifold is compact there is nothing to prove.

Theorem 10.2. *Let M be a topological manifold. Then there exists a collection $\{G_j\}_{j\in J}$ of subsets of M with $J = \{1, \dots, N\}$ finite or $J = \mathbb{Z}^{>0}$ countable such that for all $j \in J$ we have*

(i) *G_j is open and $\overline{G_j}$ is compact,*
(ii) *$\overline{G_j} \subset G_{j+1}$,*
(iii) *$M = \bigcup_{j\in J} G_j$.*

Proof. Choose a countable basis for the topology of M and throw out the basis sets whose closure is not compact. Then the remaining sets $B_1, B_2, \dots$ form a basis. (If $U \subset M$ is open and $p \in U$, choose an open set $U' \subset U$ which contains p and has compact closure. Then U' is a union of sets in the original basis, but each of those sets has compact closure, since $\overline{U'}$ is compact, so these sets are in $\{B_i\}_{i=1,2,\dots}$. Repeat for all $p \in U$.)

Set $G_1 = B_1$. Suppose $G_1, \dots, G_j$ are defined. Inductively define $G_{j+1} = B_1 \cup B_2 \cup \cdots \cup B_k$, where k is the smallest positive integer such that $\overline{G_j} \subset B_1 \cup B_2 \cup \cdots \cup B_k$. ∎

[8] A topological space M is *locally compact* if for all $p \in M$ there exists an open set U and a compact set C such that $p \in U \subset C$. If M is locally compact *Hausdorff*, then $\overline{U}$ is compact: in a locally compact Hausdorff space every point p has an open neighborhood with compact closure. (A compact subset of a Hausdorff space is closed, so if we choose $U \subset C$ as in the first sentence of this footnote, then $\overline{U} \subset C$ and hence $\overline{U}$ is compact.) A topological manifold is locally compact Hausdorff: choose U to be the inverse image of a ball in affine space under a local homeomorphism.

(10.3) *Topological manifolds are paracompact.* We begin with open covers and refinements.

Definition 10.4. Let M be a topological space. Let $\{U_\alpha\}_{\alpha\in A}$ and $\{V_\beta\}_{\beta\in B}$ be sets of open subsets of M.

(i) $\{U_\alpha\}_{\alpha\in A}$ is an *open cover* of M if $\bigcup\limits_{\alpha\in A} U_\alpha = M$.

(ii) $\{V_\beta\}_{\beta\in B}$ is a *subcover* of $\{U_\alpha\}_{\alpha\in A}$ if there exists an injective function $r\colon B\to A$ such that $V_\beta = U_{r(\beta)}$ for all $\beta\in B$.

(iii) A *refinement* of $\{U_\alpha\}_{\alpha\in A}$ is an open cover $\{V_\beta\}_{\beta\in B}$ together with a function $r\colon B\to A$ such that $V_\beta\subset U_{r(\beta)}$ for all $\beta\in B$.

(iv) The collection $\{U_\alpha\}_{\alpha\in A}$ is *locally finite* if for all $p\in M$ there exists an open neighborhood $W\subset M$ such that $\{\alpha\in A : W\cap U_\alpha\neq\emptyset\}$ is finite.

(v) M is *paracompact* if every open cover of M has an open locally finite refinement.

Theorem 10.5. *Let M be a topological manifold. Then M is paracompact. In fact, every open cover has a* countable *open locally finite refinement.*

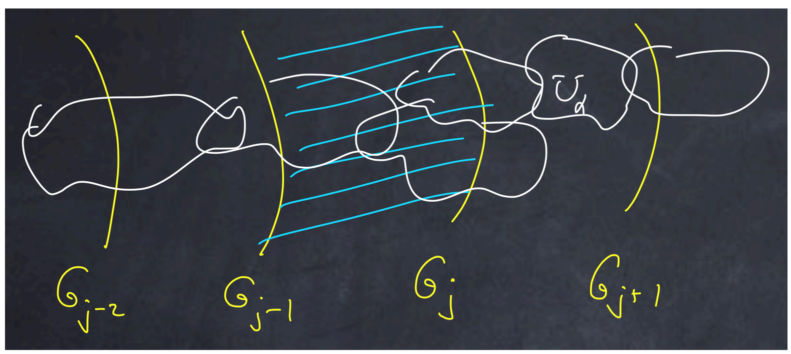


FIGURE 17. Construction of a locally finite refinement

Proof. Fix an exhaustion $\{G_1, G_2, \dots\}$ of M by open sets with compact closure as in Theorem 10.2. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. For each $j\geqslant 3$ the compact set $\overline{G_j}\backslash G_{j-1}$ is a subset of the open set $G_{j+1}\backslash\overline{G_{j-2}}$. Choose a finite subset $F_j\subset A$ such that $\mathcal{U}_j = \{U_\alpha\cap(G_{j+1}\backslash\overline{G_{j-2}})\}_{\alpha\in F_j}$ cover the compact set $\overline{G_j}\backslash G_{j-1}$. Also, choose a finite subset $F_2\subset A$ such that $\mathcal{U}_2 = \{U_\alpha\cap G_3\}_{\alpha\in F_2}$ cover the compact set $\overline{G_3}$. Then $\mathcal{U}_2\cup\bigcup_{j\geqslant 3}\mathcal{U}_j$ is the desired locally finite refinement. □

Partitions of unity

(10.6) *Bump functions.* Recall the definition of the support of a function.

Definition 10.7. Let M be a topological space and $\rho\colon M\to\mathbb{R}$ a continuous function. The *support* of ρ is the closed set

$$\operatorname{supp}\rho = \overline{\rho^{-1}(\mathbb{R}^{\neq 0})}. \tag{10.8}$$

Lemma 10.9. *Let n be a positive integer and let $\mathbb{A}^n$ be standard n-dimensional affine space with coordinates $x^1, \dots, x^n$. Then there exists a smooth function $\rho\colon \mathbb{A}^n \to \mathbb{R}$ such that*

$$\rho(x^1, \dots, x^n) = \begin{cases} 1, & |x^i| \leqslant 1 \text{ for all } i \in \{1, \dots, n\}; \\ 0, & |x^i| \geqslant 2 \text{ for some } i \in \{1, \dots, n\}. \end{cases} \tag{10.10}$$

Introduce the notation

$$C(r) = \bigl\{ (x^1, \dots, x^n) \in \mathbb{A}^n : |x^i| < r \text{ for all } i \in \{1, \dots, n\} \bigr\}. \tag{10.11}$$

Then (10.10) asserts that $\rho \equiv 1$ on $\overline{C(1)}$ and $\rho \equiv 0$ on $\mathbb{A}^n \backslash C(2)$. We sketch the proof.

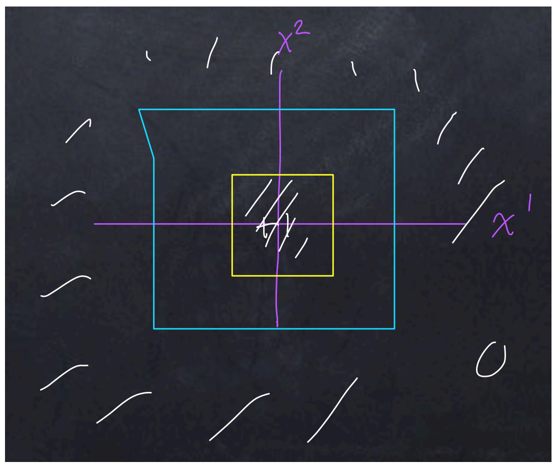


FIGURE 18. A bump function

Proof. Define a succession of C^∞ functions $f_i\colon \mathbb{R} \to \mathbb{R}$ as follows. The first is

$$f_1(x) = \begin{cases} e^{-1/x^2}, & x > 0; \\ 0, & x \leqslant 0. \end{cases} \tag{10.12}$$

Set $f_2(x) = f_1(x) f_1(1-x)$, so that $\operatorname{supp} f_2 = [0,1]$, and then define

$$f_3(x) = \frac{\int_{-\infty}^{x} f_2(t)\,dt}{\int_{-\infty}^{\infty} f_2(t)\,dt}\,. \tag{10.13}$$

Then f_3 is monotonic nondecreasing, $f_3(x) = 0$ if $x \leqslant 0$, and $f_3(x) = 1$ if $x \geqslant 1$. Now set $f_4(x) = f_3(2+x) f_3(2-x)$; it satisfies (10.10) for $n = 1$. For the general case, define

$$\rho(x^1, \dots, x^n) = \prod_{j=1}^{n} f_4(x^j). \tag{10.14}$$

□

(10.15) *Existence of partitions of unity.*

Definition 10.16. Let M be a smooth manifold.

(i) A *partition of unity* $\{\rho_i\}_{i\in I}$ is a set of C^∞ functions $\rho_i\colon M \to \mathbb{R}$ such that
  (a) $\{\operatorname{supp}\rho_i\}_{i\in I}$ is locally finite
  (b) $\rho_i \geqslant 0$
  (c) $\sum\limits_{i\in I} \rho_i(p) = 1$ for all $p \in M$

(ii) If $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M, then $\{\rho_i\}_{i\in I}$ is *subordinate* to $\{U_\alpha\}_{\alpha\in A}$ if there exists a function $r\colon I \to A$ such that $\operatorname{supp}\rho_i \subset U_{r(i)}$ for all $i \in I$.

(iii) If $I = A$ and $r = \mathrm{id}_A$, then we say $\{\rho_i\}_{i\in I}$ is *subordinate with the same index set.*

The locally finite support condition in (i)(a) means the sum in (i)(c) is locally finite, so in particular defines a C^∞ function.

Theorem 10.17. *Let M be a smooth manifold equipped with an open cover $\{U_\alpha\}_{\alpha\in A}$.*

(1) *There exists a countable partition of unity $\{\rho_i\}_{i\in I}$ subordinate to $\{U_\alpha\}_{\alpha\in A}$ such that* $\operatorname{supp}\rho_i$ *is compact for all $i \in I$.*

(2) *There exists a partition of unity $\{\varphi_\alpha\}_{\alpha\in A}$ subordinate to $\{U_\alpha\}_{\alpha\in A}$ with the same index set such that at most countably many φ_α are not identically zero.*

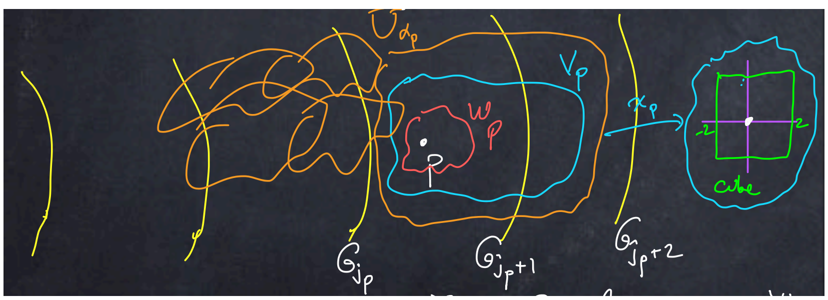


FIGURE 19. Construction of a partition of unity

Proof. Fix an exhaustion $\{G_j\}_{j\in J}$ of M by open sets with compact closure as in Theorem 10.2. For $p \in M$ let j_p be the largest positive integer such that $p \in M\backslash\overline{G_{j_p}}$. Choose $\alpha_p \in A$ such that $p \in U_{\alpha_p}$, and then choose a standard chart (V_p, x_p) such that: (i) $x_p(p) = 0$, (ii) $V_p \subset U_{\alpha_p} \cap (G_{j_p+2}\backslash\overline{G_{j_p}})$, and (iii) $x_p(V_p)$ contains the closed cube $\overline{C(2)}$. Transport the bump function (10.10) via x_p to a smooth function $\psi_p\colon M \to \mathbb{R}$, extending by zero on the complement of V_p. Note that ψ_p has compact support. Define $W_p = x_p^{-1}\bigl(C(1)\bigr)$; then $\psi_p \equiv 1$ on the open set W_p. For each $j \in J$ choose finitely many p with $j_p = j$ such that the corresponding open sets W_p cover the compact set $\overline{G_{j+1}}\backslash G_j$. Enumerate the functions obtained (for all j) as $\psi_1, \psi_2, \dots,$ indexed by a set I which is finite or countable. By construction $\{\operatorname{supp}\psi_i\}_{i\in I}$ is a locally finite cover of M by compact sets. Define

$$\rho_i = \frac{\psi_i}{\sum\limits_{i\in I}\psi_i}, \qquad i \in I. \tag{10.18}$$

The denominator is a finite sum in a neighborhood of each $p \in M$, so is a smooth function, and it is positive everywhere. Then $\operatorname{supp} \rho_i = \operatorname{supp} \psi_i$ is a compact subset of U_{α_p} if the function ψ_i corresponds to the point p. It follows that $\{\rho_i\}_{i \in I}$ satisfies the conditions in (1).

Let $r\colon I \to A$ be the refinement function determined by $\operatorname{supp} \rho_i \subset U_{r(i)}$ (where $r(i) = \alpha_p$ a few lines up). Define

$$\varphi_\alpha = \sum_{i \in r^{-1}(\alpha)} \rho_i, \qquad \alpha \in A. \tag{10.19}$$

Then $\{\varphi_\alpha\}_{\alpha \in A}$ is a partition of unity which satisfies the conditions in (2). ☐

Example 10.20. Consider $M = \mathbb{R}$ with open cover $\{M\}$ consisting of a single set. The partition of unity in Theorem 10.17(2) is the single constant function with value one. By contrast, the construction of Theorem 10.17(1) gives a partition of unity consisting of countably many functions with compact support.

Applications

(10.21) *Bump functions on a manifold.* The following construction is often useful.

Corollary 10.22. *Let M be a smooth manifold with subsets $C \subset U \subset M$ such that C is closed and U is open. Then there exists a smooth function $f\colon M \to \mathbb{R}$ such that*

(i) $0 \leqslant f \leqslant 1$,
(ii) $f\big|_C \equiv 1$,
(iii) $\operatorname{supp} f \subset U$.

Proof. Choose a partition of unity $\{\varphi_U, \varphi_{M\backslash C}\}$ subordinate to the open cover $\{U, M\backslash C\}$ of M and with the same index set. Then $f = \varphi_U$ is the desired function. ☐

(10.23) *Proper functions.* Recall that a continuous function between topological spaces is *proper* if the inverse image of every compact set is compact. On a compact space, a constant function is proper. The following result shows that noncompact manifolds also admit proper functions.

Corollary 10.24. *Let M be a smooth manifold. Then there exists a proper function $f\colon M \to \mathbb{R}$.*

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be the set of open subsets of M with compact closure; it is an open cover of M. Choose a partition of unity $\{\rho_i\}_{i \in I}$ subordinate to this cover as in Theorem 10.17(1): each $\operatorname{supp} \rho_i$ is compact and I is at most countable. We can and do write $I = \{1, \dots, N\}$ for some $N \in \mathbb{Z}^{>0}$ or $I = \mathbb{Z}^{>0}$. Define

$$f = \sum_{i \in I} i\rho_i. \tag{10.25}$$

Then f is a positive function. If $p \in M$ satisfies $f(p) \leqslant j$, then since $\sum_{i\in I}\rho_i(p) = 1$, it follows from (10.25) that $\rho_i(p) \neq 0$ for some $i \leqslant j$. Hence

$$f^{-1}\bigl([-j,j]\bigr) \subset \bigcup_{i=1}^{j} \operatorname{supp}\rho_i. \tag{10.26}$$

Since every compact subset of $\mathbb{R}$ is contained in some interval $[-j,j]$, it follows that f is proper. □

Lecture 11: Whitney embedding theorem; transversality

Most of this lecture concerns the Whitney embedding theorem, which states that every abstract smooth manifold can be realized as a submanifold of affine space. More precisely, it tells that if the manifold has dimension n, then it embeds in an affine space of dimension $2n$. That result, the "hard" Whitney theorem, is not proved here. Rather, we prove the "easy" Whitney theorem that an n-manifold embeds in an affine space of dimension $2n+1$. The proof proceeds in stages. First we demonstrate that a compact manifold embeds in some affine space. The proof uses C^∞ cutoff functions. Then we prove that an n-dimensional submanifold of affine space, compact or not, embeds into $\mathbb{A}^{2n+1}$. Finally, we use these results to prove the theorem for general (noncompact) manifolds.

In the last part of the lecture we introduce transversality, a central concept for the next part of the course.

Embeddings of compact manifolds

(11.1) *Connectivity.* Observe that a function $f\colon M \to \mathbb{A}^N$ is an ordered N-tuple $(f^1,\dots,f^N)$ of real-valued functions $f^i\colon M \to \mathbb{R}$. If f defines an embedding, then by adjoining more real-valued functions to the N-tuple we obtain an embedding into a higher dimensional affine space. Now suppose we prove that every compact *connected* manifold of a given dimension n embeds in some affine space. If M is an arbitrary compact n-manifold, then it has a finite set of components $\{M_1,\dots,M_k\}$. By hypothesis we can choose embeddings $g^i\colon M_i \to \mathbb{A}^{N_i}$, and then adjoin constant functions as necessary to each g^i obtain a map $g\colon M \to \mathbb{A}^N$ for $N = \max_i N_i$. Then g is an immersion but may not be injective. Adjoin one more real-valued function ρ to g, namely the locally constant function with value j on M_j. Then $(g,\rho)\colon M \to \mathbb{A}^{N+1}$ is an injective immersion, hence an embedding since M is compact.

Remark 11.2. The proof of Theorem 11.11 below applies directly to compact manifolds which need not be connected.

(11.3) *Examples.* We report on the embedding question in specific cases.

Example 11.4 (dimension one)**.** A compact connected 1-manifold is diffeomorphic to S^1, a fact we will prove in a subsequent lecture. There is no embedding $S^1 \hookrightarrow \mathbb{A}^1$: a smooth real-valued function on S^1 achieves a maximum, and the map fails to be an immersion where the maximum is achieved. Of course, S^1 embeds in $\mathbb{A}^2$.

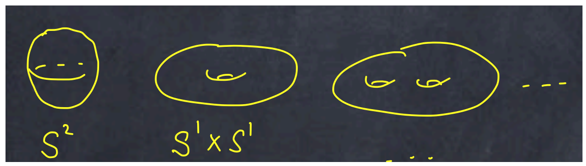


FIGURE 20. A family of compact connected 2-manifolds

Example 11.5 (dimension two)**.** The classification of surfaces, which we do not prove in this course, states that every compact connected 2-manifold is diffeomorphic to a manifold in one of two families. The first is depicted in Figure 20, where for each $g \in \mathbb{Z}^{\geqslant 0}$ there is a surface Σ_g, the "2-sphere with g holes". There exist embeddings $\Sigma_g \hookrightarrow \mathbb{A}^3$, but not embeddings into an affine space of lower dimension. There is a second infinite family whose first member is $\mathbb{RP}^2$, the real projective plane. (The Klein bottle is also in this family.) These manifolds do not admit embeddings into $\mathbb{A}^3$, though again we will not prove this assertion. On the other hand, the function

$$\begin{aligned} f\colon \mathbb{RP}^2 &\longrightarrow \mathbb{A}^4 \\ [x,y,z] &\longmapsto \frac{1}{x^2+y^2+z^2}\,(x^2,xy,yz,xz) \end{aligned} \tag{11.6}$$

is an embedding. (A point of $\mathbb{RP}^2$ is an equivalence class of ordered triples of real numbers, not all zero, under the equivalence relation $[x,y,z] = [\lambda x, \lambda y, \lambda z]$ for $\lambda \in \mathbb{R}^\times = \mathbb{R}^{\neq 0}$.) Observe too that the function

$$\begin{aligned} f\colon \mathbb{RP}^2 &\longrightarrow \mathbb{A}^3 \\ [x,y,z] &\longmapsto \frac{1}{x^2+y^2+z^2}\,(xy,yz,xz) \end{aligned} \tag{11.7}$$

is an immersion (which is not injective).

Given a manifold M we can ask for the minimal N such that M embeds (or immerses) into $\mathbb{A}^N$. The Whitney theorem gives an upper bound to N. The actual minimum is difficult to determine.

Example 11.8 (real projective spaces)**.** The question of a minimal N such that an embedding $\mathbb{RP}^n \hookrightarrow \mathbb{A}^N$ exists has been intensively studied. Here are the results in low dimensions:

(11.9)

n	N
1	2
2	4
3	5
4	8
5	9
6	11
7	12

(See `https://www.lehigh.edu/~dmd1/immtable` for an extensive table.) As you see, the hard Whitney upper bound $N = 2n$ is beat in several cases.

(11.10) *Injective immersions and embeddings.* Recall (Definition 6.18) that an embedding $f\colon M \to N$ of manifolds is an injective immersion which is a homeomorphism onto its image $f(M) \subset N$ (with the induced topology). Suppose that f is an injective immersion. If M is compact, and $C \subset M$ is a closed subset, then C is compact, so too is $f(C)$, and so $f(C) \subset f(M)$ is closed. (A compact subset of a Hausdorff topological space is closed.) Therefore, to produce an embedding of a compact manifold it suffices to produce an injective immersion.

Theorem 11.11. *Let M be a compact smooth manifold. Then for some $N \in \mathbb{Z}^{>0}$ there exists an embedding $f\colon M \to \mathbb{A}^N$.*

Proof. Assume without loss of generality that M has constant dimension n. (See **(11.1)**.) For $r \in \mathbb{R}^{>0}$, let $B(r) \subset \mathbb{A}^n$ denote the open ball of radius r about 0. Apply Corollary 10.22 to produce a C^∞ function $\chi\colon \mathbb{A}^n \to \mathbb{R}$ such that

$$\begin{aligned} 0 \leqslant \chi \leqslant 1 &\qquad \text{on } \mathbb{A}^n \\ \chi \equiv 1 &\qquad \text{on } \overline{B(1)} \\ \chi \equiv 0 &\qquad \text{on } \mathbb{A}^n \backslash B(2) \end{aligned} \tag{11.12}$$

Cover M by a finite set $\{(U_\alpha, x_\alpha)\}_{\alpha\in A}$ of standard charts (in other words, $x_\alpha\colon U_\alpha \to \mathbb{A}^n$) such that $B(2) \subset x_\alpha(U_\alpha) \subset \mathbb{A}^n$ for all $\alpha \in A$ and $\bigl\{x_\alpha^{-1}\bigl(B(1)\bigr)\bigr\}_{\alpha\in A}$ is an open cover of M. (Composition of an arbitrary standard chart (U, x) with a translation and homothety yields a chart (U, x') which satisfies $B(2) \subset x'(U) \subset \mathbb{A}^n$. A compact manifold can be covered with finitely many such charts.) For each $\alpha \in A$, $i \in \{1, \dots, n\}$, use the cutoff function χ to define global functions $\tilde{x}^i_\alpha, \rho_\alpha\colon M \to \mathbb{R}$:

$$\begin{aligned} \tilde{x}^i_\alpha &= \begin{cases} (\chi \circ x_\alpha)\, x^i_\alpha, & \text{on } U_\alpha; \\ 0, & \text{on } M\backslash x_\alpha^{-1}\bigl(B(2)\bigr). \end{cases} \\ \rho_\alpha &= \begin{cases} \chi \circ x_\alpha, & \text{on } U_\alpha; \\ 0, & \text{on } M\backslash x_\alpha^{-1}\bigl(B(2)\bigr). \end{cases} \end{aligned} \tag{11.13}$$

These are smooth functions on M, each constructed as a pair of smooth functions defined on open subsets of M such that the functions agree on the intersection. Assemble these into a single function $f\colon M \to \mathbb{A}^{(n+1)\#A}$:

$$f = \{(\rho_\alpha, \tilde{x}^1_\alpha, \dots, \tilde{x}^n_\alpha)\}_{\alpha\in A}. \tag{11.14}$$

We claim that f is an injective immersion, from which it follows that f is an embedding, since M is compact. For $\alpha \in A$, set $B_\alpha = \rho_\alpha^{-1}(1)$. Since $x_\alpha^{-1}\big(B(1)\big) \subset \rho_\alpha^{-1}(1)$, we conclude that $\{B_\alpha\}_{\alpha\in A}$ is an open cover of M. Now if $p \in B_\alpha$, then $d\tilde{x}^1_\alpha(p), \dots, d\tilde{x}^n_\alpha(p)$ are linearly independent. Hence f is an immersion. If $p, q \in B_\alpha$, then $\tilde{x}^1_\alpha, \dots, \tilde{x}^n_\alpha$ separate p and q. If $p \in B_\alpha$ and $q \notin B_\alpha$, then $\rho_\alpha(p) = 1$ and $\rho_\alpha(q) \neq 1$. Hence f is injective. □

Cutting down the dimension

In this subsection, M need not be compact. We prove that a submanifold M of an affine space A can be projected to a submanifold of a quotient affine space of smaller dimension as long as $2\dim M + 1 < \dim A$.

(11.15) *Quotient affine spaces.* Let A be an affine space over a vector space V, and suppose $W \subset V$ is a linear subspace. Then W acts freely on A by translation, and as usual we denote the quotient—the set of orbits of the action—as A/W. The translation action $V \times A \to A$ descends to an action $V \times A/W \to A/W$ of V on the quotient. The subspace W acts trivially, so we obtain an action $V/W \times A/W \to A/W$. It is easy to verify that this action is free, hence A/W has the structure of an affine space over V/W.

Theorem 11.16. *Let M be an n-dimensional manifold which is embedded into a finite dimensional affine space. Then*

(1) *M admits an immersion into $\mathbb{A}^{2n}$, and*
(2) *M admits an embedding into $\mathbb{A}^{2n+1}$.*

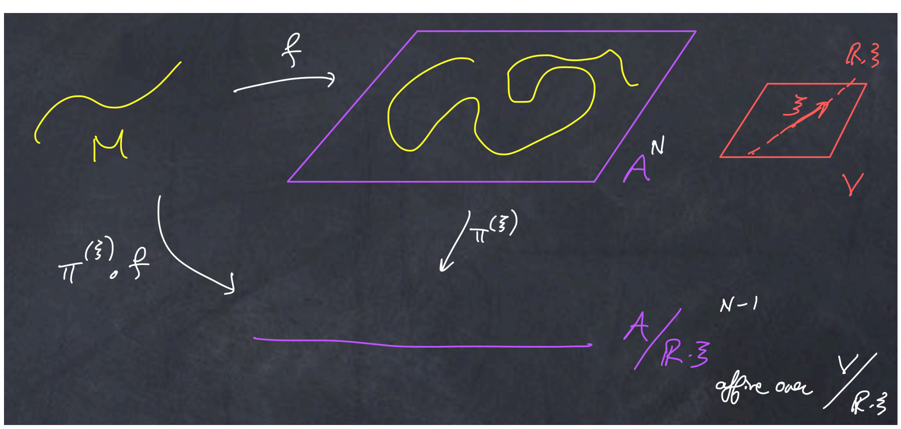


FIGURE 21. Reducing the dimension of an embedding

Proof. Let $f\colon M\to A$ be an embedding into an affine space A over a finite dimensional real vector space V. For $\xi\in V^{\neq 0}$ let

$$\pi^{(\xi)}\colon A\longrightarrow A/\mathbb{R}\cdot\xi \tag{11.17}$$

be the affine projection. Its differential is constant on A; it is the linear projection $V\to V/\mathbb{R}\cdot\xi$ with kernel $\mathbb{R}\cdot\xi$. Hence the composition

$$\pi^{(\xi)}\circ f\colon M\longrightarrow A/\mathbb{R}\cdot\xi \tag{11.18}$$

is an immersion—its differential $d(\pi^{(\xi)}\circ f)=d\pi^{(\xi)}\circ df$ is injective—if and only if $\mathbb{R}\cdot\xi$ is not in the image of the composition

$$F_1\colon TM\xrightarrow{\;df\;}A\times V\xrightarrow{\;\mathrm{pr}_2\;}V. \tag{11.19}$$

Note that the image is a union of subspaces, so it is invariant under scalar multiplication. By Sard's theorem the map F_1 has a dense set of regular values, and if $2n<\dim V$, then any regular value is not in the image. Choose $\xi\in V^{\neq 0}$ not in the image; then (11.18) is an immersion. Repeat until we arrive at an immersion into an affine space of dimension $2n$, thus proving (1).

For the composite (11.18) to be injective, we must choose ξ so that for all $p_0,p_1\in M$ the displacement vector $f(p_1)-f(p_0)$ is not in the linear span of ξ. Define

$$\begin{aligned} F_2\colon M\times M\times\mathbb{R}&\longrightarrow V\\ p_0\ ,\ p_1\ ,\ t&\longmapsto t\bigl[f(p_1)-f(p_0)\bigr]\end{aligned} \tag{11.20}$$

As long as $2n+1<\dim V$ we can choose $\xi\in V^{\neq 0}$ not in the image of F_2, and by Corollary 8.3 we can find a nonzero vector ξ which is not in the image of either F_1 or F_2. In that case, (11.18) is an injective immersion.

If M is compact, then (11.18) is an embedding since M is compact—see **(11.10)**—and so (2) follows. If M is not compact, the preceding gives an injective immersion $f\colon M\to\mathbb{E}^{2n+1}$ into standard Euclidean space (affine space with the Euclidean metric). Replace f by its composition with a diffeomorphism[9] $\mathbb{E}^{2n+1}\to B(1)$ onto the unit ball $B(1)\subset\mathbb{E}^{2k+1}$ with center $0\in\mathbb{E}^{2n+1}$ to achieve $f(M)\subset B(1)$. Choose a proper function $\rho\colon M\to\mathbb{R}$ (see Corollary 10.24), and we can assume that ρ takes only positive values. (The function (10.25) in the proof of Corollary 10.24 has that property.) Set

$$g=(f,\rho)\colon M\longrightarrow\mathbb{E}^{2n+2}. \tag{11.21}$$

The argument above justifies the existence of $\xi\in\mathbb{R}^{2n+2}$ such that $\pi^{(\xi)}\circ g\colon M\to\mathbb{E}^{2n+2}/\mathbb{R}\cdot\xi$ is an injective immersion and $\xi\notin\mathbb{R}\cdot(0,\dots,0,1)$. Choose such a vector ξ. Identify the quotient $\mathbb{E}^{2n+2}/\mathbb{R}\cdot\xi$

[9] Fix $0\in\mathbb{E}^{2n+1}$, identify the complement of $\{0\}$ with $\mathbb{R}^{>0}_r\times S^{2n}$, and then map $(r,\sigma)\mapsto\bigl(r/(1+r^2),\sigma\bigr)$.

with the orthogonal complement $E \subset \mathbb{E}^{2n+2}$ through $0 \in \mathbb{E}^{2n+2}$ to the affine line $0+\mathbb{R}\cdot\xi$. We claim that for all $r>0$ there exists $s>0$ such that

$$(11.22)\qquad \big(\pi^{(\xi)}\circ g\big)^{-1}\big(\overline{B(r)}\big) \subset \rho^{-1}\big([0,s]\big),$$

where $B(r)\subset E$ is the ball about 0 of radius r. Since ρ is proper, $\rho^{-1}\big([0,s]\big)$ is compact, and hence the closed subset in (11.22) is also compact. It follows that $\pi^{(\xi)}\circ g$ is a proper injective immersion, hence an embedding (as you proved on a homework assignment), which is the statement of (2). It remains to prove the claim.

If the claim is false for some $r>0$, choose a sequence $\{p_i\}\subset M$ such that $(\pi^{(\xi)}\circ g)(p_i)\in\overline{B(r)}$ for all i and $\rho(p_i)\to\infty$. Consider

$$(11.23)\qquad \eta_i = \frac{1}{\rho(p_i)}\big[g(p_i)-\pi^{(\xi)}g(p_i)\big] \;\in \mathbb{R}\cdot\xi.$$

Identifying $\mathbb{E}^{2n+k}$ with $\mathbb{R}^{2n+k}$, $k=1,2$, we write

$$(11.24)\qquad \frac{1}{\rho(p_i)}g(p_i) = \big(\frac{1}{\rho(p_i)}f(p_i);1\big) \longrightarrow (0;1)$$

as $i\to\infty$. Also, $\big(1/\rho(p_i)\big)\pi^{(\xi)}g(p_i)\to 0$ as $i\to\infty$. Hence $\eta_i\to(0;1)$, and since $\mathbb{R}\cdot\xi\subset\mathbb{E}^{2n+2}$ is closed we must have $(0;1)\in\mathbb{R}\cdot\xi$. But that contradicts our choice of ξ. ☐

Noncompact manifolds

We complete the proof of the easy Whitney embedding theorem with the following.

Theorem 11.25. *Let M be a smooth manifold. Then there exists an embedding of M into a finite dimensional real affine space.*

Once this is proved, apply Theorem 11.16(2) to obtain the Whitney theorem.

Lemma 11.26. *Let M be a smooth manifold which admits a finite atlas. Then there exists an embedding of M into a finite dimensional real affine space.*

Proof. Follow the proof of Theorem 11.11 to construct an injective immersion (11.14). Now adjoin a proper function, as in (11.21) to obtain a proper injective immersion, i.e., an embedding. ☐

Proof of Theorem 11.25. Construct an exhaustion of M by a sequence $\{G_j\}_{j\in J}$ of nested open sets with compact closure, as in Theorem 10.2. Define

$$(11.27)\qquad \begin{aligned} M_1 &= G_2\\ M_2 &= G_3\\ M_j &= G_{j+1}\backslash\overline{G_{j-2}}, \qquad j\geqslant 3;\end{aligned}$$

then each M_j is an open submanifold of M. Note $M_j \cap M_{j+3} = \emptyset$ for all $j \in J$. Also, M_j admits a finite atlas. Namely, for each $p \in \overline{M_j} = \overline{G_{j+1}} \setminus G_{j-2}$ choose a chart on an open neighborhood $U_p \subset M$ of p. For a finite subset $F_j \subset \overline{M_j}$ the collection $\{U_p\}_{p\in F_j}$ covers the compact set $\overline{M_j}$. Thus we obtain a finite atlas of M_j of charts with domain $U_p \cap M_j$, $p \in F_j$. Now apply Theorem 11.25 and Theorem 11.16 to construct an embedding $f_j\colon M_j \to \mathbb{A}^{2n+1}$ for each $j \in J$. Choose a partition of unity $\{\rho_j, \chi_j\}$ on M subordinate to the open cover $\bigl\{M_j\,,\ M\setminus(\overline{G_j}\setminus G_{j-1})\bigr\}$. Then $M\setminus \operatorname{supp}\chi_j$ is an open set which contains $\overline{G_j}\setminus G_{j-1}$, and $\rho_j \equiv 1$ on $M\setminus\operatorname{supp}\chi_j$. Set $\tilde f_j = \rho_j f_j\colon M \to \mathbb{A}^{2n+1}$. Then $\tilde f_j = f_j$ on $M\setminus\operatorname{supp}\chi_j$, hence $\tilde f_j$ restricts to an injective map on $\overline{G_j}\setminus G_{j-1}$, and $d(\tilde f_j)_p$ is injective for all $p \in \overline{G_j}\setminus G_{j-1}$.

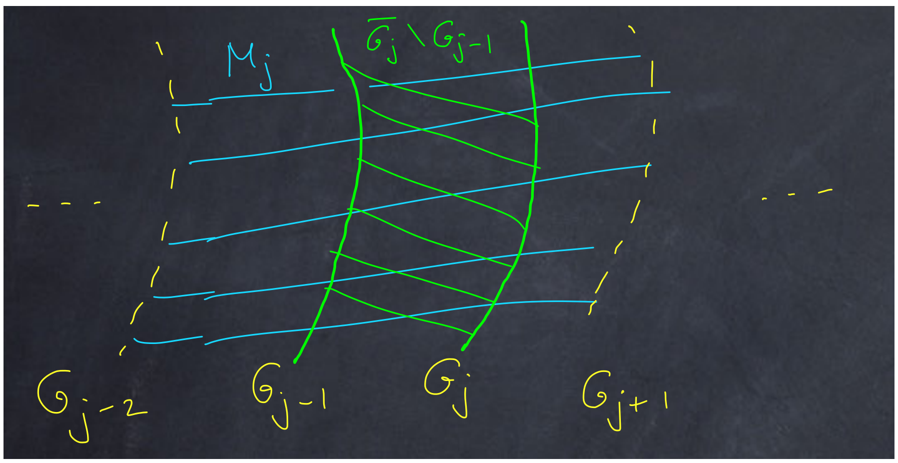


FIGURE 22. The compact subset $\overline{G_j}\setminus G_{j-1}$ of $M_j = G_{j+1}\setminus\overline{G_{j-2}}$

Define

$$
\begin{aligned}
M^{(0)} &= \bigsqcup_{j\equiv 0 \pmod 3} M_j \\
M^{(1)} &= \bigsqcup_{j\equiv 1 \pmod 3} M_j \\
M^{(2)} &= \bigsqcup_{j\equiv 2 \pmod 3} M_j
\end{aligned}
\tag{11.28}
$$

For $i \in \{0,1,2\}$, let $f^{(i)}\colon M^{(i)} \to \mathbb{A}^{2n+2}$ be the function whose first $2n+1$ coordinates are given by the disjoint union of the $\tilde f_j$ and whose last coordinate is $j\rho_j$ on M_j. Then $f^{(i)}$ restricts to an injective map on each $\overline{G_j}\setminus G_{j-1}$ and its differential is injective at each point of $\overline{G_j}\setminus G_{j-1}$. Since ρ_j has compact support in M_j, the function $\tilde f_j$ has value $0 \in \mathbb{A}^{2n+2}$ outside a compact subset of M_j, so extends to a global function $M \to \mathbb{A}^{2n+2}$. Hence $f^{(i)}$ also extends to a global function $\tilde f^{(i)}\colon M \to \mathbb{A}^{2n+2}$. Let $\rho\colon M \to \mathbb{R}$ be a proper function (Corollary 10.24). Finally, set

$$
f = \bigl(f^{(0)}, f^{(1)}, f^{(2)}, \rho\bigr)\colon M \longrightarrow \mathbb{A}^{6n+7}. \tag{11.29}
$$

Then f is a proper injective immersion, hence is an embedding. ∎

Transversality

(11.30) *Transversality for linear maps.* Let $T\colon V \to W$ be a linear map between vector spaces and $U \subset W$ a subspace. Then we say T *is transverse to* U, written $T \pitchfork U$, if and only if the subspaces $T(V)$ and U span W:

$$W = T(V) + U. \tag{11.31}$$

This is equivalent to the condition that the composition

$$V \xrightarrow{T} W \longrightarrow W/U \tag{11.32}$$

be surjective, where the second map is projection onto the quotient.

(11.33) *Nonlinear maps.* Transversality is defined via linearization; it is a local condition.

Definition 11.34. Let X, Y be smooth manifolds, $Z \subset Y$ a submanifold, $f\colon X \to Y$ a smooth map, and $p \in X$ such that $f(p) \in Z$. Then f *is transverse to* Z *at* p, written $f \pitchfork_p Z$ if

$$T_{f(p)}Y = df_p(T_pX) + T_{f(p)}Z. \tag{11.35}$$

We say f *is transverse to* Z, written $f \pitchfork Z$, if $f \pitchfork_p Z$ for all $p \in X$ such that $f(p) \in Z$.

The nonlinear transversality condition (11.35) is the linear transversality condition $df_p \pitchfork T_{f(p)}Z$.

Remark 11.36.

(1) For $q \in Y$ we have $f \pitchfork \{q\}$ if and only if q is a regular value of f.
(2) Any map $f\colon X \to Y$ satisfies $f \pitchfork Y$.
(3) If $\dim X + \dim Z < \dim Y$, then $f \pitchfork Z$ if and only if $f(X) \cap Z = \emptyset$.
(4) If $Z_1, Z_2 \subset Y$ are submanifolds, and $f_i\colon Z_i \to Y$ is the inclusion map, then we say $Z_1 \pitchfork Z_2$ if and only if $f_1 \pitchfork Z_2$. This is a symmetric relation: $f_1 \pitchfork Z_2$ if and only if $f_2 \pitchfork Z_1$.

(11.37) *Transverse inverse image of a submanifold.* Recall from Theorem 7.10 that if $f\colon X \to Y$ is a smooth map and $q \in Y$ is a regular value, then $f^{-1}(q) \subset X$ is a submanifold. The next theorem gives a sufficient condition for the inverse image of a submanifold to be a submanifold.

Theorem 11.38. *Let X, Y be smooth manifolds, $Z \subset Y$ a submanifold, and $f\colon X \to Y$ a smooth map. Assume $f \pitchfork Z$. Then $W := f^{-1}(Z) \subset X$ is a submanifold. Furthermore, if $p \in X$ satisfies $f(p) \in Z$, then*

(1) $T_pW = df_p^{-1}\bigl(T_{f(p)}Z\bigr)$.
(2) *df_p induces an isomorphism of normal spaces $\nu_p(W \subset X) \to \nu_{f(p)}(Z \subset Y)$.*
(3) $\operatorname{codim}_p(W \subset X) = \operatorname{codim}_{f(p)}(Z \subset Y)$.

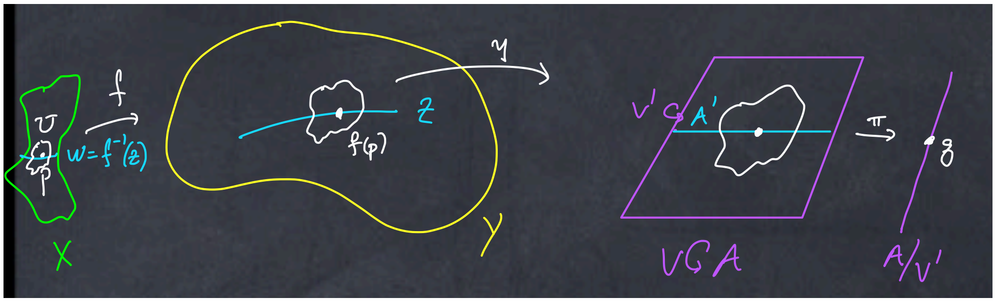


FIGURE 23. Local reduction: transversality ⟶ regular value

Proof. Each statement is local. Choose a submanifold chart (U_Y, y) on Y with $f(p) \in U_Y$, and suppose $y\colon U_Y \to A$, where A is an affine space over a vector space V. Furthermore, let $A' \subset A$ be an affine subspace so that $y^{-1}(A') = U_Y \cap Z$. Suppose $V' \subset V$ is the subspace of translations which preserve A'. Let $\pi\colon A \to A/V'$ be projection onto the quotient affine space, and let $q \in A/V'$ be the image of $A' \subset A$ under π. Since $f \pitchfork_p Z$ it follows that $d(\pi \circ y \circ f)_p$ is surjective. Since surjectivity is an open condition, choose an open neighborhood $U_X \subset X$ of p so that $\pi \circ y \circ f\big|_{U_X}\colon U_X \to A/V'$ is a submersion; in particular, $q \in A/V'$ is a regular value and $(\pi \circ y \circ f\big|_{U_X})^{-1}(q) = W \cap U_X$. Now apply Theorem 7.10. ☐

Lecture 12: Stability under deformations

There are two subjects in this lecture: transversality, which we did not get to in the last lecture and which is discussed in the notes for Lecture 11; and stability of special types of maps under deformation. Theorem 11.38 is the main result about transversality at this stage—the transverse inverse image of a submanifold is a submanifold—and it is a simple generalization of Theorem 7.10, which states that the inverse image of a regular value—the transverse inverse image of a point—is a submanifold. In these notes we treat smooth families of maps which go between smooth families of manifolds, i.e., between total spaces of fiber bundles. It is simpler at first to focus on the case when the manifolds are fixed, as in Remark 12.7, and only the map varies, and that is the approach I will take in lecture. But the arguments which prove the key Theorem 12.17 are the same; only the setup is different.

Stability

(12.1) *Introduction.* Recall that we produced a local normal form for a smooth function whose differential has maximal rank (Theorem 6.15). We also proved in Lemma 6.3 that maximal rank is an open condition for linear maps between fixed vector spaces. Now we consider families of maps between manifolds, and we can allow the manifolds to vary as well. We prove that not only are these local conditions—and the local condition of transversality to a submanifold—open conditions, but so too are related global conditions. This openness, or stability under deformation, is an important feature when constructing topological invariants, which we do in the next few lectures.

(12.2) *Smooth families of manifolds and maps.* A parametrized family of geometric objects lives over a parameter space, which in smooth geometry is a smooth manifold S. The nicest smooth families of manifolds are fiber bundles: they are locally trivial (Definition 8.24). The tangent spaces then form a locally trivial family of vector spaces: a vector bundle. Thus suppose $\pi_{\mathcal{X}}\colon \mathcal{X}\to S$ is a fiber bundle. Since $\pi_{\mathcal{X}}$ has surjective differential, the kernels of the differential have locally constant dimension and, using the local triviality of $T\mathcal{X}$, we deduce that they form a vector bundle. Hence over the total space $\mathcal{X}$ we obtain a short exact sequence of vector bundles

$$0 \longrightarrow \ker d\pi_{\mathcal{X}} \longrightarrow T\mathcal{X} \longrightarrow \pi_{\mathcal{X}}^*TS \longrightarrow 0 \tag{12.3}$$

The kernel of the differential—which consists of "vertical tangent vectors"—is the tangent bundle along the fibers, or *relative tangent bundle*, and it is denoted

$$T(\mathcal{X}/S) = \ker d\pi_{\mathcal{X}}. \tag{12.4}$$

See Figure 24 for an illustration.

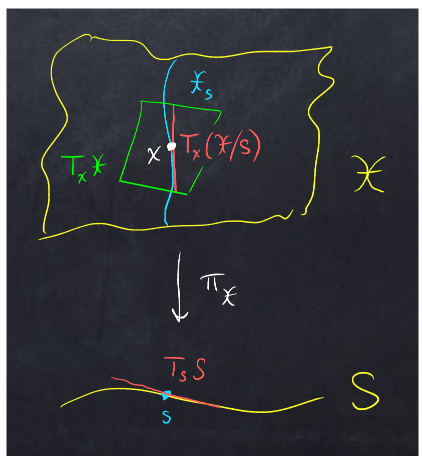


FIGURE 24. The tangent bundle along the fibers, or relative tangent bundle

A smooth family of maps, then, is a smooth map between the total spaces of two fiber bundles which commutes with the projections to the base. It fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\;F\;} & \mathcal{Y} \\ & {\scriptstyle\pi_{\mathcal{X}}}\searrow \quad \swarrow{\scriptstyle\pi_{\mathcal{Y}}} & \\ & S & \end{array} \tag{12.5}$$

in which $\pi_{\mathcal{X}}$ and $\pi_{\mathcal{Y}}$ are fiber bundles and F is a smooth map. For each $s\in S$ the restriction

$$F_s\colon \mathcal{X}_s \longrightarrow \mathcal{Y}_s \tag{12.6}$$

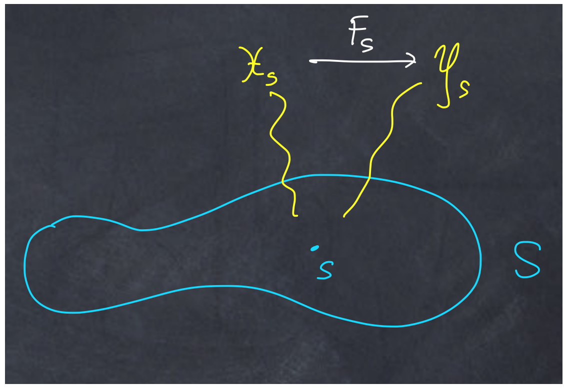


FIGURE 25. A smooth family of maps

of F to $\mathcal{X}_s = \pi_{\mathcal{X}}^{-1}(s)$ is the map at the parameter value s; see Figure 25.

Remark 12.7. If $\pi_{\mathcal{X}}, \pi_{\mathcal{Y}}$ are product fiber bundles with fibers X, Y, then (12.5) is replaced with a smooth map $S \times X \to Y$.

(12.8) *The differential as a family of linear maps.* A special case of (12.5) is a family of linear maps between locally trivial families of vector spaces, i.e., between vector bundles. For example, suppose $f\colon X \to Y$ is a smooth map between smooth manifolds. At each $x \in X$ the differential is a linear map $df_x\colon T_xX \to T_{f(x)}Y$, and these linear maps fit together into a map of vector bundles over X:

$$df\colon TX \longrightarrow f^*TY. \tag{12.9}$$

In the case of a family (12.5) of smooth maps over a base S, the differential $dF\colon T\mathcal{X} \to F^*T\mathcal{Y}$ restricts to the vertical tangent bundles to give a family of vector bundle maps $T\mathcal{X}_s \to T\mathcal{Y}_s$ parametrized by S, i.e., a family of linear maps $T(\mathcal{X}/S)_x \to T(\mathcal{Y}/S)_{F(x)}$ parametrized by $\mathcal{X}$:

$$\begin{array}{ccc} T(\mathcal{X}/S) & \xrightarrow{\;dF^{\mathrm{vert}}\;} & F^*T(\mathcal{Y}/S) \\ & {\scriptstyle \pi_{\mathcal{X}}}\searrow \quad \swarrow{\scriptstyle \pi_{\mathcal{Y}}} & \\ & \mathcal{X} & \end{array} \tag{12.10}$$

(12.11) *Maximal rank condition with variable vector spaces.* Recall from Lemma 6.3 that maximal rank maps between fixed vector spaces are an open subset of all linear maps.

Lemma 12.12. *Let*

$$\begin{array}{ccc} E' & \xrightarrow{\;T\;} & E \\ & {\scriptstyle \pi'}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & M & \end{array} \tag{12.13}$$

be a family of linear maps between vector bundles. Then

(12.14) $$\{m \in M : T_m \colon E'_m \longrightarrow E_m \text{ has maximal rank}\}$$

is an open subset of M.

Proof. Let $m \in M$ be a point at which T_m has maximal rank. Choose an open neighborhood $U \subset M$ of m together with local trivializations of π' and π over U. We obtain a square of fiberwise linear maps over U (the projections to U are not drawn):

(12.15) $$\begin{array}{ccc} U \times E'_m & \longrightarrow & U \times E_m \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ E'\big|_U & \xrightarrow{\ T\ } & E\big|_U \end{array}$$

Then $\psi^{-1} \circ T \circ \varphi$ defines a smooth map $L \colon U \to \operatorname{Hom}(E'_m, E_m)$, and for $m' \in U$, the linear map $T_{m'}$ has maximal rank if and only if $L(m')$ does. Since maximal rank maps in $\operatorname{Hom}(E'_m, E_m)$ are an open set (Lemma 6.3), it follows that (12.14) contains an open neighborhood of m. □

(12.16) *Stability theorem.* We prove the stability theorem for maps of fiber bundles; the special case with fixed fibers (as in Remark 12.7) occurs often.

Theorem 12.17. *Let S be a smooth manifold; $\pi_{\mathcal{X}} \colon \mathcal{X} \to S$ and $\pi_{\mathcal{Y}} \colon \mathcal{Y} \to S$ smooth fiber bundles; let $\pi_{\mathcal{Z}} \colon \mathcal{Z} \to S$ be a smooth sub-fiber bundle of $\pi_{\mathcal{Y}}$ such that each fiber $\mathcal{Z}_s \subset \mathcal{Y}_s$ is a* closed *submanifold; and let $F \colon \mathcal{X} \to \mathcal{Y}$ be a smooth map of fiber bundles as in (12.5). Assume $\pi_{\mathcal{X}}$ is proper. Suppose for some $s_0 \in S$ the map $F_{s_0} \colon \mathcal{X}_{s_0} \to \mathcal{Y}_{s_0}$ is one of*

(i) *a local diffeomorphism*
(ii) *an immersion*
(iii) *a submersion*
(iv) *transverse to $\mathcal{Z}_{s_0}$*
(v) *an injective immersion*
(vi) *an embedding*
(vii) *a diffeomorphism*

Then there exists an open neighborhood $V \subset S$ of s_0 such that F_s satisfies the same condition for all $s \in V$.

A few explanations are in order. First, the conclusion in case (iv) is that $F_s \pitchfork \mathcal{Z}_s$ for $s \in W$. Also, the statement that $\pi_{\mathcal{Z}}$ is a sub-fiber bundle of $\pi_{\mathcal{Y}}$ means: $\mathcal{Z} \subset \mathcal{Y}$ is a submanifold, $\pi_{\mathcal{Z}}$ is the restriction of $\pi_{\mathcal{Y}}$ to $\mathcal{Z}$, and there is a cover of S by simultaneous local trivializations of $\pi_{\mathcal{Z}}$ and $\pi_{\mathcal{Y}}$ to a constant submanifold over open subsets U:

(12.18) $$\begin{array}{ccc} U \times (Z \subset Y) & \xrightarrow{\ \varphi\ } & \big(\pi_{\mathcal{Z}}^{-1}(U) \subset \pi_{\mathcal{Y}}^{-1}(U)\big) \\ & {\scriptstyle \mathrm{pr}_1}\searrow \quad \swarrow{\scriptstyle \pi_{\mathcal{Y}}} & \\ & U & \end{array}$$

The manifolds Z and Y are allowed to depend on U. Our assumption is that, in addition, $Z \subset Y$ is a *closed* submanifold. Finally, the properness assumption implies in particular that each $\mathcal{X}_s$ is a compact manifold. That compactness implies openness[10] (here in parameter space) is a general principle by the first argument what follows.[11]

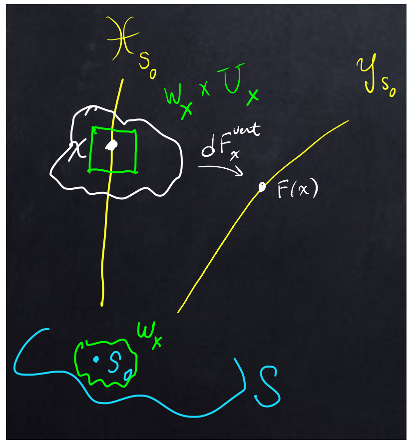


FIGURE 26. Compactness implies openness

Proof. We prove (i)–(iii) simultaneously. Choose simultaneous local trivializations of $\pi_{\mathcal{X}}$ and $\pi_{\mathcal{Y}}$ about s_0. For each $x \in \mathcal{X}_{s_0}$ the vertical differential $dF^{\text{vert}}\colon T_x(\mathcal{X}/S) \to T_{F(x)}(\mathcal{Y}/S)$ has maximal rank. Since maximal rank is an open condition, choose a product open set $V_x \times U_x$, relative to the local trivialization of $\pi_{\mathcal{X}}$, so that dF^{vert} has maximal rank in $V_x \times U_x$. By compactness choose a finite subset $F \subset \mathcal{X}_{s_0}$ so that $\bigcup_{x\in F} U_x = \mathcal{X}_{s_0}$. Then $V = \bigcap_{x\in F} V_x \subset S$ is the desired open set.

For (iv), observe first that the subset $\mathcal{W} := F^{-1}(\mathcal{Z}) \subset \mathcal{X}$ is closed, since $\mathcal{Z} \subset \mathcal{Y}$ is closed. Hence if $\mathcal{X}_{s_0} \cap \mathcal{W} = \emptyset$, we can cover $\mathcal{X}_{s_0} \subset \mathcal{X}$ by a finite collection of product open sets $V_i \times U_i$ in the complement of $\mathcal{W}$, and then as before $V = \bigcap_i V_i \subset S$ is the desired open set. Otherwise, for each $x \in \mathcal{X}_{s_0} \cap \mathcal{W}$, choose a submanifold chart about $F(x)$ and use the maneuver in the proof of Theorem 11.38 to convert the transversality condition to the submersion condition. Since the latter is open, by part (iii), we can find a product open neighborhood $V_x \times U_x$ of x such that at all points of $\mathcal{W}$ in that neighborhood the map F is transverse to $\mathcal{Z}$. Cover $\mathcal{X}_{s_0}$ by a finite number of such product neighborhoods and also product neighborhoods which do not intersect $\mathcal{W}$. Finish as before by intersecting the open subsets of S.

For (v), assume $V \subset S$ is a neighborhood of s_0 such that F_s is an immersion for $s \in V$. We claim F_s is injective for all s in a possibly smaller open neighborhood of s_0. If not, choose a sequence $\{s_n\} \subset S$ such that $s_n \to s_0$ as $n \to \infty$, and choose sequences $\{x_n\}, \{x'_n\} \subset \mathcal{X}$ such that $\pi_{\mathcal{X}}(x_n) = \pi_{\mathcal{X}}(x'_n) = s_n$ and $F(x_n) = F(x'_n)$ for all n. Since the fibers of $\pi_{\mathcal{X}}$ are compact, we can and do choose convergent subsequences $\{x_{n_k}\} \subset \{x_n\}$ and $\{x'_{n_k}\} \subset \{x'_n\}$ (defined by the same function $n\colon \mathbb{Z}^{>0} \to \mathbb{Z}^{>0}$, $k \mapsto n_k$), say $x_{n_k} \to x_0$ and $x'_{n_k} \to x'_0$. But since F_{s_0} is injective, $x_0 = x'_0$. We

[10] Recall the proof that the complement of a compact subset of a Hausdorff space—e.g. a metric space—is open.

[11] The first part of the proof is a bit more complicated in the case of a fiber bundle than for fixed manifolds, as in Remark 12.7. In that case, we are interested in the partial differential of F in the X-direction, which at a fixed point $x \in X$ is a family of linear maps $T_xX \to T_{F(s,x)}Y$ parametrized by S. The domain is a constant vector space, but the codomain varies with $s \in S$: those vector spaces fit together to a vector bundle over S.

claim that $dF_{x_0}\colon T_{x_0}\mathcal{X} \to T_{F(x_0)}\mathcal{Y}$ is injective. Namely, relative to the short exact sequence (12.3) we have[12]

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T_{x_0}(\mathcal{X}/S) & \longrightarrow & T_{x_0}\mathcal{X} & \longrightarrow & T_{s_0}S & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle dF^{\text{vert}}_{x_0}} & & \Big\downarrow{\scriptstyle dF_{x_0}} & & \Big\downarrow{\scriptstyle \text{id}} & & \\
0 & \longrightarrow & T_{F(x_0)}(\mathcal{Y}/S) & \longrightarrow & T_{F(x_0)}\mathcal{Y} & \longrightarrow & T_{s_0}S & \longrightarrow & 0
\end{array} \tag{12.19}
$$

The vertical map on the subspaces is injective, since F_{s_0} is an immersion. It follows from (12.19) that dF_{x_0} is also injective. Now the local normal form Theorem 6.15 shows that F is injective in a neighborhood of x_0, which contradicts the existence of the sequences $\{x_{n_k}\}, \{x'_{n_k}\}$.

Part (vi) follows from (v) since a proper injective immersion is an embedding.

For (vii), choose V connected so that F_s is an injective local diffeomorphism for all $s \in V$. (The existence of V follows from (i) and (v).) Shrink V if necessary to choose simultaneous local trivializations of $\pi_{\mathcal{X}}, \pi_{\mathcal{Y}}$, so we may assume constant fibers X, Y. We may also assume, after further shrinkage if necessary, that V is connected. Fix a component $Y_0 \subset Y$ and choose the corresponding component $X_0 \subset X$ with $F_{s_0}(X_0) = Y_0$. Now any local diffeomorphism is an open map, so $F_s(X_0) \subset Y$ is open for all $s \in V$. Furthermore, X_0 is compact so $F_s(X_0) \subset Y$ is closed. Given $s \in S$, choose a path $t \mapsto s_t$ from s_0 to s in V. Then for any $x \in X_0$, that path $t \mapsto F_{s_t}(x)$ connects $F_{s_0}(x) \in Y_0$ to $F_s(x) \in Y$. It follows that $F_s(X_0) \subset Y_0$. Since Y_0 is a component, and any open and closed subset is a union of components, we conclude $F_s(X_0) = Y_0$. Therefore, F_s is surjective.[13] $\square$

Remark 12.20. For fixed manifolds X, Y we can form the set $C^\infty(X, Y)$ of smooth maps $X \to Y$. Just as there are different topologies on the set of continuous maps between topological spaces, so too are there different (Whitney) topologies on the space of C^∞ maps. With the correct topology, Theorem 12.17 asserts that if X is compact, then the various subsets are *open*. In fact, the formulation in terms of finite dimensional families of maps is a convenient technique to avoid working directly with infinite dimensional function spaces. (We work instead *functorially* with a variable base S.) The slogan stability=openness has a flip side: approximation=density. We will discuss some approximation theorems in future lectures.

Lecture 13: Manifolds with boundary

As background we begin with calculus on a half-space. The key points are Definition 13.10 of smoothness and Lemma 13.12 which states that the differential is well-defined. Lemma 13.15 allows us to define the interior and boundary of a manifold with boundary unambiguously. Then we define

[12] If we choose a splitting, then this is a triangular decomposition of a linear transformation. A diagram chase shows that relative to such a decomposition, a linear transformation is maximal rank iff it is maximal rank on both the sub and the quotient.

[13] In this argument we use the fact that manifolds, which are locally Euclidean, are also locally path connected. Therefore, the partition of a manifold into components is the same as its partition into path components.

a manifold with boundary using the half-space model, just as we defined a manifold using affine space as a model. We revisit basic notions—tangent space and submanifold—for manifolds with boundary. The discussion of submanifolds is particularly important: again there is a local model which we allow (and alternatives which occur but which do not merit the moniker 'submanifold'). Finally, we modify some of the basic theorems we have proved for manifolds to manifolds with boundary.

We conclude this introduction with some motivation and remarks.

(13.1) *Motivation.* We offer three motivations to generalize the concept of a manifold to a manifold with boundary. First, we will shortly construct topological invariants and will prove that they are invariant under homotopy. A smooth homotopy of maps $X \to Y$ is a smooth map $[0,1] \times X \to Y$. But $[0,1] \subset \mathbb{A}^1$ is not locally Euclidean, so is not a manifold, and neither is $[0,1] \times X$. Both are manifolds with boundary. Second, we often do calculus on subsets of affine space which are manifolds with boundary, for example closed intervals in $\mathbb{R}$ or closed disks in $\mathbb{A}^2$. Those should be included in our calculus on curved spaces. Finally, in homology theory one studies cycles a in a topological space and the notion of a homology between cycles a_0 and a_1: a homology is a chain b such that $\partial b = a_1 - a_0$. Note the theory is vacuous if the space is a single point. There is a similar smooth notion called *bordism* in which a topological space is replaced by a smooth manifold M, a cycle by a smooth map $Y \to M$ in which Y is a compact manifold, and a bordism from $Y_0 \to M$ to $Y_1 \to M$ is a smooth map $X \to M$ where X is a compact manifold with boundary $\partial X = Y_0 \amalg Y_1$ and the map restricts on the boundary to the given maps. There the theory is nontrivial even if $M = \mathrm{pt}$. For example, there does not exist a compact 3-manifold with boundary X such that $\partial X = \mathbb{RP}^2$.

Remark 13.2. It is tempting to use 'manifold-with-boundary' rather than 'manifold with boundary' to emphasize that 'manifold with boundary' is a single concept; 'with boundary' is not a modifier to 'manifold'. A manifold is in particular a manifold with boundary, so the manifold with boundary concept generalizes the manifold concept we have been studying heretofore. If manifolds with boundary are around, then sometimes 'manifold without boundary' is used for 'manifold'. (The hyphens are not standard, and besides an n-manifold is a manifold of constant dimension n, which would lead to 'n-manifold-with-boundary' which transgresses most laws of acceptability.)

Remark 13.3. When manifolds with boundary are admitted into the game, we use the term *closed manifold* for a compact manifold without boundary. (The term *open manifold* is less common. One possible meaning is a manifold without boundary, each of whose components is not compact.)

Calculus on closed affine half-spaces

(13.4) *Local model.* Recall that the local model for a topological manifold is a finite dimensional affine space: topological manifolds are locally affine. For manifolds with boundary the local model is a closed half of an affine space.

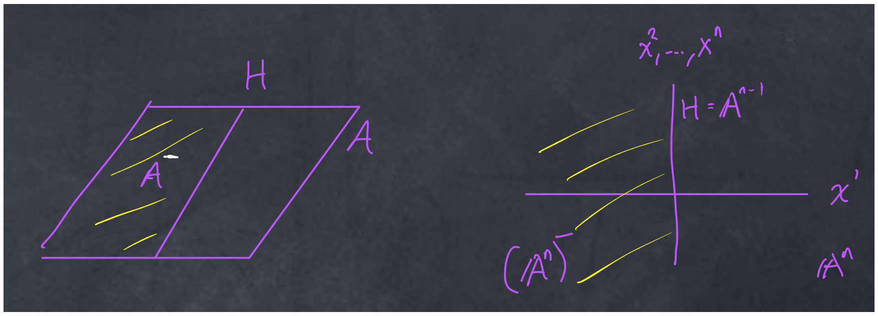


FIGURE 30. Local model for a manifold with boundary

Definition 13.5. Let A be a finite dimensional affine space, $H \subset A$ an affine hyperplane, and $A^- \subset A$ the closure of one component of $A \backslash H$. Then A^- is a *closed affine half-space.*

An affine hyperplane H is the zero set of a nonconstant affine function $f\colon A \to \mathbb{R}$, and $A^- = \{a \in A : f(a) \leqslant 0\}$. So the data in Definition 13.5 can be taken to be the pair (A, f). Of course, A^- inherits a topology by dint of being a subspace of A. Then $\partial A^- = H$, and A^- partitions as

$$A^- = \operatorname{Int} A^- \amalg \partial A^-. \tag{13.6}$$

(13.7) *Standard local model.* For each $n \in \mathbb{Z}^{>0}$ there is a standard model: $A = \mathbb{A}^n$ with standard affine coordinates $x^1, \ldots, x^n$; the hyperplane is $H = \{x^1 = 0\}$; and we choose

$$A^- = (\mathbb{A}^n)^- = \{x^1 \leqslant 0\}. \tag{13.8}$$

The choice of the first (as opposed to last) coordinate, and of the nonpositive (as opposed to nonnegative) half-space is deliberate, as we explain below.

(13.9) *Smooth functions and diffeomorphisms on closed affine half-spaces.* Use A^- as a shorthand for the triple of data (A, H, A^-) in Definition 13.5. The setting for calculus is two closed affine half-spaces A^-, B^-, an open subset $U \subset A^-$, and a function $f\colon U \to B^-$. Let the vector space of translations of A, B be V, W, and let V', W' be the subspaces of translations which preserve the hyperplanes $H \subset A$ and $K \subset B$.

Definition 13.10. Let $p \in U$. We say f *is* C^∞ at p if there exists an open neighborhood $\widetilde{U} \subset A$ of p and a C^∞ function $\tilde{f}\colon \widetilde{U} \to B$ such that $\tilde{f}\big|_{U \cap \widetilde{U}}$ equals the composition

$$U \cap \widetilde{U} \xrightarrow{f|_{U \cap \widetilde{U}}} B^- \longrightarrow B. \tag{13.11}$$

Lemma 13.12. *The linear transformation $d\tilde{f}_p\colon V \to W$ is independent of the extension $\tilde{f}$ of f.*

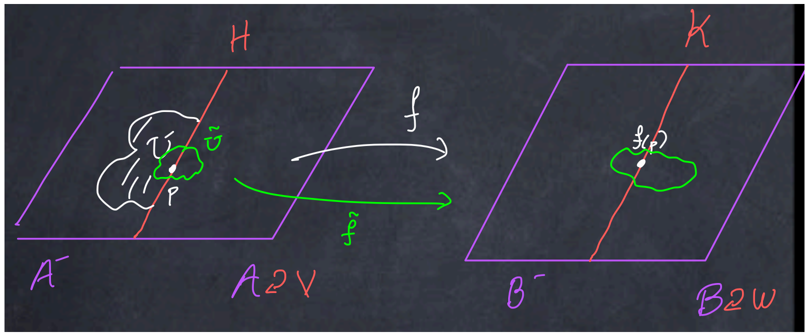


FIGURE 31. Local smoothness at the boundary

Proof. If $p \in U \cap \operatorname{Int} A^-$ then $df_p\colon V \to W$ is defined and equals $d\tilde{f}_p$ for any extension. If $p \in U \cap H$, then since the differential is continuous,

$$d\tilde{f}_p = \lim_{p' \to p} d\tilde{f}_{p'} = \lim_{\substack{p' \to p \\ p' \in U \cap \operatorname{Int} A^-}} df_p. \tag{13.13}$$

□

Remark 13.14. Lemma 13.12 extends to all higher derivatives: the *infinite jet* of a C^∞ function f is well-defined, even at points of the boundary H.

The other statement we need from calculus is that a diffeomorphism preserves the partition (13.6). We continue with the setup of this subsection (**13.9**).

Lemma 13.15. *Suppose $f\colon U \to B^-$ is a diffeomorphism onto its image. Then*

$$f(U \cap H) \subset K, \tag{13.16}$$

$$f(U \cap \operatorname{Int} A^-) \subset \operatorname{Int} B^-. \tag{13.17}$$

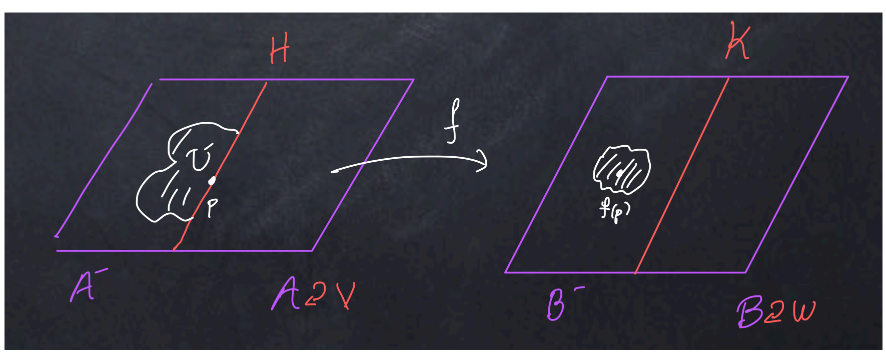


FIGURE 32. Diffeomorphisms preserve the boundary

Proof. Suppose $p \in U \cap H$ and $\tilde{f}\colon \widetilde{U} \to B$ is an extension of f near p, where $\widetilde{U} \subset A$ is an open neighborhood of p. Since $d\tilde{f}_p$ is invertible, the inverse function theorem implies that possibly after reducing $\widetilde{U}$ to a smaller open neighborhood, the map $\tilde{f}\colon \widetilde{U} \to B$ is a diffeomorphism onto its image $\tilde{f}(\widetilde{U}) \subset B$. Assume $f(p) \notin K$, and reduce $\widetilde{U}$ further so that $\tilde{f}(\widetilde{U}) \subset \operatorname{Int} B^-$. Furthermore, since f is a diffeomorphism of $U \subset A^-$ onto $f(U) \subset B^-$, there is an open neighborhood of $f(p)$ in $\operatorname{Int} B^- \subset B^- \subset B$ contained in $f(U)$. Upon reducing $\widetilde{U}$ even further, then, we may assume $\tilde{f}(\widetilde{U}) \subset f(U)$. But then $\tilde{f}^{-1}\colon \tilde{f}(\widetilde{U}) \to A$ carries an open neighborhood of $f(p)$ in B to an open neighborhood of p in A, from which we deduce that there exists an open neighborhood of p in A that is contained in U. However, every open neighborhood of p in A contains points of $A \backslash A^-$, which contradicts $U \subset A^-$.

Assertion (13.17) follows from (13.16) applied to both f and f^{-1}. □

Basic definitions

A manifold with boundary is defined using the local model **(13.4)** and its standard variant **(13.7)**.

Definition 13.18.

1. Let X be a topological space. Then X is a *topological manifold with boundary* if it is Hausdorff, second countable, and locally homeomorphic to a closed affine half-space. A *chart* on X is a pair (U, x) consisting of an open subset $U \subset X$ and a homeomorphism $x\colon U \to A^-$ into a closed affine half-space.
2. Let X be a topological manifold with boundary. An *atlas* is a covering of X by charts with C^∞ overlaps, as in Definition 2.9.
3. A *smooth manifold with boundary* is a topological manifold with boundary equipped with a maximal atlas of standard charts.

Smoothness of the overlaps is defined in **(13.9)**.

(13.19) *Partition of a manifold with boundary.* Recall from (13.6) that a closed affine half-space is partitioned into its interior and its boundary. Lemma 13.15 implies that the overlap functions on a smooth manifold X with boundary preserve this partition. Therefore, there is a global partition

$$X = \operatorname{Int} X \amalg \partial X. \tag{13.20}$$

Proposition 13.21.

1. $\operatorname{Int} X$ *is a smooth manifold.*
2. ∂X *is a smooth manifold.*

If (a component of) X has dimension n, then $\dim \operatorname{Int} X = n$ and $\dim \partial X = n - 1$.

Proof. If $x\colon U \to A^-$ is a chart on an open subset $U \subset X$ with values in a closed affine half-space A^-, set $U' = U \cap \operatorname{Int} X$ and

$$x'\colon U' \xrightarrow{\;x|_{U'}\;} \operatorname{Int} A \longrightarrow A \tag{13.22}$$

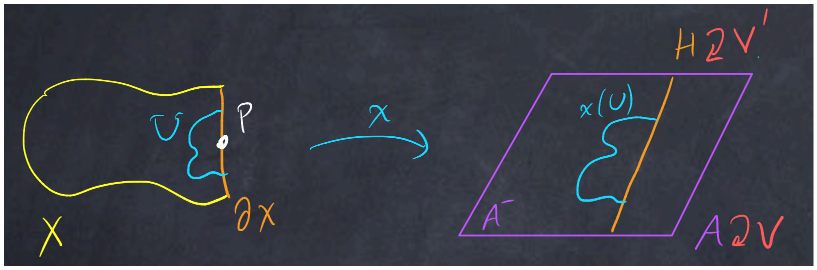


FIGURE 33. A chart on X induces charts on $\mathrm{Int}\, X$ and ∂X

Then (U', x') is a chart on $\mathrm{Int}\, X$. Execute this construction on each chart of an atlas on X to produce an atlas on $\mathrm{Int}\, X$ and so prove (1). Similarly, set $U'' = U \cap \partial X$. By the definition of the partition (13.20), the restriction of x to $U'' \subset U$ factors to a map

$$x'' : U'' \xrightarrow{x|_{U''}} H, \tag{13.23}$$

where $H = \partial A$. Then (U'', x'') is a chart on ∂X. The overlaps between charts are smooth by Lemma 13.15, so we obtain an atlas on ∂X, which proves (2). □

Example 13.24. A closed ball is a manifold with boundary. Namely, for any $n \in \mathbb{Z}^{>0}$ set

$$D^n = \{(x^1, \dots, x^n) \in \mathbb{A}^n : (x^1)^2 + \cdots + (x^n)^2 \leqslant 1\}. \tag{13.25}$$

Then

$$\begin{aligned} \mathrm{Int}\, D^n = B^n &= \{(x^1, \dots, x^n) \in \mathbb{A}^n : (x^1)^2 + \cdots + (x^n)^2 < 1\}, \\ \partial D^n = S^{n-1} &= \{(x^1, \dots, x^n) \in \mathbb{A}^n : (x^1)^2 + \cdots + (x^n)^2 = 1\}. \end{aligned} \tag{13.26}$$

(13.27) *Cartesian products.* Let X be a manifold with boundary and Y a manifold (no boundary). Then $X \times Y$ is a manifold with boundary, and $\partial(X \times Y) = \partial X \times Y$. An atlas for $X \times Y$ can be constructed using Cartesian products of charts of X and charts of Y over atlases for X and Y.

The tangent space

The tangent space T_pX to a manifold with boundary X at a point $p \in X$ is defined exactly as in the case with no boundary (Lecture 3). If (U, x) is a chart about p with values in a closed affine half-space A^-, then there is an isomorphism

$$T_pX \xrightarrow[\cong]{(U,x)} V, \tag{13.28}$$

where V is the vector space of translations of the affine space A; see (3.41). The tangent spaces glue to a vector bundle $TX \to X$.

(13.29) *Tangent space at a boundary point.* The definition of T_pX holds for $p \in \partial X$; there is no drop in dimension of the tangent space at a boundary point. However, there is a canonical subspace of T_pX of codimension one, namely the tangent space $T_p(\partial X)$ to the boundary. In a boundary chart (U, x), as depicted in Figure 33, the isomorphism (13.28) extends to a commutative diagram

(13.30)
$$\begin{array}{ccc} T_p(\partial X) & \longrightarrow & V' \\ \big\downarrow & & \big\downarrow \\ T_pX & \longrightarrow & V \end{array}$$

where $V' \subset V$ is the subspace of translations that preserve the hyperplane $H \subset A$. In a standard chart—one in which $A = \mathbb{A}^n$ and $H = \{x^1 = 0\}$—the vector fields $\frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n}$ span $T_p(\partial X)$, as depicted in Figure 34. The tangent bundle of the boundary is a subbundle of the tangent bundle restricted to the boundary:

(13.31)
$$\begin{array}{ccc} T(\partial X) & \hookrightarrow & TX\big|_{\partial X} \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

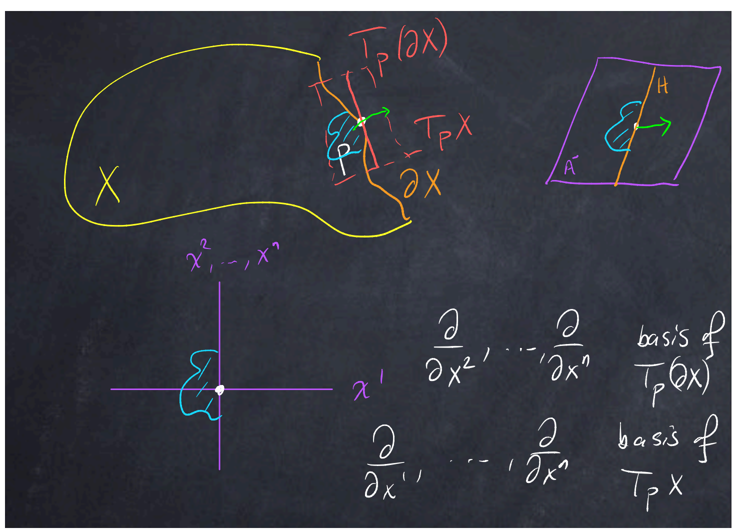


FIGURE 34. The tangent space at a boundary point in a standard chart

(13.32) *Normal space at the boundary.* At $p \in \partial X$ we define the *normal line* to the boundary as

(13.33)
$$\nu_p = \nu_p(\partial X \subset X) = T_pX/T_p(\partial X).$$

The normal line carries a canonical orientation.

Definition 13.34. Let L be a real line, i.e., a real one-dimensional vector space. An *orientation* of L is a choice of component of $L\backslash\{0\}$.

There are two components of $L\backslash\{0\}$, hence two orientations. For the normal line ν_p we choose the component of *outward* normals, so tangents to smooth motions $\gamma\colon (-\delta, 0] \to X$ such that $\gamma(0) = p$ and $\gamma(t) \in X\backslash\partial X$ for $t < 0$.

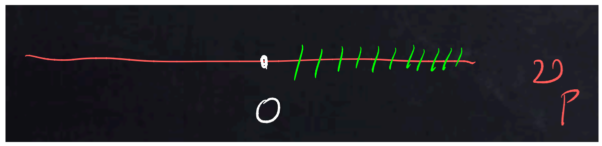


Figure 35. Orientation of the normal line

There is a short exact sequence of vector spaces

$$0 \longrightarrow T_p(\partial X) \longrightarrow T_pX \longrightarrow \nu_p \longrightarrow 0 \tag{13.35}$$

In a standard boundary chart with local coordinates $x^1, \dots, x^n$, the basis of T_pX is

$$\underbrace{\frac{\partial}{\partial x^1}}_{\text{basis of } \nu_p}, \underbrace{\frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n}}_{\text{basis of } T_p(\partial X)}. \tag{13.36}$$

Remark 13.37. The order of the basis is an example of a general principle: Quotient Before Sub. This rule will recur when we discuss orientations in more generality. It is a convention which experience shows is a good one.

Remark 13.38. Another relevant convention has the acronym ONF, which stands for Outward Normal First, something One Never Forgets.

Submanifolds

Figure 36. Disallowed local models

(13.39) *Non-examples.* Once we introduce manifolds with boundary there are several new possibilities for a local model of a submanifold. Recall (Definition 6.19) the local model of a submanifold of a manifold (no boundaries) is an affine subspace of an affine space. One could conceivably allow a manifold with boundary to be a submanifold of a manifold, as in (i) in Figure 36, but we do not allow it. (Nonetheless, that configuration occurs; we simply do not use the term 'submanifold' to describe it.) In the second drawing, we do not allow (ii) or (iii), which are analogous to (i). Nor do we allow (iv), in that case because the subset is not transverse to the boundary.

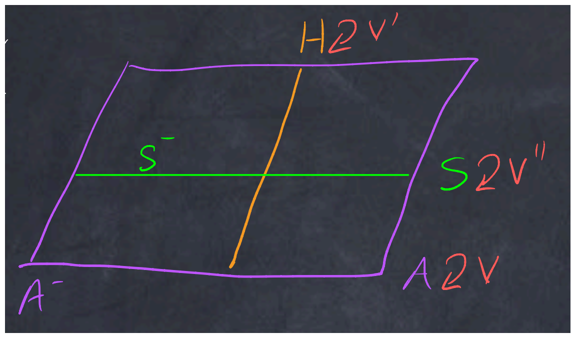


FIGURE 37. Local model of a submanifold of a manifold with boundary

(13.40) *Local model.* The local model (Figure 37) we use is the following data: A is an affine space, $H \subset A$ is a codimension one affine subspace, A^- is the closure of a chosen component of $A\backslash H$, and $S \subset A$ is an affine subspace such that $S \pitchfork H$. Define $S^- = S \cap A^-$. If V is the vector space of translations of A, and $V', V'' \subset V$ the subspaces of translations which preserve H, S, respectively, then the transversality condition is $V = V' + V''$. For the standard local model, choose $A = \mathbb{A}^n$, $A^- = (\mathbb{A}^n)^- = \{x^1 \leqslant 0\}$, and $S = \{x^{n-k+1} = \cdots = x^n = 0\}$ where k is the codimension.

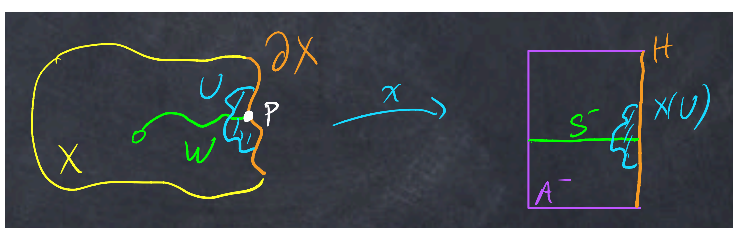


FIGURE 38. A submanifold chart of a manifold with boundary

Definition 13.41. Let X be a manifold with boundary and $W \subset X$ a subset. Then W is a *submanifold* if for each $p \in W$ there exists a chart (U, x) of X about p with codomain a local model (A^-, S) such that $x(W \cap U) \subset S^-$ and $x(\partial W \cap U) \subset S^- \cap H$.

A submanifold chart is depicted in Figure 38. The local model enforces

$$\begin{aligned} \partial W &= W \cap \partial X \\ W &\pitchfork \partial X \end{aligned} \tag{13.42}$$

Remark 13.43. Sometimes what we are calling 'submanifold' is called a 'neat submanifold'. Since this is the only notion of submanifold we use for a manifold with boundary, we omit 'neat'.

Submanifolds of a manifold with boundary via pullback

One convenient method to construct submanifolds of manifolds (no boundary) is via inverse image of a regular value (Theorem 7.10) or transverse pullback of a submanifold (Theorem 11.38). Similar results hold for a submanifold with boundary, as we prove in this section. The first theorem is closely related to these results.

Theorem 13.44. *Let X be a manifold. Suppose $f\colon X \to \mathbb{R}$ is a smooth function, and $c \in \mathbb{R}$ is a regular value of f. Then $f^{-1}(\mathbb{R}^{\leqslant c})$ is a manifold with boundary $f^{-1}(c)$.*

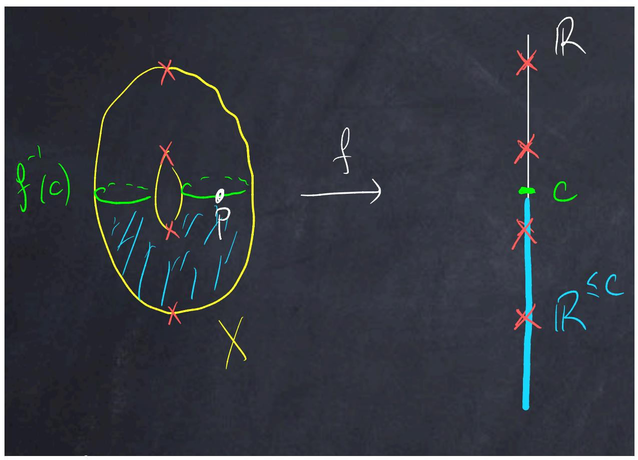


FIGURE 39. A manifold with boundary cut out by a real-valued function

Notice that $f^{-1}(\mathbb{R}^{\geqslant c}) \subset X$ is not a submanifold; the local model is (i) in Figure 36.

Remark 13.45. In Figure 39 the red exes depict critical points and critical values. In Morse Theory one studies how the inverse image of a regular value changes as we cross a critical value. Here—the poster child of elementary Morse theory—as the regular value increases from $-\infty$ to $+\infty$, the inverse image undergoes a sequence of surgeries:

$$\emptyset \rightsquigarrow S^1 \rightsquigarrow S^1 \amalg S^1 \rightsquigarrow S^1 \rightsquigarrow \emptyset. \tag{13.46}$$

Also, if $c_0 < c_1$ are regular values, then $f^{-1}\big((c_0, c_1)\big)$ is a bordism from $f^{-1}(c_0)$ to $f^{-1}(c_1)$.

Example 13.47. The closed ball in Example 13.24 can be constructed via the function

$$\begin{aligned} f\colon \mathbb{A}^n &\longrightarrow \mathbb{R} \\ (x^1,\dots,x^n) &\longmapsto (x^1)^2 + \cdots + (x^n)^2 \end{aligned} \tag{13.48}$$

using the regular value $c = 1$.

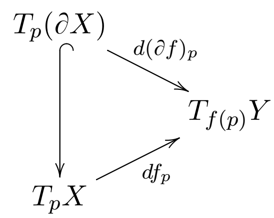
Proof of Theorem 13.44. The interior $f^{-1}(\mathbb{R}^{<c}) \subset X$ is an open subset, so a submanifold. The boundary $f^{-1}(c) \subset X$ is a submanifold since c is a regular value. But it remains to find boundary charts for $p \in f^{-1}(c)$. Apply Proposition 6.14(2) to construct a chart of X on an open neighborhood $U \subset X$ of p with local coordinate functions $x^1 = f - c, x^2, \dots, x^n$. The restriction to $U \cap \{x^1 \leqslant 0\}$ is a standard boundary chart for $f^{-1}(\mathbb{R}^{\leqslant c})$. The overlap between two such charts is smooth since it is the restriction of an overlap of charts of X. □

(13.49) *Notation.* Let X be a manifold with boundary, Y a manifold, and $f\colon X \to Y$ a function. We denote the restriction of f to ∂X as $\partial f\colon \partial X \to Y$.

Proposition 13.50. *Let X be a manifold with boundary, Y a manifold, and $f\colon X \to Y$ a smooth function. Then the set of simultaneous regular values of $f, \partial f$ is dense in Y.*

Proof. A regular point $p \in \partial X$ of ∂f is also a regular point of f: if $d(\partial f)_p$ is surjective in

(13.51)



then so too is df_p. Now use Corollary 8.3 to deduce that the set of simultaneous regular values of

(13.52)
$$\begin{aligned} f\big|_{\operatorname{Int} X}\colon \operatorname{Int} X &\longrightarrow Y \\ \partial f\colon \quad \partial X &\longrightarrow Y \end{aligned}$$

is dense. □

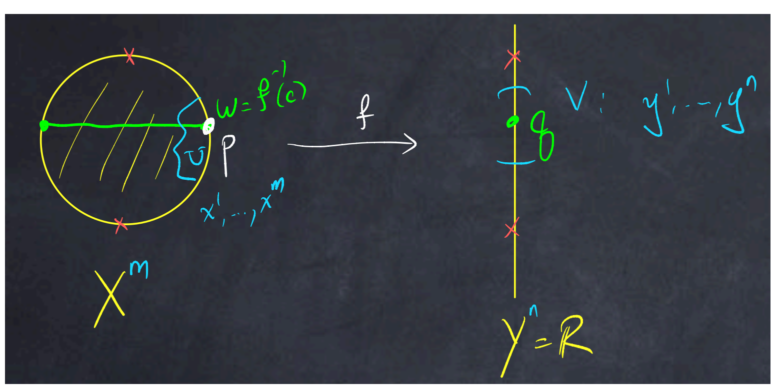


FIGURE 40. A submanifold with boundary cut out by a regular value

Theorem 13.53. *Let X be a manifold with boundary, Y a manifold, $f\colon X \to Y$ a smooth function, and suppose $q \in Y$ is a regular value of f and ∂f. Then $W := f^{-1}(q) \subset X$ is a submanifold. Furthermore, the differential of f induces an isomorphism*

$$\nu_p(W \subset X) \xrightarrow[\cong]{df_p} T_{f(p)}Y. \tag{13.54}$$

In particular, $\operatorname{codim}_p(W \subset X) = \dim_{f(p)} Y$.

Proof. For $p \in W \cap \operatorname{Int} X$ use Theorem 7.10 to construct a submanifold chart. Suppose $p \in W \cap \partial X$. Choose a chart $(V; y^1, \dots, y^n)$ about q, and suppose $y^1(p) = \cdots = y^n(p) = 0$. Then choose a boundary chart $(U; x^1, \dots, x^m)$ about p such that $f(U) \subset V$. Then we claim the differentials of $x^1, f^*y^1, \dots, f^*y^m$ are linearly independent at p. For if $a\, dx^1_p + b_i\, d(f^*y^i)_p = 0$ is a linear relation, then evaluating on $T_p(\partial X)$ we conclude that $b_i\, d(f^*y^i)_p = 0$ restricted to $T_p(\partial X)$. Since q is a regular value of ∂f, we conclude that this is the trivial linear relation: $b_i = 0$ for all i. Since $dx_p \neq 0$, we conclude $a = 0$ and the differentials are linearly independent as claimed. Now apply Proposition 6.14(2) to construct a chart $x^1, \tilde{x}^2, \dots, \tilde{x}^{m-n}, f^*y^1, \dots, f^*y^n$ of X on an open subset of U. This is a standard submanifold chart; see Figure 41. $\square$

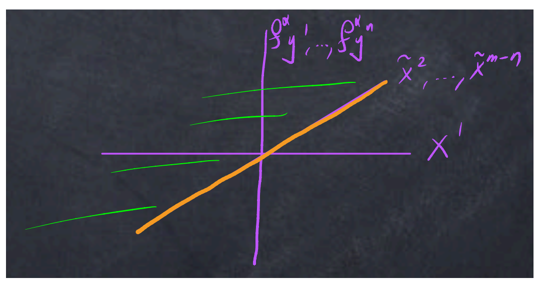


FIGURE 41. Image of the submanifold chart

The technique used in the proof of Theorem 11.38 applies to manifolds with boundary to prove the following.

Theorem 13.55. *Let X be a manifold with boundary, Y a manifold, $Z \subset Y$ a submanifold, $f\colon X \to Y$ a smooth function, and suppose $f, \partial f \pitchfork Z$. Then $W := f^{-1}(Z) \subset X$ is a submanifold. Furthermore, if $p \in X$ satisfies $f(p) \in Z$, then*

(1) $T_pW = df_p^{-1}\big(T_{f(p)}Z\big)$.
(2) *df_p induces an isomorphism of normal spaces*

$$\nu_p(W \subset X) \xrightarrow[\cong]{df_p} \nu_{f(p)}(Z \subset Y). \tag{13.56}$$

(3) $\operatorname{codim}_p(W \subset X) = \operatorname{codim}_{f(p)}(Z \subset Y)$.

Lecture 14: Classification of 1-manifolds with boundary

Introduction

In this lecture we prove the following important result.

Theorem 14.1. *Let X be a nonempty connected one-dimensional manifold with boundary. Then*

$$X \approx S^1 \text{ or } [0,1] \text{ or } \mathbb{R} \text{ or } [0,1). \tag{14.2}$$

(The '$\approx$' symbol means 'is diffeomorphic to'.)

Corollary 14.3. *If X be a compact one-dimensional manifold with boundary, then $\#\partial X$ is even.*

Proof. The boundary ∂X is a compact 0-manifold, so it is a finite set of points. By Theorem 14.1, X is a finite union of circles and closed intervals, each of which has an even number of boundary points (zero and two, respectively). □

Remark 14.4. We will construct counting invariants—degrees, intersection numbers, etc.—which depend on choices. To compare the count for different choices we construct a bordism between the choices, and the difference of the counts for two different choices is the count of boundary points of a compact 1-manifold with boundary. Corollary 14.3 is used to conclude that the count modulo 2 is independent of the choices. Later we introduce orientations to refine the mod 2 invariants to integer invariants.

(14.5) *Classification of low-dimensional manifolds.* The classification result Theorem 14.1 uses the simplicity of dimension one. There are classification results in higher dimensions, but unsurprisingly they are more complicated. We give a flavor of some results, and restrict to compact connected manifolds.

In dimension two there is a complete classification of compact connected manifolds. They come in two families, which are enumerated in Example 1.21.

In dimension three the situation is much more complicated. If we add the hypothesis that the manifold be simply connected, as well as compact, then it has the homotopy type of S^3. The Poincaré Conjecture, a theorem proved by Grigori Perelman in 2002, asserts that a compact manifold with the homotopy type of S^3 is diffeomorphic to S^3. There are also structure theorems for compact 3-manifolds not homotopy equivalent to the 3-sphere. The most profound is Bill Thurston's Geometrization Conjecture, again a theorem proved by Perelman.

In dimension four the classification of compact manifolds is intractable since any finitely presented group can be the fundamental group of such a manifold, and there is no algorithm to distinguish finitely presented groups. Even if we restrict to simply connected compact 4-manifolds, there is no easy classification. The measure of our ignorance is the fact that the 4-dimensional Poincaré Conjecture is open: it is not known if a closed 4-manifold homotopy equivalent to S^4 is diffeomorphic to S^4. On the other hand, for simply connected compact topological manifolds the situation is under control due to work of Mike Freedman in the early 1980s. He proved that the homeomorphism type is determined by the homotopy type and, in "half" of the cases, a mod 2 invariant due to Rob Kirby and Larry Siebenmann. In particular, he proved that a homotopy 4-sphere is homeomorphic to the 4-sphere, which is the topological version of the Poincaré conjecture. When we come to smooth structures on these simply connected compact topological manifolds, there are new phenomena not seen in lower dimensions. Namely, there exist such manifolds which admit no smooth structure, and there exist others for which there are multiple nondiffeomorphic smooth structures. (In dimensions one, two, and three every topological manifold admits a unique smooth structure.) Surprisingly, affine space $\mathbb{A}^4$ admits uncountably many inequivalent smooth structures. Also, the *smooth* version of the 4-dimensional Poincaré conjecture is very much open. It states that a compact smooth 4-manifold which is homotopy equivalent to S^4 is diffeomorphic to S^4.

(14.6) *Local geometric structures in differential topology.* The general technique we use to prove Theorem 14.1 is one that appears often: a local geometric structure is introduced, and it is used to prove a global result. Sometimes this strategy is implemented by solving a differential equation. Sometimes it involves proving that some quantity is independent of the choice of a particular geometric structure on the manifold. In the case at hand, we introduce a Riemannian metric—a notion of lengths of tangent vectors—on our given 1-manifold X and then write an ordinary differential equation for a motion $f\colon \mathbb{R} \to X$ to have unit speed, i.e., velocity of length one at all times. The function f is used to get hold of the diffeomorphism type of X.

Remark 14.7. An alternative approach is to introduce a Morse function $f\colon X \to \mathbb{R}$, i.e., a function with finitely many nondegenerate critical points. (Nondegeneracy means the second derivative of f at a critical point is nonzero.) Then simple surgeries are used to eliminate most or all of the critical points, which again gives control of the global structure of X. Note the duality of the two approaches: in our approach we map the standard manifold $\mathbb{R}$ in whereas in the alternative one maps out to the standard model.

(14.8) *Higher dimensions.* The general strategy outlined in **(14.6)** is used to prove some of the classification results described in **(14.5)**. For example, one approach to the classification of compact 2-manifolds is via Morse Theory, as in Remark 14.7. The approach pioneered by Richard Hamilton to prove the Poincaré conjecture in dimension three is to introduce a Riemannian metric and study the Ricci flow equation, a nonlinear evolution equation of heat type for the initial metric. Perelman was able to complete the proof by showing how to continue the evolution after singularities developed. The endpoint of the evolution is a simpler Riemannian metric which reveals the global structure. In dimension four one also introduces a Riemannian metric, but now that is fixed and one writes a nonlinear elliptic partial differential equation for extrinsic objects: connections. There is a *moduli space* of solutions which is used to detect obstructions to smoothness or to construct invariants of the smooth structure (which must then be proved independent of the choice of Riemannian metric). This *self-duality equation* arose in quantum field theory, and was explored in a geometric context by several mathematicians (Atiyah, Singer, Hitchin, Drinfeld, Manin, Ward, ...), leading to Simon Donaldson's applications to the topology of 4-manifolds.

(14.9) *Outline of this lecture.* Returning to the task at hand, we begin with two background topics: Riemannian metrics and ordinary differential equations (ODEs) on manifolds. Both are important in differential geometry and differential topology.[14] After a little maneuver—passage to a double cover—we introduce the basic ODE. Then we use a maximal solution to gain control of the global structure of X and complete the proof of Theorem 14.1.

Riemannian metrics

(14.10) *Inner products.* Let V be a real vector space. An *inner product* or *metric* on V is a function

$$g\colon V\times V\longrightarrow \mathbb{R} \tag{14.11}$$

which for all $N\in\mathbb{Z}^{>0}$, $\xi,\eta,\xi_i,\eta_j\in V$, $1\leqslant i,j\leqslant N$, and $a^i,b^j\in\mathbb{R}$ satisfies

$$g(a^i\xi_i,b^j\eta_j)=a^ib^jg(\xi_i,\eta_j) \tag{14.12}$$

$$g(\xi,\eta)=g(\eta,\xi) \tag{14.13}$$

$$g(\xi,\xi)>0\qquad \text{if } \xi\neq 0. \tag{14.14}$$

The first condition (14.12)—*bilinearity*—is linear in g and cuts a linear subspace out of the space of all functions (14.11). The second condition (14.13)—*symmetry*—is also linear in g. The last condition (14.14) is not linear in g.

Definition 14.15. Let A be an affine space. A subset $C\subset A$ is *convex* if for all $p_0,p_1\in S$ the line segment $\{(1-t)p_0+tp_1 : 0\leqslant t\leqslant 1\}$ is contained in S. A subset $C\subset A$ is a *cone* if for all $p\in S$ the ray $\{tp : t>0\}$ is contained in S.

Condition (14.14) implies that the space $\operatorname{Inn} V$ of inner products on V is a *convex cone* inside the vector space $\operatorname{Sym}^2 V^*$ of symmetric bilinear forms; see Figure 39. The reader should work out the actual picture for $\dim V=2$.

An inner product on V determines a norm on V:

$$\|\xi\|=\sqrt{g(\xi,\xi)},\qquad \xi\in V. \tag{14.16}$$

Remark 14.17. We have not assumed that V is finite dimensional, but in the sequel we only consider inner products on finite dimensional vector spaces.

[14] One might ask: What is the relationship between 'geometry' and 'topology'. First comment: There is no sharp divide, and in some sense this is one of many false dichotomies in mathematics. One viewpoint is that geometry is the general study of shapes, and topology is a branch of geometry which concerns more global properties. From another perspective—Klein's *Erlangen Program*—a *type* of geometry is specified by its symmetry group and that type of geometry is the study of properties invariant under the group. Very roughly, then, topology is the case when the symmetry group is infinite dimensional.

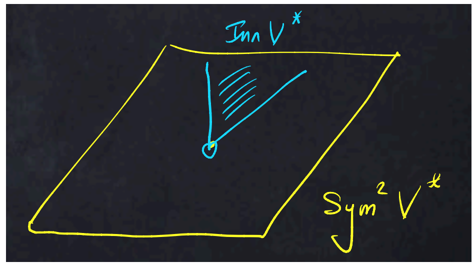


FIGURE 39. The space of inner products on a real vector space

(14.18) *Inner products on a vector bundle.* We consider inner products, or metrics, on a smooth family of vector spaces.

Definition 14.19. Let $\pi\colon E \to S$ be a real vector bundle. A *metric* on π is a smoothly varying family of inner products on the fibers.

This is wishy-washy until we explain the meaning of 'smoothly varying'. The simplest statement is that for every local trivialization

$$\begin{array}{ccc} U \times V & \xrightarrow{\varphi} & \pi^{-1}(U) \\ & {\scriptstyle \mathrm{pr}_1}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & U & \end{array} \tag{14.20}$$

the inner products on E_s, $s \in U$, transport to a *smooth* function $U \to \operatorname{Inn} V \subset \operatorname{Sym}^2 V^*$.

Remark 14.21. Alternatively, the data of a metric on π—the parametrized version of (14.11)—is a function $E \times_S E \to \mathbb{R}$ on the fiber product of π with itself, and we demand that this function be smooth. Yet another formulation: Associated to π is a fiber bundle $\operatorname{Inn} E \to S$ whose fiber at $s \in S$ is the space of inner products on E_s. Then Definition 14.19 amounts to a smooth section of $\operatorname{Inn} E \to S$.

Proposition 14.22. *Let $\pi\colon E \to S$ be a real vector bundle. Then there exists a metric on π.*

Proof. Suppose $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is a cover of S by open sets equipped with a local trivialization

$$\begin{array}{ccc} U \times V_\alpha & \xrightarrow{\varphi_\alpha} & \pi^{-1}(U) \\ & {\scriptstyle \mathrm{pr}_1}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & U & \end{array} \tag{14.23}$$

For each $\alpha \in \mathcal{A}$ choose an inner product g_α on V_α. Let $\{\rho_\alpha\}_{\alpha \in \mathcal{A}}$ be a partition of unity subordinate to $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ with the same index set $\mathcal{A}$. Then $g = \sum_{\alpha \in \mathcal{A}} \rho_\alpha g_\alpha$ is a metric on π. ☐

Remark 14.24.

(1) The key point is that inner products are a convex set, so can be averaged against a partition of unity. The same argument proves the existence of sections of any fiber bundle whose fibers are convex subsets of a fiber bundle of affine spaces.

(2) A bit more argument proves that the space of metrics is contractible.

(14.25) *Riemannian metrics.* The discussion in **(14.18)** applies in particular to the tangent bundle $\pi\colon TX \to X$ to a smooth manifold X.

Definition 14.26. Let X be a smooth manifold. A *Riemannian metric* on X is a metric on the tangent bundle $\pi\colon TX \to X$.

Proposition 14.22 implies that any smooth manifold admits a Riemannian metric. There is a rich geometry of Riemannian manifolds, but in this class we only use Riemannian metrics as auxiliary devices, as discussed in **(14.6)**.

Ordinary differential equations on manifolds

An application of the contraction mapping fixed point theorem guarantees the existence and uniqueness of local solutions to an ODE on affine space. That theorem is proved in Lecture 17 of the notes on multivariable analysis that I handed out at the beginning of the course. There is also a global theorem in Lecture 18 of those notes. These are basic theorems and you will do well to study them if you have not seen them before. In this section I use those results to produce a geometric formulation on smooth manifolds. Theorem 14.32 below is easily proved from the theorems on affine space.

(14.27) *Integral curves.* The following definition is illustrated in Figure 40.

Definition 14.28. Let X be a smooth manifold.

(1) A *vector field* on X is a smooth section of the tangent bundle $TX \to X$.
(2) Let ξ be a vector field on X and $J \subset \mathbb{R}$ an open interval. A smooth motion $\gamma\colon J \to X$ on X is an *integral curve* of ξ if

$$\dot\gamma(t) = \xi_{\gamma(t)}, \qquad t \in J. \tag{14.29}$$

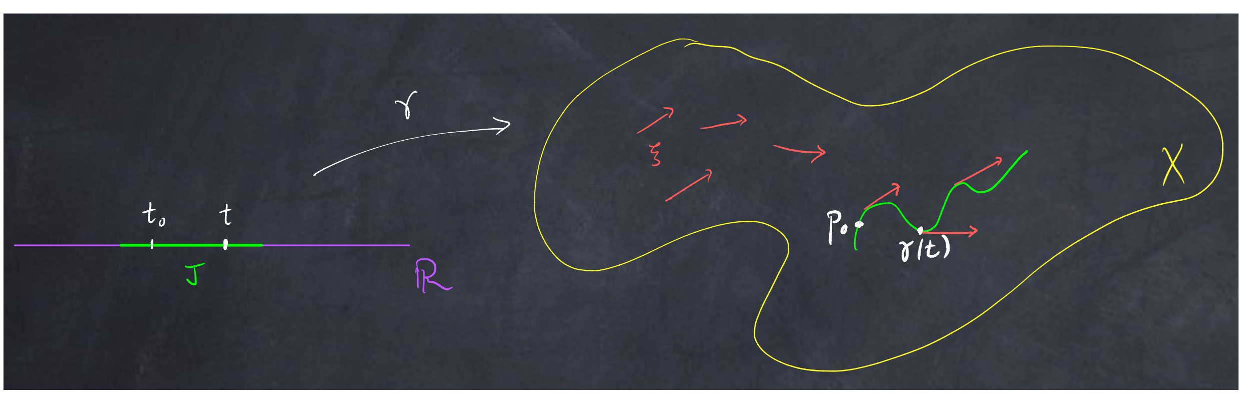


FIGURE 40. A vector field and integral curve

In the figure an initial time $t_0 \in J$ and initial position $p_0 \in X$ are depicted. Equation (14.29) is an ordinary differential equation, as one can see in a local coordinate system.

Remark 14.30. There is a generalization to time-varying vector fields, but in fact that is a special case of a fixed vector field. Namely, consider the trivial fiber bundle $\rho\colon J \times X \to J$. Then a time-varying vector field on X is a smooth section of the relative tangent bundle (12.4) whose fiber at $(t, p) \in J \times X$ is T_pX. Since the relative tangent bundle is a subbundle of $T(J \times X)$, we can regard this as a vector field on $J \times X$. An integral curve for this vector field is a solution of (14.29) where now the right hand side can also vary with $t \in J$.

The "device" we use here illustrates two important general techniques: (1) Encode parameters in a space, and (2) Reduce a complicated setup to a familiar one on a more complicated space.

(14.31) *Existence and uniqueness.* The main theorem asserts the existence and uniqueness of integral curves, both locally and globally.

Theorem 14.32. *Let ξ be a smooth vector field on a smooth manifold X. Fix $t_0 \in \mathbb{R}$ and $p_0 \in X$.*

(1) *There exists an open interval $J \subset \mathbb{R}$ containing t_0 and a smooth function $\gamma\colon J \to X$ such that*

$$\begin{aligned} \dot\gamma(t) &= \xi_{\gamma(t)}, \qquad t \in J \\ \gamma(t_0) &= p_0. \end{aligned} \tag{14.33}$$

Furthermore, any two solutions agree on the intersection of their domains.

(2) *There exists an open interval $J_{\max} \subset \mathbb{R}$ containing t_0 and a solution $\gamma_{\max}\colon J_{\max} \to X$ to (14.33) such that any solution is contained in $\gamma_{\max}$.*

The containment in (2) makes sense if we identify a function with its graph, so for any solution $\gamma\colon J \to X$ we have $J \subset J_{\max}$ and $\gamma = \gamma_{\max}\big|_J$.

Unit speed parametrization

We have one more preliminary to the proof of Theorem 14.1. Henceforth, X is a connected 1-dimensional manifold with boundary. Apply Proposition 14.22 to introduce a Riemannian metric g on X, which we fix for the remainder of the lecture.

(14.34) *Setting up the ODE.* We aim to construct a motion $f\colon J \to X$ on a maximal interval $J \subset \mathbb{R}$ which has *unit speed*: the norm of the velocity $\dot f(t)$ is one for all $t \in J$. It is instructive to write this condition with respect to a local coordinate $x\colon U \to \mathbb{R}$ on an open subset $U \subset X$. Assume $f(J) \subset U$, or cut down J until it is so. Let $G\colon x(U) \to \mathbb{R}$ be the positive function $G(x) = \|\frac{\partial}{\partial x}\|^2$. Then the composite function $h = x \circ f\colon J \to \mathbb{R}$ satisfies the equation

$$\dot h(t)^2 = \frac{1}{G(x)}, \tag{14.35}$$

and so

$$\dot h(t) = \frac{\pm 1}{\sqrt{G(x)}}. \tag{14.36}$$

The sign ambiguity means we have two ODEs, not one. Geometrically, we have at each $p \in X$ two opposite tangent vectors $\pm\eta \in T_pX$ with unit norm $\|\eta\| = 1$. To write a single ODE we must pick out a single unit norm vector at each point. We can make such a choice locally, but we would have to prove that there exists a global choice; it is possible that one local choice, continued to be continuous, can come back to the opposite choice. Rather than disentangle the global issue, we use another "device" that is often the better route: Construct a new space which encodes both choices.

(14.37) *The orientation double cover.* Consider the function

$$\begin{aligned} TX &\longrightarrow \mathbb{R} \\ \eta &\longmapsto g(\eta,\eta) \end{aligned} \tag{14.38}$$

We claim that $1 \in \mathbb{R}$ is a regular value. Namely, if $\eta \in T_pX$ has unit norm, then

$$\frac{d}{dt}\bigg|_{t=1} g(t\eta, t\eta) = 2 \neq 0, \tag{14.39}$$

so the differential of (14.38) is surjective at η. Let $\widetilde{X} \subset TX$ be the inverse image of 1; it is the space of unit norm tangent vectors. The restriction

$$\pi\colon \widetilde{X} \longrightarrow X \tag{14.40}$$

of the tangent bundle $TX \to X$ is a double cover: there are two vectors of unit norm in each tangent space.

Remark 14.41. Since X is 1-dimensional, the tangent bundle $TX \to X$ is a real line bundle: the fibers are 1-dimensional real vector spaces. Recall Definition 13.34 that an orientation of a fiber T_pX is a choice of component of nonzero vectors. A unit norm vector is nonzero, and there is a unique unit norm vector in each component.[15] Hence the double cover (14.40) can be identified with the *orientation double cover*, the double cover[16] of X whose fiber consists of the two orientations of T_pX.

(14.42) *The vector field* ξ. There is a tautological vector field ξ on the manifold $\widetilde{X}$. Namely, a point $\tilde{p} \in \widetilde{X}$ is a point $p = \pi(\tilde{p})$ of X together with a unit tangent vector $\eta \in T_pX$. The double cover π induces an isomorphism of tangent spaces $d\pi_{\tilde{p}}\colon T_{\tilde{p}}\widetilde{X} \to T_pX$, and the value of ξ at $\tilde{p}$ is the vector in $T_{\tilde{p}}\widetilde{X}$ which projects to η. (It is natural to conflate $\tilde{p}$ and η, since $\tilde{p} \in \widetilde{X} \subset TX$.)

[15] Alert! When you read an assertion like this, pause to work out a proof for yourself. Develop this habit of mind—it is important to being a mathematician. In this case if L is a line with an inner product, and $\zeta \in L$ is a nonzero vector, then ζ is a basis and any other vector equals $t\zeta$ for a unique $t \in \mathbb{R}$. The equation $1 = g(t\zeta, t\zeta) = t^2 g(\zeta,\zeta)$ has two solutions $\pm t$ which lead to opposite unit vectors $\eta = \pm t\zeta$. If $\gamma\colon [0,1] \to L$ is a continuous path which connects them, apply the intermediate value theorem to the continuous function $t \mapsto g(\gamma(t), \eta)$ to conclude that $\gamma(t) = 0$ for some $t \in [0,1]$. Therefore, η and $-\eta$ lie in opposite components of $L\backslash\{0\}$. The point of this footnote is not so much the proof, but the state of mind required to read and do mathematics.

[16] I'm sure an alert went off and you are now constructing this double cover and proving the local triviality!

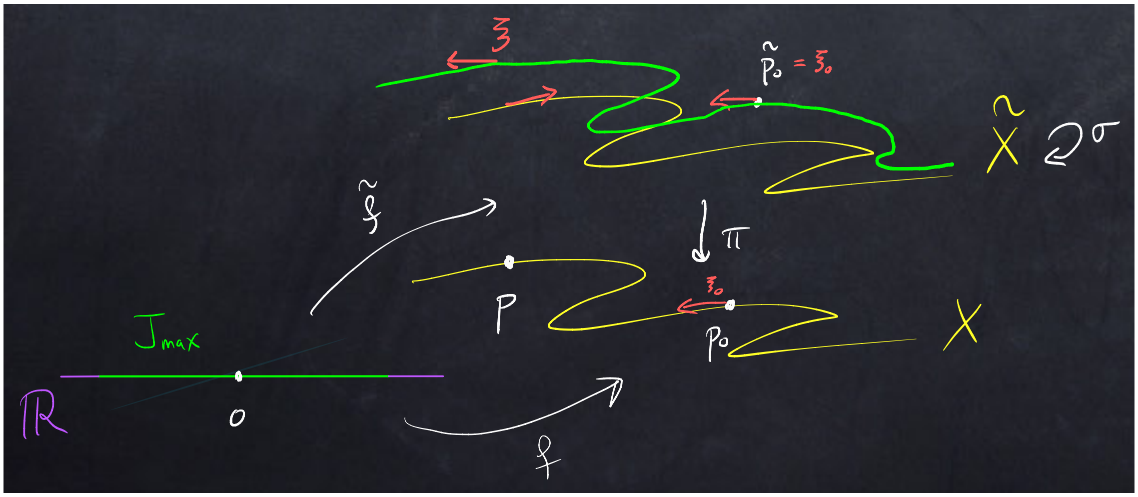


FIGURE 41. The integral curve $\tilde{f}$ and its projection f

Proof of Theorem 14.1

After these preliminaries, we are ready for the main task in this lecture. Let X be a nonempty connected 1-manifold. Choose a Riemannian metric on X and let ξ be the vector field on the orientation double cover which was constructed in (14.42).

(14.43) *Construction of a function f into X.* Choose $\tilde{p}_0 \in \tilde{X}$ and let $p_0 = \pi(\tilde{p}_0) \in X$ be its projection. Apply Theorem 14.32 to construct an open interval $J_{\max} \subset \mathbb{R}$ and a smooth function $\tilde{f}\colon J_{\max} \to \tilde{X}$ which is the maximal integral curve of ξ that satisfies the initial condition $\tilde{f}(0) = \tilde{p}_0$. Set $f = \pi \circ \tilde{f}\colon J_{\max} \to X$. Note that both $\tilde{f}$ and f are local diffeomorphisms: the differential is an isomorphism since ξ is nonvanishing. We use the map f to gain global control over the diffeomorphism type of X. The proof breaks up into two main cases according to whether $J_{\max} = \mathbb{R}$ or $J_{\max}$ is a proper subinterval.

(14.44) $J_{\max} = \mathbb{R}$. As a first step we prove surjectivity.

Proposition 14.45. *Assume $J_{\max} = \mathbb{R}$. Then f is surjective.*

Proof. Since f is a local diffeomorphism, it is an open map and so $f(\mathbb{R}) \subset X$ is open. We claim that $f(\mathbb{R}) \subset X$ is also closed, and then since $f(\mathbb{R})$ is nonempty and X is connected, it follows that $f(\mathbb{R}) = X$. Thus suppose $\{t_n\} \subset \mathbb{R}$ is a sequence such that there exists $q \in X$ with $f(t_n) \to q$ as $n \to \infty$. Choose a chart (U, x) about q and let $K \subset U$ be a compact subset which contains $x(q)$ in its interior. After possibly omitting a finite number of terms of the sequence, we may assume that $\{f(t_n)\} \subset K$. Let $\gamma\colon I \to \mathbb{R}$ be the composition of $x\colon U \to \mathbb{R}$ and the restriction of f to an open interval $I \subset J_{\max}$ which contains $\{t_n\}$. Define $G\colon x(K) \to \mathbb{R}$ by $G(x) = \|\frac{\partial}{\partial x}\|^2$. Since $x(K)$ is compact, we can and do choose $C > 0$ such that $G(x) \leqslant C$ for $x \in x(K)$. Hence

$$|\dot{\gamma}(t)| = \frac{1}{\sqrt{G\big(\gamma(t)\big)}} \geqslant \frac{1}{\sqrt{C}}, \qquad t \in I. \tag{14.46}$$

Now for $n, m \in \mathbb{Z}$ we have

$$|\gamma(t_n) - \gamma(t_m)| = \left| \int_{t_m}^{t_n} dt\, \dot{\gamma}(t) \right| \geqslant \frac{1}{\sqrt{C}}\, |t_n - t_m|, \tag{14.47}$$

which implies

$$|t_n - t_m| \leqslant \sqrt{C}\, |\gamma(t_n) - \gamma(t_m)|. \tag{14.48}$$

Since $\gamma(t_n) \to x(q)$, the sequence $\{\gamma(t_n)\}$ is Cauchy, hence by (14.48) so too is the sequence $\{t_n\}$. Hence there exists $t_0 \in J_{\max} = \mathbb{R}$ such that $t_n \to t_0$. It follows that $f(t_0) = \lim\limits_{n\to\infty} f(t_n) = q$. This proves the surjectivity of f. $\square$

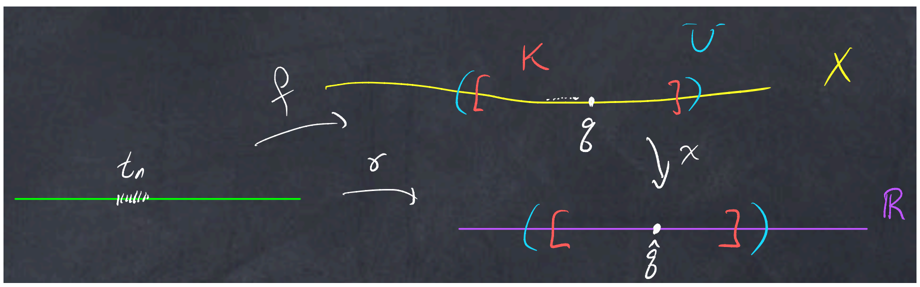


FIGURE 42. Proof that $f(\mathbb{R}) \subset X$ is closed if $J_{\max} = \mathbb{R}$

Corollary 14.49. *If $J_{\max} = \mathbb{R}$ and f is injective, then $X \approx \mathbb{R}$.*

Proof. f is a bijective local diffeomorphism, hence a global diffeomorphism. $\square$

Proposition 14.50. *If $J_{\max} = \mathbb{R}$ and f is not injective, then $X \approx S^1$.*

Proof. Choose $t_0, t_1 \in \mathbb{R}$ such that $f(t_0) = f(t_1)$. Let $\sigma\colon \widetilde{X} \to \widetilde{X}$ be the involution (nonidentity deck transformation) of the double cover $\pi\colon \widetilde{X} \to X$. Then either $\tilde{f}(t_0) = \tilde{f}(t_1)$ or $\tilde{f}(t_0) = \sigma\tilde{f}(t_1)$. In the former case, the motion $t \mapsto \tilde{f}(t + t_1 - t_0)$ is an integral curve of ξ which maps $t_0 \mapsto \tilde{f}(t_1) = \tilde{f}(t_0)$, hence the uniqueness statement in Theorem 14.32 implies $\tilde{f}(t) = \tilde{f}(t + t_1 - t_0)$ for all $t \in \mathbb{R}$. It follows that $\tilde{f}(0) = \tilde{f}(t_1 - t_0)$, and so $f(0) = f(t_1 - t_0) = p_0$. Consider $f^{-1}(p_0) \subset \mathbb{R}$. It is a discrete subset, since f is a local diffeomorphism. Let $T \in \mathbb{R}$ be the minimal positive element of $f^{-1}(p_0)$. The uniqueness argument implies $f(t + T) = f(t)$ for all $t \in \mathbb{R}$, from which the map f factors:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\;f\;} & X \\ \downarrow & \nearrow_{\bar{f}} & \\ \mathbb{R}/(\mathbb{Z}\cdot T) & & \end{array} \tag{14.51}$$

Here $\mathbb{Z} \cdot T \subset \mathbb{R}$ is the subgroup generated by T, and $\mathbb{R}/(\mathbb{Z} \cdot T) \approx S^1$. Minimality of T implies that $\bar{f}$ is injective. It is also surjective and a local diffeomorphism, hence a global diffeomorphism.

It remains to rule out $\tilde{f}(t_0) = \sigma\tilde{f}(t_1)$, a situation depicted in Figure 43. If so, then the motions $t \mapsto \tilde{f}(t_0 + t)$ and $t \mapsto \sigma\tilde{f}(t_1 - t)$ are integral curves of ξ which agree at $t = 0$, hence are equal. Set $t = (t_1 - t_0)/2$ to deduce $\tilde{f}(\frac{t_0+t_1}{2}) = \sigma\tilde{f}(\frac{t_0+t_1}{2})$, which is absurd. $\square$

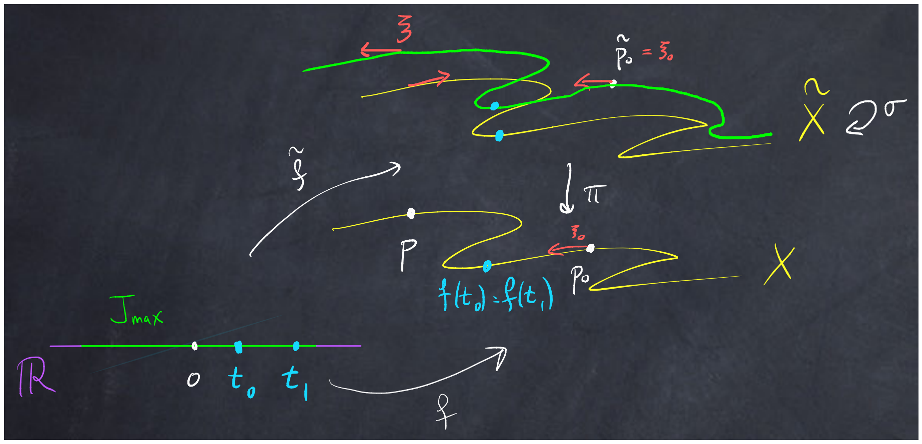


FIGURE 43. Ruling out distinct lifts of $f(t_0) = f(t_1)$

(14.52) $J_{\max} \neq \mathbb{R}$. If the maximal open interval of definition of the integral curve $\tilde{f}$ in (14.43) is a proper subset of $\mathbb{R}$, then $J_{\max} = (a, b)$ or (a, ∞) or $(-\infty, b)$ for some $a, b \in \mathbb{R}$. We treat the three cases simultaneously.

Proposition 14.53. *$f\colon J_{\max} \to X$ extends to $\bar{f}\colon \overline{J_{\max}} \to X$ and $\bar{f}(\overline{J_{\max}} \backslash J_{\max}) \subset \partial X$.*

Proof. If $\tilde{p} \in \overline{\tilde{f}(J_{\max})} \backslash \tilde{f}(J_{\max})$, then the proof of Proposition 14.45 shows that there exists a sequence $\{t_n\} \in J_{\max}$ such that $t_n \to a$ or $t_n \to b$ and $\lim_{n\to\infty} f(t_n) = \pi(\tilde{p})$. Use this to define[17] $\bar{f}$. If $\pi(\tilde{p}) \subset \operatorname{Int} X$, then $\tilde{p} \in \operatorname{Int} \widetilde{X}$ and any local integral curve which maps the appropriate endpoint of $\overline{J_{\max}}$ (a or b) to $\tilde{p}$ patches with $\tilde{f}$ to obtain an integral curve of ξ on an open interval which strictly contains $J_{\max}$. This contradicts the maximality of $J_{\max}$. ∎

Proposition 14.54. *$\bar{f}\colon \overline{J_{\max}} \to X$ is a diffeomorphism. Therefore, $X \approx [0, 1]$ or $X \approx [0, 1)$.*

Proof. It follows from Proposition 14.53 that $f(J_{\max}) \subset X$ is closed. The image is open, since f is a local diffeomorphism, and now since X is connected we deduce that $\bar{f}$ is surjective. Injectivity of $\bar{f}$ follows as in the proof of Proposition 14.50: if f is not injective, then it is periodic and the domain $J_{\max}$ can be extended to $\mathbb{R}$. The bijective local diffeomorphism $\bar{f}$ is a global diffeomorphism. ∎

Theorem 14.1 follows by combining Corollary 14.49, Proposition 14.50, and Proposition 14.54.

Lecture 15: Brouwer fixed point theorem; mod 2 degree; transversality in families

After building many foundations we capitalize on our work and prove some theorems in topology. The first is a nonretraction theorem, which has as a corollary the Brouwer fixed point theorem.

[17] The careful reader will note that we must prove that $\bar{f}$ is smooth. To do so, choose a boundary chart of $\widetilde{X}$ at $\tilde{p}$ (we prove next that $\tilde{p} \in \partial\widetilde{X}$), smoothly extend the vector field ξ in the chart, and hence smoothly extend the integral curve.

Next, we attempt to define the mod 2 degree of a map. We have most of the tools to do so, but fall short at a crucial stage. Telling this tale now motivates the work on transversality we begin at the end of this lecture.

Nonretraction; a fixed point theorem

(15.1) *A nonretraction theorem.* A *retraction* of a set X onto a subset $A \subset X$ is a left inverse of the inclusion map $i\colon A \hookrightarrow X$, i.e., a map $r\colon X \to A$ such that $r\big|_A = \mathrm{id}_A$.[18] The following is a non-retraction theorem.

Theorem 15.2. *Let X be a compact manifold with boundary. Then there does not exist a retraction $r\colon X \to \partial X$.*

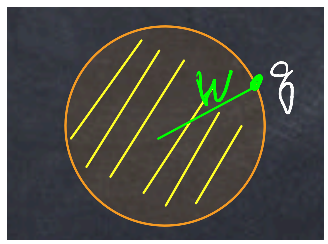


FIGURE 44. The inverse image of a point under a putative retraction

Proof. Suppose $r\colon X \to \partial X$ is a retraction. Use Sard's theorem (Theorem 8.1) to choose a regular value $q \in \partial X$ of r. Then $W := r^{-1}(q) \subset X$ is a 1-dimensional submanifold, by Theorem 13.53. In particular,

$$\partial W = W \cap \partial X = \{w \in \partial X : \partial r(w) = q\} = \{q\}, \tag{15.3}$$

since $\partial r = \mathrm{id}_{\partial X}$. This contradicts Corollary 14.3. □

Remark 15.4. Compactness is crucial. For example, if Y is a closed manifold, then $X = (-1, 0] \times Y$ retracts onto its boundary $\partial X = \{0\} \times Y$.

(15.5) *A fixed point theorem.* Let $f\colon X \to X$ be a map from a set to itself. A point $p \in X$ with $f(p) = p$ is a *fixed point* of f. Recall the closed ball $D^n \subset \mathbb{A}^n$ from Example 13.24.

Corollary 15.6 (Brouwer fixed point theorem)**.** *Suppose $f\colon D^n \to D^n$ is a smooth map. Then there exists $p \in D^n$ such that $f(p) = p$.*

The following lovely proof is due to Morris Hirsch.

[18] More generally, a retraction of *any* map is a left inverse. Dually, a section of any map is a right inverse. Note that if $r \circ i = \mathrm{id}$, then i is injective and r is surjective.

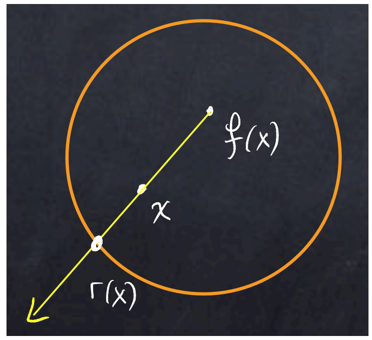


FIGURE 45. Proof of the Brouwer fixed point theorem

Proof. If $f\colon D^n \to D^n$ has no fixed point, then define a retraction $r\colon D^n \to \partial D^n = S^{n-1}$ which sends $x \in D^n$ to the intersection of the ray emanating from $f(x)$ through x with ∂D^n, as in Figure 45. But Theorem 15.2 rules out the existence of r.

To prove that r is smooth, consider

$$\begin{aligned} F\colon D^n \times \mathbb{R}^{>0} &\longrightarrow \mathbb{R} \\ x\ ,\ t &\longmapsto \mathrm{dist}_{\mathbb{A}^n}\bigl[0\,,\,x+t\bigl(x-f(x)\bigr)\bigr]^2 \end{aligned} \tag{15.7}$$

The implicit function theorem implies that the function $t(x)$ defined by $F\bigl(x,t(x)\bigr)=1$ is smooth—check that $\partial F/\partial t \neq 0$ at points where $F(x,t)=1$—hence $r(x)=x+t(x)\bigl(x-f(x)\bigr)$ is smooth. $\square$

Remark 15.8. The case $n=1$ of Corollary 15.6 has an elementary proof: apply the intermediate value theorem to $f(x)-x$. This case is illustrated in Figure 46, which indicates how the fixed point set of a map $f\colon X\to X$ can be expressed as the intersection in $X\times X$ of the graph of f and the diagonal $\Delta=\{(x,x) : x\in X\}$.

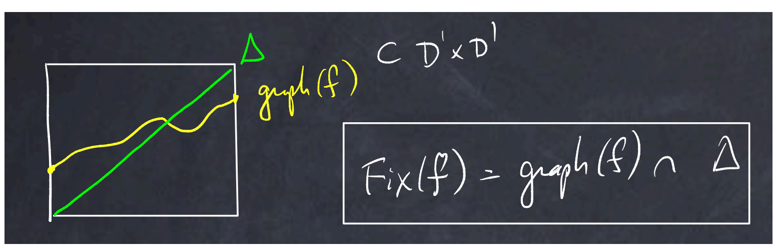


FIGURE 46. Brouwer fixed point theorem on D^1

Mod 2 degree: first attempt

(15.9) *Setup.* The degree of a map is defined in the following situation. Fix a positive integer n. Let X be a compact n-manifold and Y a connected n-manifold. Notice the explicit assumption that $\dim X=\dim Y$. Suppose $f\colon X\to Y$ is a smooth map. If $q\in Y$ is a regular value, then $f^{-1}(q)$ is a 0-dimensional submanifold of X, hence a finite set of points, since X is compact. The degree counts the number of points in the inverse image.

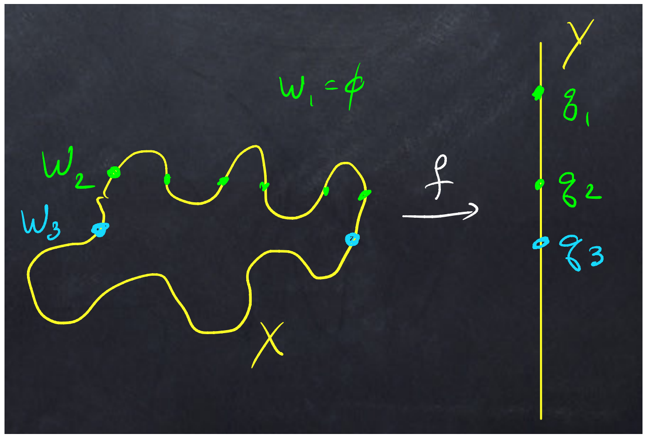


FIGURE 47. The mod 2 degree is independent of the regular value q

(15.10) *Dependence on the regular value.* Figure 47 illustrates that $\#f^{-1}(q)$ depends on the regular value q. In this figure, f is orthogonal projection from the plane curve X onto the vertical line Y, and q_1, q_2, q_3 are regular values. As we move $q_1 \to q_2 \to q_3$ the cardinality of the inverse image changes: $0 \to 6 \to 2$. So while the count is not independent of the regular value, its reduction modulo two is, and that is true in general. We can see intuitively what happens by examining the inverse images W_1 and W_2 of the closed intervals $[q_1, q_2]$ and $[q_2, q_3]$. By an easy adaptation of Theorem 13.44, W_1 and W_2 are 1-dimensional manifolds with boundary, and they are compact since X is compact. In fact, W_1 is a bordism from $f^{-1}(q_1)$ to $f^{-1}(q_2)$, and W_2 is a bordism from $f^{-1}(q_2)$ to $f^{-1}(q_3)$. The observed fact that the number of inverse image points modulo two is unchanged when passing through a critical value follows from Corollary 14.3: the number of boundary points of a compact 1-manifold with boundary is even. It is useful to think of a movie where time is $t \in [q_1, q_2]$ and we watch $f^{-1}(t)$ as t evolves. We can observe the birth of pairs of inverse image points as we pass through critical values. Continuing the movie for $t \in [q_2, q_3]$ we can see annihilations, or deaths, or pairs as we pass through critical values. These birth and death singularities are the essence of Morse/Cerf Theory.

Remark 15.11. We will introduce *orientations* later in the course, and then count points with sign to obtain an integer invariant (which in this example is zero).

(15.12) *Main theorem; partial proof.* We begin with a definition.

Definition 15.13. A *smooth homotopy* of maps $X \to Y$ between manifolds (no boundary) is a smooth map $F\colon [0,1] \times X \to Y$.

Write $F_t\colon X \to Y$ for the restriction of F to $\{t\} \times X$.

Theorem 15.14. *Fix $n \in \mathbb{Z}^{>0}$ and let X be a compact n-manifold, Y a connected n-manifold, and $f\colon X \to Y$ a smooth map.*

(1) *The mod 2 cardinality $\#f^{-1}(q) \pmod 2$ of the inverse image of a regular value $q \in Y$ is independent of q.*

(2) *If* $F\colon [0,1]\times X \to Y$ *is a smooth homotopy of maps, and* $q\in Y$ *a simultaneous regular value of* F*,* F_0*, and* F_1*, then* $\#F_0^{-1}(q) \equiv \#F_1^{-1}(q) \pmod 2$.

Statement (1) is the well-definedness of the mod 2 degree $\deg_2 f \in \mathbb{Z}/2\mathbb{Z}$, and (2) implies that $\deg_2 f$ is a smooth homotopy invariant. For the latter, we observe by Sard's theorem (Corollary 8.3) that simultaneous regular values of F, F_0, F_1 exist.

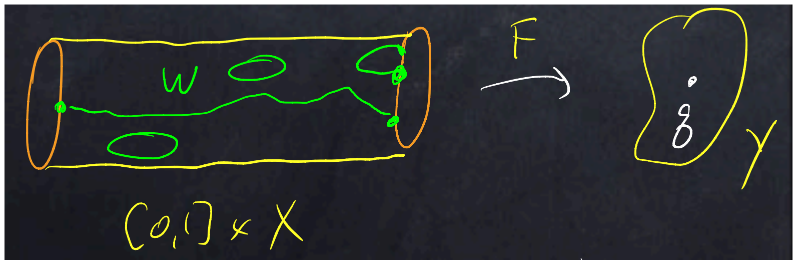


FIGURE 48. Homotopy invariance of $\deg_2 f$

Proof of (2). Observe that $\partial\bigl([0,1]\times X\bigr) = \{0\}\times X \amalg \{1\}\times X$, and so $\partial F = F_0 \amalg F_1$. Theorem 13.53 implies that $W := F^{-1}(q)$ is a 1-dimensional submanifold of $[0,1]\times X$—see Figure 48—and that

$$\begin{aligned} \partial W &= W\cap\bigl(\{0\}\times X\bigr) \;\amalg\; W\cap\bigl(\{1\}\times X\bigr) \\ &= \quad \{0\}\times F_0^{-1}(q) \quad \amalg \quad \{1\}\times F_1^{-1}(q). \end{aligned} \tag{15.15}$$

Since $\#\partial W$ is even, it follows that $\#F_0^{-1}(q) \equiv F_1^{-1}(q) \pmod 2$. □

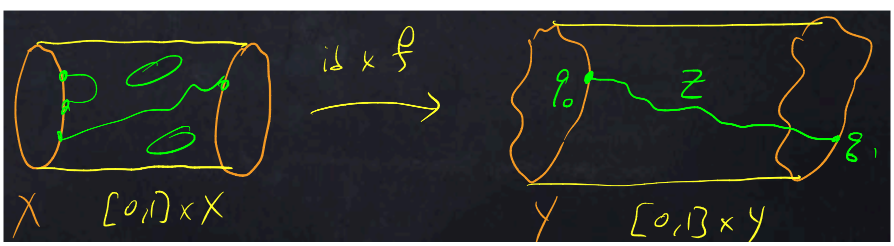


FIGURE 49. Independence of $\#f^{-1}(q) \pmod 2$ as the regular value q varies

(15.16) *Attempted proof of (1).* Suppose $q_0, q_1 \in Y$ are regular values. Since Y is connected, we can and do choose a motion $t\mapsto q_t$ from q_0 to q_1. Its graph is a closed submanifold $Z\subset [0,1]\times Y$; see Figure 49. Consider $\mathrm{id}_{[0,1]}\times f\colon [0,1]\times X\to[0,1]\times Y$. If $(\mathrm{id}_{[0,1]}\times f) \pitchfork Z$, then by Theorem 13.55 the inverse image of Z is a 1-dimensional submanifold of $[0,1]\times X$. Now we can argue as in (15.15) to prove (1). However, there is no guarantee of the needed transversality $(\mathrm{id}_{[0,1]}\times f) \pitchfork Z$. Our task, carried out over the next few lectures, is to prove that we can *perturb* $\mathrm{id}_{[0,1]}\times f$ to be transverse to Z. Furthermore, since it is already transverse when restricted to the boundary, since q_0, q_1 are assumed to be regular values of f, we want to perturb without changing the restriction to the boundary. It is precisely this approximation theorems that we will prove.

Remark 15.17. In Lecture 12 we proved that transversality is a stable condition, so transverse maps form an open subset in a suitable mapping space. The approximation theorem is a companion result which says that transverse maps form a dense subset in a suitable mapping space.

Families of maps and transversality

As a first step toward the approximation theorem, we consider a family of maps such that the entire family is transverse to a submanifold. The construction of such families is the subject of the next lecture. Here we assume we have such a family and prove the density of transverse maps in the family. For the first version of the theorem, we omit boundaries.

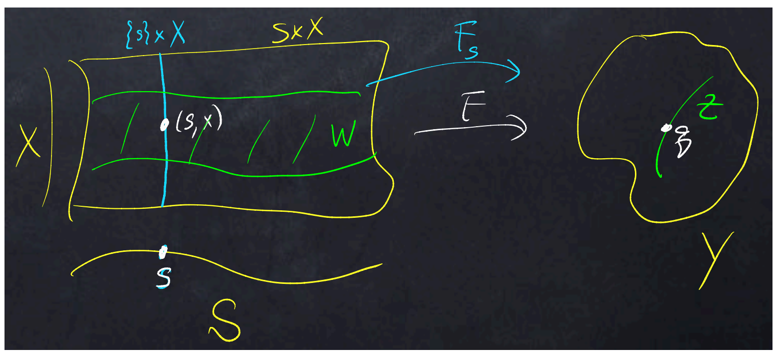


FIGURE 50. A transverse family of maps $X \to Y$ parametrized by S

Theorem 15.18. *Let X, Y, S be smooth manifolds, and $Z \subset Y$ a submanifold. Suppose $F\colon S \times X \to Y$ is a smooth map. If $F \pitchfork Z$, then for a dense set of $s \in S$ we have $F_s \pitchfork Z$.*

Here, for $s \in S$ we set

$$\begin{aligned} F_s\colon X &\longrightarrow Y \\ p &\longmapsto F(s,p) \end{aligned} \tag{15.19}$$

Proof. By Theorem 11.38 the inverse image $W := F^{-1}(Z) \subset S \times X$ is a submanifold. Let $\pi\colon W \to S$ be the restriction of the projection $\mathrm{pr}_1\colon S \times X \to S$:

$$\begin{array}{ccc} W & \hookrightarrow & S \times X \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & S & \end{array} \tag{15.20}$$

Then we claim that $s \in S$ is a regular value of π if and only if $F_s \pitchfork Z$. The theorem then follows from Sard, since regular values of π are dense in S.

The claim is then a linear algebra assertion about the differentials. Fix $(s,p)\in W\subset S\times X$ and consider the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \big\downarrow & & & & \\
 & & & & T_pX & \xrightarrow[\boxed{2}]{d(F_s)_p} & T_qY/T_qZ & & \\
 & & & & \big\downarrow & & \big\Vert & & \\
0 & \longrightarrow & T_{(s,p)}W & \longrightarrow & T_sS\oplus T_pX & \xrightarrow{dF_{(s,p)}} & T_qY/T_qZ & \longrightarrow & 0 \\
 & & \boxed{1}\big\downarrow d\pi_{(s,p)} & & \big\downarrow \mathrm{pr}_1 & & & & \\
 & & T_sS & = & T_sS & & & & \\
 & & & & \big\downarrow & & & & \\
 & & & & 0 & & & &
\end{array} \tag{15.21}$$

The claim is that $\boxed{1}$ is surjective iff $\boxed{2}$ is surjective. In the diagram, the two squares commute by (15.20) and (15.19). The middle row is exact since $F\pitchfork Z$. The middle column is exact by the definition of direct sum. These properties mean that (15.21) is symmetric about the diagonal from NW to SE, and that symmetry means that $\boxed{2}$ surjective $\Longrightarrow$ $\boxed{1}$ surjective follows from $\boxed{1}$ surjective $\Longrightarrow$ $\boxed{2}$ surjective. The latter is proved by a 4-step diagram chase:

(i) Fix $\eta\in T_qY/T_qZ$.
(ii) By exactness of the middle row, choose $\zeta\in T_sS$ and $\xi\in T_pX$ such that

$$dF_{(s,p)}\colon \zeta+\xi\longmapsto\eta. \tag{15.22}$$

(iii) Since we assume $\boxed{1}$ is surjective, choose $\lambda\in T_{(s,p)}W$ such that

$$d\pi_{(s,p)}\colon \lambda\longmapsto\zeta. \tag{15.23}$$

(iv) Then $\zeta+\xi-\lambda\in T_pX$ maps to $\eta\in T_qY/T_qZ$ via $d(F_s)_p$.

This proves the surjectivity of $\boxed{2}$. ∎

An almost identical argument proves the variation of Theorem 15.18 when X has boundary.

Theorem 15.24. *Let X be a smooth manifold with boundary, Y,S smooth manifolds, and $Z\subset Y$ a submanifold. Suppose $F\colon S\times X\to Y$ is a smooth map. If $F,\partial F\pitchfork Z$, then for a dense set of $s\in S$ we have $F_s,\partial F_s\pitchfork Z$.*

Lecture 16: Perturbing a map to achieve transversality

Introduction

In Lecture 15 we attempted to define a topological invariant—the mod 2 index of a map between equidimensional manifolds—but we discovered in **(15.16)** that our current toolkit is not sufficient. Namely, we need to be able to perturb a smooth map to be transversal to a given submanifold. Furthermore, if transversality has already been achieved on a subset, then we'd like a controlled perturbation which does not move the map on the good subset. We took the first step at the end of the last lecture with Theorem 15.18 (and its variation Theorem 15.24 for manifolds with boundary). It asserts that if a family of maps is transverse *as a family*, then the generic map in the family is transverse. We are reduced (or rather emboldened) to constructing transverse families. We take that up first for a submanifold $Z \subset A$ of affine space, where translation provides a ready family of perturbations, and then to a submanifold $Z \subset Y$ of a general manifold Y. The families we construct are transverse to the ambient manifold Y, hence *a fortiori* they are transverse to any submanifold $Z \subset Y$. So we quickly drop Z. The main tool is a *tubular neighborhood theorem* for submanifolds of affine space (Theorem 16.8).

We begin the next lecture with controlled perturbations.

(16.1) *Setup.* Throughout this lecture X is a smooth manifold with boundary, Y is a smooth manifold, $Z \subset Y$ is a submanifold, and $f\colon X \to Y$ is a smooth map. (In the next lecture we assume that $Z \subset Y$ is *closed*, but that is not necessary in this lecture.) What we need to define topological invariants is that $f \mathrel{\overline{\pitchfork}} Z$ and $\partial f \mathrel{\overline{\pitchfork}} Z$. The goal is to prove that we can homotop f to achieve this transversality.

Remark 16.2. In Theorem 12.17 we proved that transversality is stable under deformation. (For this result we *do* need that $Z \subset Y$ be closed.) This means that in a suitable topology on the space of smooth maps $X \to Y$, the subspace of maps transverse to Z is open. The results in this lecture show that the subspace of transverse maps is dense. We do not develop the topology of mapping spaces in these lectures.

Perturbations in affine targets

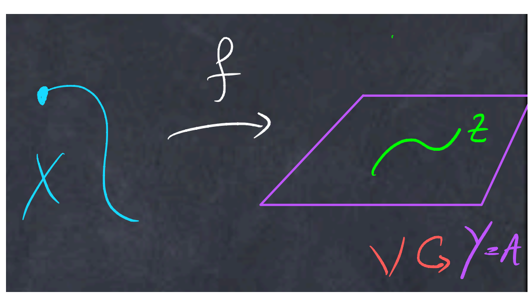


FIGURE 51. A map with affine codomain

To begin, consider the special case in which Y is an affine space A with vector space of translations V. We use the transitive group of translations to perturb. Namely, define

$$\begin{aligned} F\colon V \times X &\longrightarrow A \\ \xi \;,\; p &\longmapsto f(p) + \xi \end{aligned} \tag{16.3}$$

This is a family of maps parametrized by $S = V$, and at the center of the family is the original map $F_0 = f$. Note that $\partial(V \times X) = V \times \partial X$.

Proposition 16.4. *F and ∂F are submersions. In particular, $F, \partial F \pitchfork Z$.*

Proof. For any $\xi \in V$ and $p \in X$, the differential $dF_{(\xi,p)}\colon V \oplus T_pX \to V$ restricts to id_V on $V \oplus \{0\}$, so is surjective. The same holds for the differential of ∂F. ☐

Statement of theorems

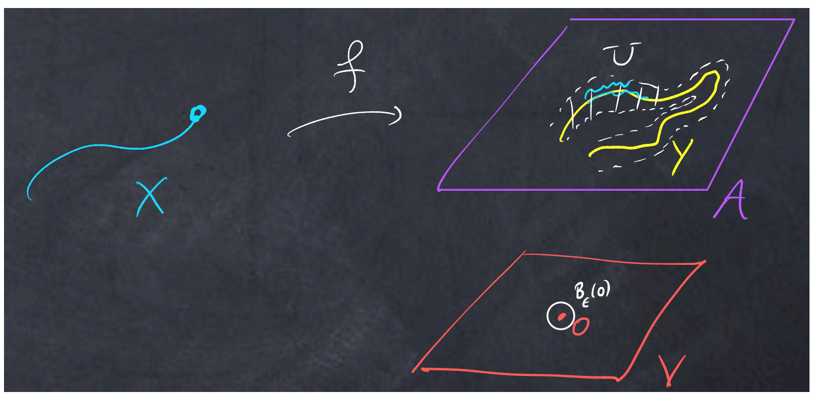


FIGURE 52. Perturbation by a uniform translation followed by projection

(16.5) *Strategy.* Now suppose Y is an arbitrary smooth manifold. By Theorem 11.25 we may assume that Y is a submanifold of a finite dimensional affine space. The idea, then, is to perturb the composition $X \xrightarrow{f} Y \hookrightarrow A$ as in (16.3), control that the image of the perturbation lies in a small neighborhood $U \subset A$ of Y, and then compose with a submersion $\pi\colon U \to Y$. In Figure 52 we illustrate a small uniform perturbation. In general, though, the size of the perturbation may depend where we are on Y, since a noncompact manifold embedded in affine space may not have a neighborhood of uniform size. The task, then, is to first construct U and π, and then to control the size of the translation to ensure that the perturbed maps have image in U.

Fix a norm on V. For $r > 0$ and $p \in Y$ let $B_r(p) \subset A$ be the ball of radius r about p.

Definition 16.6. For a smooth function $\epsilon\colon Y \to \mathbb{R}^{>0}$, let

$$Y^\epsilon = \bigcup_{q \in Y} B_{\epsilon(q)}(q). \tag{16.7}$$

Y^ϵ is an open subset of A.

Theorem 16.8. *Let A be a finite dimensional real affine space and $Y \subset A$ a submanifold.*

(1) *There exists an open neighborhood $U \subset A$ and a submersion $\pi\colon U \to Y$.*
(2) *For any open neighborhood $U \subset A$ of Y, there exists a smooth function $\epsilon\colon Y \to \mathbb{R}^{>0}$ such that $Y^\epsilon \subset U$. If Y is compact, then we can choose ϵ to be a constant function.*

A continuous positive function on a compact space has a positive minimum, hence the last assertion.

The rest of the lecture is devoted to the proof of Theorem 16.8. For now we extract the statements about transversality we need.

Corollary 16.9. *Let X be a manifold with boundary, A a finite dimensional real affine space with vector space V of translations, $Y \subset A$ a submanifold, and $f\colon X \to Y$ a smooth map. Choose U, π, ϵ as in Theorem 16.8. Let $B_1(0) \subset V$ be the unit ball. Then the map*

$$\begin{aligned} F\colon B_1(0) \times X &\longrightarrow Y \\ \xi\ ,\ p &\longmapsto \pi\Big(f(p) + \epsilon\big(f(p)\big)\xi\Big) \end{aligned} \tag{16.10}$$

is a submersion.

Proof. The first differential of F at (ξ, p) is the composition of the surjective map $d\pi$ with the invertible homothety which scales by $\epsilon\big(f(p)\big)$, hence is surjective. $\square$

Corollary 16.11. *Let X be a manifold with boundary, Y a smooth manifold, $Z \subset Y$ a submanifold, and $f\colon X \to Y$ a smooth map. Then there exists a smooth homotopy $H\colon [0,1] \times X \to Y$ such that $H_0 = f$ and $H_1, \partial H_1 \pitchfork Z$.*

Proof. Embed Y in a finite dimensional real affine space A and construct the family of maps (16.10). Use Theorem 15.24 to choose $\xi \in B_1(0)$ so that $F_\xi, \partial F_\xi \pitchfork Z$. Then define $i\colon [0,1] \to B_1(0)$ by $i(t) = t\xi$ and set $H = F \circ (i \times \mathrm{id}_X)$. $\square$

Splittings of the normal bundle

(16.12) *Splittings of a short exact sequence of vector spaces.* Let V, V', V'' be a vector spaces and

$$0 \longrightarrow V' \xrightarrow{\ i\ } V \xrightarrow{\ j\ } V'' \longrightarrow 0 \tag{16.13}$$

a short exact sequence of linear maps.

Definition 16.14. A *splitting* of (16.13) is a right inverse $s\colon V'' \to V$ to j.

In other words,

$$j \circ s = \mathrm{id}_{V''}. \tag{16.15}$$

A splitting is equivalent to a left inverse to i: either expresses V as a direct sum $V = s(V'') \oplus i(V')$. Splittings exist since linear subspaces have linear complements. Observe that (16.15) is an affine equation in s: the left hand side is linear and the right hand side is constant. From this we deduce that splittings of (16.13) form an affine space over $\mathrm{Hom}(V'', V')$.

(16.16) *Splittings of a short exact sequence of vector bundles.* We pass from vector spaces to vector bundles and use local triviality to deduce the existence of local splittings of short exact sequences. Then a partition of unity lets us pass from local to global.

Proposition 16.17. *Let Y be a smooth manifold, $\pi\colon E \to Y$, $\pi'\colon E' \to Y$, $\pi''\colon E'' \to Y$ vector bundles over Y, and*

$$0 \longrightarrow E' \xrightarrow{\;i\;} E \xrightarrow{\;j\;} E'' \longrightarrow 0 \tag{16.18}$$

a short exact sequence of linear maps. Then there exists a splitting $s\colon E'' \to E$.

Proof. Construct an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of Y together with local trivializations (9.24) of each vector bundle over each U_α. Then restricted to U_α, (16.18) becomes a short exact sequence of linear maps $i(q), j(q)$, $q \in U_\alpha$ between fixed vector spaces (16.13). Choose a complement to $i(q_0)(V') \subset V$ at some $q_0 \in U_\alpha$. It is a complement to $i(q)(V') \subset V$ for q in an open neighborhood of U_α, and by cutting down the U_α we can assume it is a complement for all $q \in U_\alpha$, and so find a splitting s_α of (16.18) restricted to U_α. Let $\{\rho_\alpha\}_{\alpha\in\mathcal{A}}$ be a partition of unity subordinate to $\{U_\alpha\}_{\alpha\in\mathcal{A}}$, and define $s = \sum\limits_{\alpha\in\mathcal{A}} \rho_\alpha s_\alpha$. Then s is the desired global splitting. □

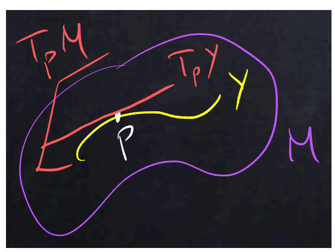


FIGURE 53. The normal space $\nu_q = T_qY/T_qM$ to $Y \subset M$ at $q \in Y$

(16.19) *Recollection of the normal bundle.* You can recollect by rereading **(9.72)** and looking at Figure 53. Note in particular the short exact sequence (9.73), which we now know has a splitting.

Proof of Theorem 16.8

With these preliminaries we turn to our main task, split into the two parts of the theorem.

Proof of (1). Let $\rho\colon \nu \to Y$ be the normal bundle to $Y \subset A$. Use Proposition 16.17 to choose a splitting s in

$$0 \longrightarrow TY \longrightarrow A \times V \underset{}{\overset{s}{\rightleftarrows}} \nu \longrightarrow 0 \tag{16.20}$$

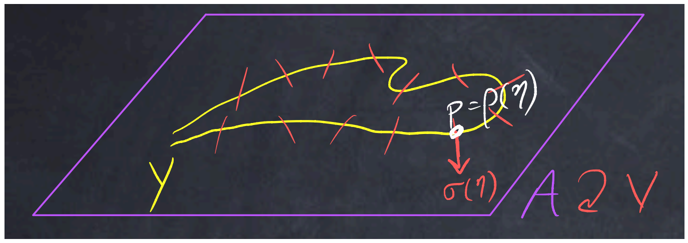


FIGURE 54. Construction of a tubular neighborhood

and set $\sigma = \mathrm{pr}_2 \circ s\colon \nu \to V$. Define

$$\text{(16.21)} \qquad \begin{aligned} h\colon \nu &\longrightarrow A \\ \eta &\longmapsto \rho(\eta) + \sigma(\eta) \end{aligned}$$

We will prove that h restricts to a diffeomorphism of a neighborhood of the zero section in ν to a neighborhood of Y in A.

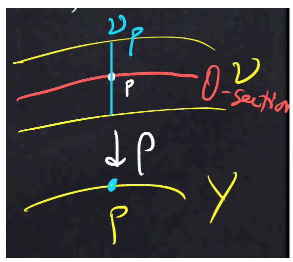


FIGURE 55. The 0-section of the normal bundle

First, for any $q \in Y$ we have $h(0_q) = q$, where $0_q \in \nu_q$ is the zero vector. Next, there is a natural isomorphism $T_qY \oplus \nu_q \xrightarrow{\cong} T_{0_q}\nu_q$ which maps $\xi \in T_qY$ to the corresponding tangent vector to the zero section at 0_q; see Figure 55. With this identification, we compute

$$\text{(16.22)} \qquad \begin{aligned} dh_{0_q}\colon T_qY \oplus \nu_q &\longrightarrow \quad V \\ \xi \,+\, \eta &\longmapsto \xi + \sigma(\eta) \end{aligned}$$

By the splitting property, this is an isomorphism. Hence h is a local diffeomorphism at 0_q. Choose an open neighborhood $W_q \subset A$ of q and a local inverse $g_q\colon W_q \to \nu$ to h. Define

$$\text{(16.23)} \qquad W = \bigcup_{q \in Y} W_q;$$

it is an open subset of A and $\{W_q\}_{q \in Y}$ is an open cover of W. Since W is a smooth manifold, it is paracompact (Theorem 10.5), so there is a countable locally finite refinement $\{W_i'\}_{i \in I}$ of $\{W_q\}_{q \in Y}$. Let $g_i\colon W_i' \to \nu$ be the local inverse to h induced from the refinement function $I \to Y$ by restricting the appropriate g_q.

The local inverses need not agree on intersections, so define

$$\widetilde{W} = \{q \in W : g_i(q) = g_j(q) \text{ if } q \in W_i' \cap W_j' \text{ for some } i, j \in I\}. \tag{16.24}$$

Then $Y \subset \widetilde{W}$ since $g_i(q) = 0_q$ for all $q \in Y$, $i \in I$. Also, for $q \in Y$ the set

$$I_q = \{i \in I : q \in W_i\} \tag{16.25}$$

is finite, from which $\bigcap_{i\in I_q} W_i'$ is an open neighborhood of q. Since h is a local diffeomorphism at q, we can and do choose an open neighborhood $U_q \subset \bigcap_{i\in I_q} W_i'$ of q on which h is invertible, and by the uniqueness of inverses we have $U_q \subset \widetilde{W}$. Set $U = \bigcup_{q\in Y} U_q$. The local inverses on U_q patch to a function $g\colon U \to \nu$ which inverts h restricted to U. Set $\pi = \rho \circ g\colon U \to Y$. Then π is the composition of a submersion and a diffeomorphism, so is a submersion. ☐

Remark 16.26. We have proved more than is stated in Theorem 16.8(1). Namely, we constructed a diffeomorphism g of U with a neighborhood $g(U)$ of the zero section in the normal bundle; under that identification π is the restriction of the normal bundle ρ to that neighborhood. This is a *tubular neighborhood*. The tubular neighborhood theorem asserts the existence of a tubular neighborhood for any submanifold $Y \subset M$ of any smooth manifold M.

The next proof once more illustrates the local-to-global technique using partitions of unity.

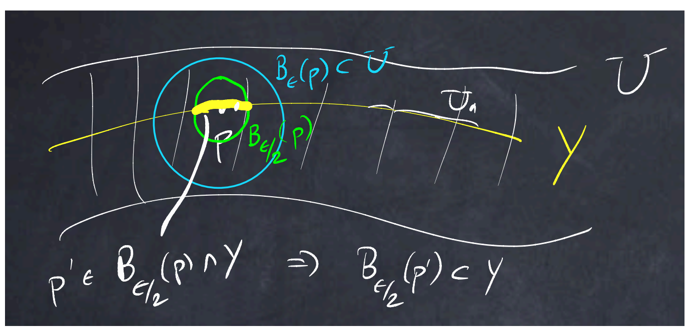


FIGURE 56. Constructing the function $\epsilon\colon Y \to \mathbb{R}^{>0}$

Proof of (2). For any $q \in Y$ there exists $\delta > 0$ such that $B_\delta(q) \subset U$, since U is open. Then for all $q' \in B_{\delta/2}(q) \cap Y$, the triangle inequality implies $B_{\delta/2}(q') \subset U$. This solves the problem locally. Construct an open covering $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ together with $\epsilon_\alpha > 0$ such that $B_{\epsilon_\alpha}(q) \subset U$ for all $q \in U_\alpha$. Let $\{\rho_\alpha\}_{\alpha\in\mathcal{A}}$ be a partition of unity subordinate to the cover, and set $\epsilon = \sum_\alpha \epsilon_\alpha\rho_\alpha$. Then $\epsilon\colon Y \to \mathbb{R}$ is a smooth function. For $q \in Y$ define the finite set $\mathcal{A}_q = \{\alpha \in \mathcal{A} : \rho_\alpha(q) \neq 0\}$, and let $\alpha_q^-, \alpha_q^+ \in \mathcal{A}_q$ be elements at which $\alpha \mapsto \epsilon_\alpha$ achieves its minimum and maximum, respectively. Then $0 < \epsilon_{\alpha_q^-} \leqslant \epsilon(q)$, which proves that ϵ is a positive function. Also, $\epsilon(q) \leqslant \epsilon_{\alpha_q^+}$ from which $B_{\epsilon(q)}(q) \subset B_{\epsilon_{\alpha_q^+}}(q) \subset U$, since $q \in U_{\alpha_q^+}$. It follows that $Y^\epsilon \subset U$. ☐

Lecture 17: Mod 2 Intersection Theory

We still have one piece of unfinished business from Lecture 16: controlled perturbation to achieve transversality. After finishing that off we recall the discussion in Lecture 15 about the mod 2 degree, which the now finished business allows us to finish off the discussion. Then we introduce mod 2 intersection number and some applications.

Controlled perturbations

(17.1) *Motivation.* Suppose X is a manifold with boundary, Y a smooth manifold, $Z \subset Y$ a submanifold, and $f\colon X \to Y$ a smooth map. To construct topological invariants we need maps f such that $f \pitchfork Z$. Corollary 16.11 tells that we can perturb (homotop) f to make it transverse to Z. But now suppose $C \subset X$ is a subset on which our given f is already transverse to Z. Then we would like to perturb f only on the complement of C, as there is no reason to move it on C. To implement this kind of control we expect to use bump functions, constructed via a partition of unity, and those we glue on *open* subsets. So we would like the transversality to persist on an open neighborhood of C. Therefore, we require that $Z \subset Y$ be a *closed* submanifold, since transversality to a closed submanifold is an open condition. Also, we require that C be a *closed* subset (arbitrary, not necessarily a submanifold) of X since asking for constancy of a homotopy on a subset is an equality that occurs on closed subsets.

Theorem 17.2. *Let X be a smooth manifold with boundary, Y a smooth manifold, $Z \subset Y$ a closed submanifold, $C \subset X$ a closed subset, and $f\colon X \to Y$ a smooth map such that $f\big|_C, \partial f\big|_{\partial X \cap C} \pitchfork Z$. Then there exists a smooth homotopy $H\colon [0,1] \times X \to Y$ such that $H_0 = f$, $H_1, \partial H_1 \pitchfork Z$, and $H_t\big|_C = f\big|_C$ for all $t \in [0,1]$.*

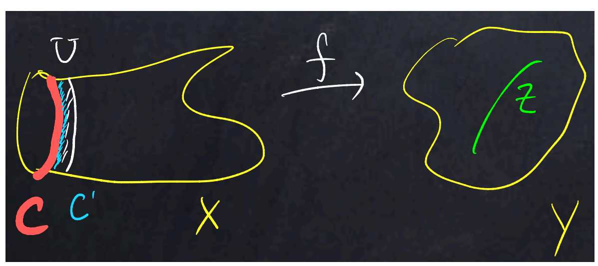


FIGURE 57. $f\big|_U, \partial f\big|_U \pitchfork Z$

Proof. Since transversality to a closed submanifold is an open condition, choose an open set $U \subset X$ which contains C such that $f\big|_U, \partial f\big|_U \pitchfork Z$. Separate the disjoint closed subsets C, $X\backslash U$ by open sets W_C, $W_{X\backslash U}$, and let $C' = X \setminus W_{X\backslash U}$. Then $C' \subset X$ is closed and satisfies $C \subset \operatorname{Int} C' \subset C' \subset U$. Let $\{\rho_U, \rho_{X\backslash C'}\}$ be a partition of unity subordinate to the open cover $\{U, X\backslash C'\}$ of X. Set $\tau = \rho^2_{X\backslash C'}$. Then $\tau\big|_C = 0$; and if $p \in X$ satisfies $\tau(p) = 0$, then $d\tau_p = 0$. Recall the perturbation F, defined in (16.10), which we use to achieve transversality without control. Define the controlled variation

$$G(\xi, p) = F\big(\tau(p)\xi, p\big), \qquad \xi \in B_1(0), \quad p \in X, \tag{17.3}$$

where recall we have embedded Y in a finite dimensional affine space and $B_1(0)$ is the unit ball in the normed vector space V of translations. Notice that if $p \in C$, then $G(\xi, p) = f(p)$ for all ξ. We claim that $G, \partial G \pitchfork Z$. Granting the claim, we argue as in the proof of Corollary 16.11 to complete the proof of Theorem 17.2.

To verify the claim, set

$$m\colon B_1(0) \times X \longrightarrow B_1(0) \times X \qquad (17.4)$$
$$(\xi, p) \longmapsto \big(\tau(p)\xi, p\big)$$

Then $G = F \circ m$. The restriction of m to $B_1(0) \times \tau^{-1}(\mathbb{R}^{>0}) \subset B_1(0) \times X$ is a diffeomorphism—its inverse can be written explicitly—and so on that subset G is the composition of a submersion and a diffeomorphism, hence is itself a submersion. In particular, it and its restriction to the boundary are transverse to Z. Now fix $p \in X$ such that $\tau(p) = 0$. Then $p \in U$ and for any $\xi \in B_1(0)$,

$$dm_{(\xi,p)}(\dot\xi, \dot p) = \big(d\tau_p(\dot p)\xi + \tau(p)\dot\xi\,,\, \dot p\big) = (0, \dot p), \qquad \dot\xi \in V, \quad \dot p \in T_pX, \qquad (17.5)$$

from which

$$dG_{(\xi,p)}(\dot\xi, \dot p) = dF_{(0,p)}(0, \dot p) = df_p(\dot p). \qquad (17.6)$$

Since $f, \partial f \pitchfork Z$ at p, we conclude $G, \partial G \pitchfork Z$ at (ξ, p). ∎

Mod 2 degree redux

(17.7) *Completion of* **(15.16)**. We resume the setup in **(15.9)**. Given $f\colon X \to Y$ we can choose a regular value $q \in Y$ and count the number of inverse image points modulo two. Theorem 15.14(1) asserts that the count is independent of the choice of regular value q, so defines an invariant $\deg_2 f \in \mathbb{Z}/2\mathbb{Z}$. In **(15.16)** we sketched a proof of Theorem 15.14(1), but we fell short since we needed a controlled perturbation to achieve the desired transversality, as depicted in Figure 49. Theorem 17.2 applies to fill the gap in the proof there, as you should check carefully.

Proposition 17.8. *Let X be a compact connected manifold of positive dimension. Then id_X is not smoothly homotopic to a constant map.*

Proof. The mod 2 degree is defined for maps $X \to X$, and $\deg_2 \mathrm{id}_X = 1$, since every point of X is a regular value with a single inverse image point. On the other hand, the constant map $X \to X$ with value $p \in X$ has any $q \neq p$ as a regular value with empty inverse image, so the mod two degree of a constant map is zero. (The assumption of positive dimension guarantees that $q \neq p$ exists.) ∎

Proposition 17.9. *Let n be a positive integer, W a compact $(n+1)$-dimensional manifold with boundary, Y a connected n-dimensional manifold, and $F\colon W \to Y$ a smooth map. Then the mod two degree of the restriction of F to the boundary vanishes:* $\deg_2 \partial F = 0$.

Proof. Let $q \in Y$ be a simultaneous regular value of $F, \partial F$. Then $F^{-1}(q) \subset W$ is a compact 1-dimensional submanifold with $\partial F^{-1}(q) = F^{-1}(q) \cap \partial W$. Now apply Corollary 14.3. □

Proposition 17.10. *Let X be a compact n-dimensional manifold. Then there exists $f\colon X \to S^n$ such that* $\deg_2 f = 1$.

Given any point of X we construct a map which wraps a ball centered at that point around the sphere and collapses the rest of X to a point. We use a bump function to smooth out what might otherwise only be a continuous map.

Proof. Fix once and for all a smooth function $\rho\colon \mathbb{R}^{\geqslant 0} \to \mathbb{R}$ such that $0 \leqslant \rho \leqslant \pi$ and

$$\rho(r) = \begin{cases} 0, & r \leqslant 1/4; \\ \pi, & r \geqslant 3/4. \end{cases} \tag{17.11}$$

Let $D_1(\pi) \subset \mathbb{R}^n$ be the closed ball of radius π, and identify $\mathbb{R}^n$ with the tangent space to the unit sphere $S^n \subset \mathbb{A}^{n+1}$ at the north pole $n = (0, \dots, 0, 1)$. Define $\phi\colon D_1(\pi) \to S^n$ to map $0 \in \mathbb{R}^n$ to the north pole n, and a nonzero vector $\xi \in D_1(\pi)$ to the endpoint of arc of length $\rho(\|\xi\|)$ along the half great circle emanating from n with tangent ξ. Note that ϕ maps all vectors of norm $\geqslant 3/4$ to the south pole $s = (0, \dots, 0, -1)$.

For any $p \in X$ choose a coordinate chart $x\colon U \to \mathbb{R}^n$ about p such that $x(p) = 0$ and $x(U) \supset D_1(\pi)$. Transport ϕ to a map of $x^{-1}\big(D_1(\pi)\big) \to S^n$ and extend ϕ to all of X by mapping the complement of $x^{-1}\big(D_1(\pi)\big)$ to the south pole s. The resulting map $f\colon X \to S^n$ is smooth, and the regular value $(1, 0, \dots, 0) \in S^n$ has a single inverse image point. □

Mod 2 intersection theory

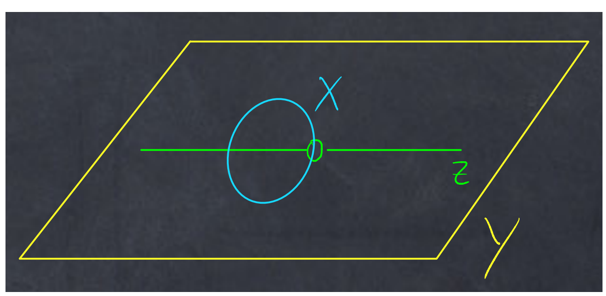


FIGURE 58.

(17.12) *Motivation for setup.* Let Y be a smooth manifold and $X, Z \subset Y$ submanifolds of complementary dimension: $\dim X + \dim Z = \dim Y$. We would like to define the intersection number of X and Z in Y by counting the elements of $X \cap Z \subset Y$. The first problem is that this intersection may contain infinitely many points. For example, let $Y = \mathbb{A}^2$ and $X = Z = \{(x, 0) : x \in \mathbb{R}\} \subset \mathbb{A}^2$. So we need to perturb one of the submanifolds, say X, to achieve a transverse intersection. Our techniques let us perturb *maps* rather than *spaces*, so we perturb the inclusion map $i_X\colon X \to Y$.

Therefore, with no cost we generalize the setup to include an arbitrary smooth map $f\colon X \to Y$. Then Corollary 16.11 implies we can homotop f to a map $g\colon X \to Y$ such that $g \pitchfork Z$, and hence $g^{-1}(Z) \subset X$ is a 0-dimensional submanifold, i.e., a discrete subset of X. But we want it to be a *finite* subset, and therefore we add the hypothesis that X be *compact*. Finally, we want the mod 2 count of points in $g^{-1}(Z)$ to be independent of the perturbation, and that requires that $Z \subset Y$ be a *closed* submanifold. For example, consider $Y = \mathbb{A}^2$, $Z = \{(x,0) : x \in \mathbb{R}^{\neq 0}\} \subset \mathbb{A}^2$, and $X = \{(x,y) : (x-1)^2 + y^2 = 1\}$. Then $\#(X \cap Z) = 1$, but any small nonzero translation of X intersects Z in 2 points; see Figure 58.

(17.13) *Setup for intersection theory.* Hence we arrive at the following collection of data:

$$\begin{aligned} &X && \text{compact manifold}\\ &Y && \text{manifold}\\ &Z \subset Y && \text{closed submanifold}\\ &f\colon X \longrightarrow Y && \text{smooth map}\\ &\dim X + \dim Z = \dim Y && \end{aligned} \tag{17.14}$$

Lemma 17.15. *Let $g_0, g_1\colon X \to Y$ be smoothly homotopic maps which satisfy $g_0, g_1 \pitchfork Z$. Then*

$$\#g_0^{-1}(Z) = \#g_1^{-1}(Z). \tag{17.16}$$

Each $g_i^{-1}(Z) \subset X$, $i = 1, 2$, is a compact 0-dimensional submanifold, hence a finite subset.

Proof. By perturbing a given smooth homotopy away from the boundary (Theorem 17.2) we may assume given a smooth homotopy $g\colon [0,1]\times X \to Y$ from g_0 to g_1 such that $g \pitchfork Z$. Since $\partial g = g_0 \amalg g_1$, we also have $\partial g \pitchfork Z$. Hence $g^{-1}(Z) \subset [0,1] \times X$ is a compact 1-dimensional submanifold with $\partial g^{-1}(Z) = g_0^{-1}(Z) \amalg g_1^{-1}(Z)$. The result now follows from Corollary 14.3. ∎

Definition 17.17. Given the setup (17.14), define the *mod 2 intersection number*

$$\#_2(f, Z) = \#g^{-1}(Z) \tag{17.18}$$

where $g \simeq f$ is any smoothly homotopic map such that $g \pitchfork Z$.

Such maps exist by Corollary 16.11; the count is independent of the choice of g by Lemma 17.15.

Remark 17.19 (Intersections of submanifolds)*.* A special case of (17.14) is when $X \subset Y$ is a compact submanifold and $f = i_X$ is the inclusion map. Then we write $\#_2(f,Z) = \#_2(X,Z)$. The situation is not symmetric: whereas X is compact and Z is only assumed closed. If, however, we also assume that Z is compact, then it is true that $\#_2(X,Z) = \#_2(Z,X)$. One can prove that by identifying each side as a mod 2 intersection number inside the Cartesian product $Y \times Y$. Namely, let $\Delta \subset Y \times Y$ be the diagonal submanifold, and then

$$\#_2^Y(X,Z) = \#_2^Y(Z,X) = \#_2^{Y\times Y}(i_X \times i_Z, \Delta), \tag{17.20}$$

where for clarity we include the ambient manifold in the notation.

(17.21) *Properties of the mod 2 intersection number.* The following properties are analogous to properties of the mod 2 degree; see Theorem 15.14(2) and Proposition 17.9.

Proposition 17.22. *Suppose given the setup (17.14).*

(1) *If $f_0 \simeq f_1$ are smoothly homotopic, then $\#_2(f_0, Z) = \#_2(f_1, Z)$.*
(2) *If W is a compact $(n+1)$-dimensional manifold with boundary $\partial W = X$, and $F\colon W \to Y$ a smooth map such that $\partial F = f$, then $\#_2(f, Z) = 0$.*

To prove (1), perturb a given homotopy to achieve transversality and apply the argument for Theorem 15.14(2). To prove (2), perturb F keeping it fixed on X to achieve transversality and then consider the inverse image of Z.

Examples, applications and variations

Example 17.23. Let $Y = S^1 \times S^1$ be a 2-torus, and consider the submanifolds $X = S^1 \times \{0\}$ and $Z = \{0\} \times S^1$. Then $\#_2(X, Z) = 1$. On the other hand, $\#_2(X, X) = \#_2(Z, Z) = 0$. These mod 2 intersection numbers organize into the 2×2 *intersection matrix* $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Example 17.24 (a nonzero self-intersection number)**.** Let $Y = \mathbb{RP}^2$ be the real projective plane, and $X = \mathbb{RP}^1 \subset \mathbb{RP}^2$ a projective line. Then $\#(X, X) = 1$. To compute this, we perturb the inclusion $i\colon \mathbb{RP}^1 \to \mathbb{RP}^2$ to achieve transversality with the given line X, something we can achieve by choosing a transverse line. In terms of $\mathbb{RP}^2 = \mathbb{P}(\mathbb{R}^3)$, a projective line is a 2-dimensional subspace of $\mathbb{R}^3$, and two transverse 2-dimensional subspaces intersect in a 1-dimensional subspace. That is, two projective lines in the projective plane intersect in a point.

Remark 17.25. Note that two transverse affine lines in an affine plane can intersect in a point or be disjoint (parallel). The lack of compactness prevents mod 2 intersection theory from working for affine lines; Example 17.24 shows how compactification produces an arena in which nontrivial topology emerges.

Remark 17.26. There is an analog of Example 17.24 in the complex projective plane $\mathbb{CP}^2$: two distinct complex projective lines intersect in a single point.

Next, we apply the mod 2 intersection number to distinguish two manifolds.

Theorem 17.27. *The 2-torus $S^1 \times S^1$ is not diffeomorphic to the 2-sphere S^2.*

Proof. If there is a diffeomorphism, then by Example 17.23 we can find two 1-dimensional submanifolds of S^2 with nonzero mod 2 intersection number. However, any map $f\colon X \to S^2$ with $\dim X = 1$ is not surjective, and via stereographic projection from a point not in the image, followed by a 1-parameter family of homotheties, we can homotop f to a constant map to a point which does not lie on any given 1-dimensional submanifold $Z \subset S^2$, hence the mod 2 intersection number vanishes. □

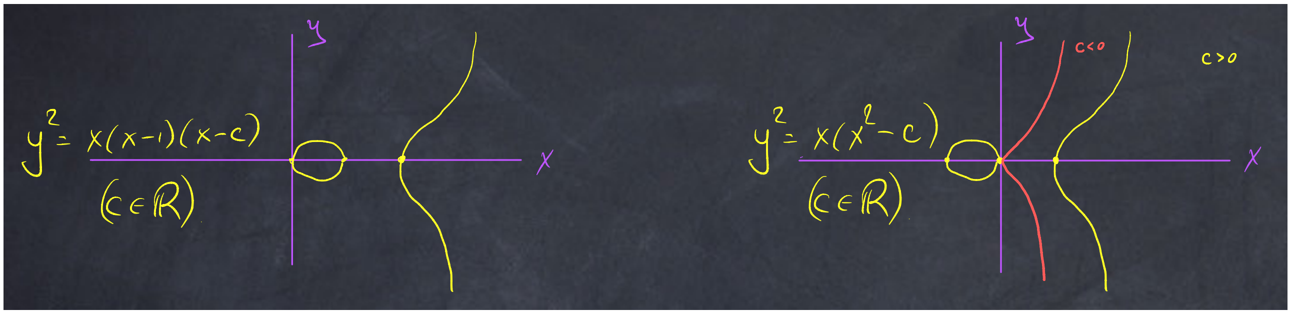


FIGURE 59. Two families of cubic plane curves

(17.28) *Cubics in* $\mathbb{RP}^2$. Now we move from lines in the real projective plane to solutions to a cubic equation: cubic curves. (We skipped over quadrics—solutions to quadratic equations—and you may want to consider what happens there in parallel.) Figure 59 shows two families of cubic curves parametrized by $c \in \mathbb{R}$. We consider the intersection with the x-axis, which is the set of roots of a real cubic equation. In the first family, for any value of c there are three real roots, though two of the roots can coincide. In the second family, for $c > 0$ there are three distinct real roots, whereas for $c < 0$ there is only one real root. As c passes from positive to negative, the real roots all come together at $c = 0$ and then two of them disappear. Of course, they do not disappear if we use complex coefficients; they become a pair of complex conjugate roots to the real cubic equation. In other words, if we consider the cubic in $\mathbb{CP}^2$ rather than $\mathbb{RP}^2$, then there are always three roots, so three intersection points with the line $y = 0$. When we come to oriented intersection theory, we will be able to count these three intersection points. But in $\mathbb{RP}^2$ we can still use the mod 2 theory to detect the intersection of a line and a cubic.

(17.29) *Universal family of cubics.* Rather than work with special one-parameter families, we can consider the universal family. Let $[x, y, z]$ denote a point of $\mathbb{RP}^2$, where $x, y, z \in \mathbb{R}$ are not all zero. The equivalence relation is $[\lambda x, \lambda y, \lambda z] = [x, y, z]$ for all $\lambda \in \mathbb{R}^{\neq 0}$. A cubic curve is the vanishing set of a homogeneous cubic polynomial

$$a_1x^3 + a_2y^3 + a_3z^3 + a_4x^2y + a_5x^2z + a_6xy^2 + a_7xz^2 + a_8y^2z + a_9yz^2 + a_{10}xyz, \tag{17.30}$$

where not all coefficients $a_1, \dots, a_{10}$ vanish. Furthermore, proportional cubics give the same vanishing set. Hence the cubics are parametrized by the projective space[19] $\mathbb{RP}^9$ with homogeneous coordinates $a_1, \dots, a_{10}$. Note that some of these cubics are degenerate. For example, the zero set of xyz is a set of three lines transverse lines in the plane (which form a triangle); the zero set of x^3 is a single triple line. But bear in mind that these lines are projective: each is diffeomorphic to a circle. Of course, the zero set of (17.30) is a closed subset of $\mathbb{RP}^2$, and since $\mathbb{RP}^2$ is compact, it too is compact. By Theorem 14.1 if it is a manifold, then it is a union of circles. As illustrated in the second family of **(17.28)**, the number of circles can change.

[19] If V is a 3-dimensional real vector space, and we replace $\mathbb{RP}^2$ by $\mathbb{P}V$, then the projective space which parametrizes the lines in $\mathbb{P}V$ is the dual projective space $\mathbb{P}V^*$, and the vector space which parametrizes the cubics is $\mathbb{P}\mathrm{Sym}^3 V^*$. We have not yet defined symmetric powers of a vector space: $\mathrm{Sym}^3 V^*$ is (isomorphic to) the vector space of symmetric trilinear functions $V \times V \times V \to \mathbb{R}$.

(17.31) *Intersection theory for the universal family.* The mod 2 intersection of a line and a cubic curve is defined, as long as the cubic curve is a submanifold. We expect that mod 2 intersection number, which is $1 \in \mathbb{Z}/2\mathbb{Z}$ from the examples shown, to be independent of the smooth cubic curve. That is true, but it is not covered by the setup (17.14) since the topology is changing. The following theorem, whose proof is a homework problem, covers this situation.

Theorem 17.32. *Let $\mathcal{X}, Y$ be smooth manifolds, S a connected smooth manifold, $Z \subset Y$ a closed submanifold, and suppose in the commutative diagram*

(17.33)
$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\;G\;} & S \times Y \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & S & \end{array}$$

the map G is an embedding and F is proper. Assume that $(\dim \mathcal{X} - \dim S) + \dim Z = \dim Y$. Suppose that $s_0, s_1 \in S$ are regular values of F. Then

(17.34)
$$\#_2^Y\bigl(G\big|_{F^{-1}(s_0)}, Z\bigr) = \#_2^Y\bigl(G\big|_{F^{-1}(s_1)}, Z\bigr),$$

where we use G to embed $F^{-1}(s_i)$ as submanifolds of Y.

To apply the theorem to the universal family of cubics, set $S = \mathbb{RP}^9$, $Y = \mathbb{RP}^2$, $Z \subset Y$ a fixed line $\mathbb{RP}^1 \subset \mathbb{RP}^2$, and define

(17.35)
$$\mathcal{X} = \Bigl\{\bigl([a_1, \dots, a_{10}], [x, y, z]\bigr) : (17.30) \text{ vanishes}\Bigr\} \subset \mathbb{RP}^9 \times \mathbb{RP}^2.$$

Then $\mathcal{X}$ is a submanifold, since the vanishing of (17.30) is a transverse condition. The maps F and G are the restrictions of projection onto a factor of $\mathbb{RP}^9 \times \mathbb{RP}^2$.

Lecture 18: Mod 2 winding number, Jordan-Brouwer, and Borsuk-Ulam

In the first part of the lecture we add the mod 2 winding number to our arsenal of mod 2 invariants, which up to now consists of the mod 2 degree and mod 2 intersection number. We then apply it to prove a generalization of the classical Jordan curve theorem. As bonus material we probably will not get to in lecture, I include an account of the Borsuk-Ulam theorem as well. The Jordan-Brouwer and Borsuk-Ulam theorems in topology can be proved in a continuous setting; our methods grounded in calculus are suitable for the smooth setting. There are approximation theorems from which one can deduce the general continuous statements from the smooth ones.

(18.1) *Setting for the lecture.* We use the following data throughout:

(18.2)

n	positive integer
A	real affine space of dimension $n+1$
V	tangent space to A, equipped with an inner product
X	compact n-dimensional manifold
$f\colon X \longrightarrow A$	smooth map

Below in the Jordan-Brouwer theorem the map f is an embedding, and we identify X with its image $f(X) \subset A$, a codimension one submanifold (hypersurface) in the affine space A. The discrete topological invariants do not depend on the inner product on V, which is a contractible choice.

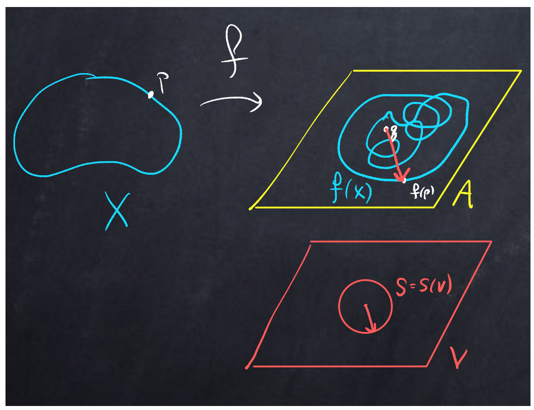


FIGURE 60. Definition of the mod 2 winding number

Mod 2 winding number

(18.3) *Definition and homotopy invariance.* Given the data (18.2), choose $q \in A\backslash f(X)$. (By Sard's theorem, $f(X) \neq A$, so q exists.) Let $S = S(V) \subset V$ be the n-sphere of unit norm vectors. Define

$$\begin{aligned} w_q\colon X &\longrightarrow S \\ p &\longmapsto \frac{f(p)-q}{\|f(p)-q\|} \end{aligned} \tag{18.4}$$

Definition 18.5. The *mod 2 winding number of f about q* is

$$W_2(f,q) = \deg_2 w_q. \tag{18.6}$$

Remark 18.7. If $n=1$ and $X=S^1$, then this is the classical case encountered in complex analysis, for example. There one writes an integral formula for the winding number. There are similar integral formulæ for the general case of Definition 18.5.

Proposition 18.8. *Given the data (18.2), the mod 2 winding number $W_2(f,q)$ depends only on the (path) component of q in $A\backslash f(X)$. Also, $W_2(f,q)$ is unchanged under smooth homotopies of f which do not contain q in the image.*

Proof. Both statements follow from the homotopy invariance of the mod 2 degree (Theorem 15.14(2)). For the first choose a path $t \mapsto q_t$ connecting two points in the complement of $f(X)$. For the second let $t \mapsto f_t$ be a homotopy such that $f_t(p) \neq q$ for all t, p. In each case we obtain a homotopy of (18.4), well-defined since the vector in the denominator is nonzero. ∎

(18.9) *Computation of the mod 2 winding number via extension.* We give two methods to compute. In the first we write f as the boundary of a map out of a compact manifold with boundary. (This is possible if X is null bordant.) In the second we express the mod 2 winding number as the mod 2 intersection number with a ray.

Theorem 18.10. *Given the data (18.2), suppose in addition that W is a compact $(n+1)$-manifold with ∂W, and $F\colon W \to A$ a smooth map with $\partial F = f$. Let $q \in A\backslash f(X)$ be a regular value of F. Then $W_2(f,q) \equiv \#F^{-1}(q) \pmod 2$.*

Notice that q is trivially a regular value of $\partial F = f$ since q is not in the image of f. Regular values in each path component of $A\backslash f(X)$ exist by Sard's theorem.

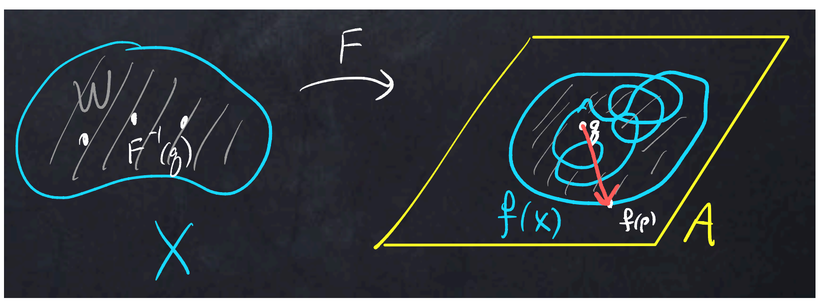


FIGURE 61. Computation of W_2 via extension and counting

Proof. If $F^{-1}(q) = \emptyset$, then the map w_q extends to W using the formula in (18.4) with f replaced by F. The vanishing of the mod 2 winding number follows from Proposition 17.9.

Hence we may assume $F^{-1}(q) = \{p_1, \dots, p_N\}$ is a nonempty finite set. Since q is a regular value, F is a local diffeomorphism at each p_i. Choose a ball $B \subset A$ containing q such that $F^{-1}(B) = \bigsqcup_i B_i$ is a disjoint union of open neighborhoods of the p_i and $F\big|_{B_i} : B_i \to B$ is a diffeomorphism. We may further assume—by shrinking B if necessary—that the closure $D_i = \overline{B_i}$ of each B_i is a manifold with boundary. Apply the first paragraph of the proof to $W \setminus \bigcup_{i=1}^N B_i$ to conclude

$$W_2(f,q) = \sum_{i=1}^{N} W_2(f_i, q), \tag{18.11}$$

where $f_i = F\big|_{\partial D_i}$. Finally, since f_i is a diffeomorphism, so too is (18.4) with $f = f_i$, and hence its mod 2 degree equals one. ∎

(18.12) *Computation of the mod 2 winding number via ray crossing.* For $q \in A$ and $\xi \in V^{\neq 0}$ a nonzero vector, define the *ray*

$$Z_q(\xi) = \{q + t\xi : t > 0\}. \tag{18.13}$$

Observe that $Z_q(\xi) \subset A\backslash\{q\}$ is a closed submanifold of the deleted affine space.

Theorem 18.14. *Given the data (18.2), fix $q \in A\backslash f(X)$ and $\xi^{\neq 0} \in V$. Let $Z = Z_q(\xi)$ be a ray in A emanating from q in the direction ξ. Then if $f \pitchfork Z$,*

$$W_2(f,q) = \#_2(f,Z), \tag{18.15}$$

where the intersection number is computed in $A\backslash\{q\}$.

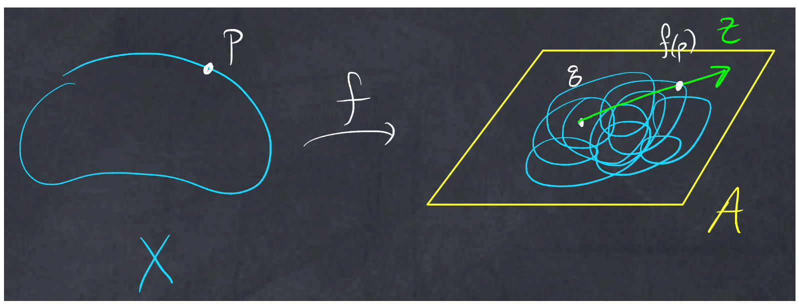


FIGURE 62. Computation of W_2 via intersection with a ray

Proof. Suppose $p \in X$ satisfies $f(p) \in Z$. Then the tangent space to the unit sphere S of V at the unit vector $w_q(p) = \dfrac{f(p)-q}{\|f(p)-q\|} \in S$ is the orthogonal complement to ξ in V. Now

$$d\left(\frac{f-q}{\|f-q\|}\right)_p = \frac{df_p}{\|f(p)-q\|} - \frac{\langle df_p, f(p)-q\rangle}{\|f(p)-q\|^3}\,[f(p)-q] \tag{18.16}$$

has image a subspace of that orthogonal complement. Note that the second term of (18.16) is a multiple of ξ. Hence the image of (18.16) is the entire orthogonal complement—i.e., p is a regular point of w_q—if and only if the image of df_p spans a complement to the span of ξ, i.e., $f \pitchfork_p Z$. Therefore ξ is a regular value of w_q iff $f \pitchfork Z$, in which case $w_q^{-1}(\xi) = f^{-1}(Z)$. Take cardinalities of both sides of this equation to deduce (18.15). ☐

It is true that $f \pitchfork Z_q(\xi)$ for $\xi \in S(V)$ outside a set of measure zero, though we do not provide a proof here. (One proof relies on modifying Theorem 15.18 to allow the submanifold $Z \subset Y$ to vary with the parameter in S.) However, if f is an embedding then we can exchange the roles of f and the ray.

Proposition 18.17. *Suppose given the data (18.2) and assume that f is an embedding. Fix $q \in A\backslash f(X)$. Then for generic $\xi \in V^{\neq 0}$ we have $f(X) \pitchfork Z_q(\xi)$.*

Here, as always, 'generic' means 'outside a set of measure zero'.

Proof. The map

$$\begin{aligned} V^{\neq 0} \times (0,\infty) &\longrightarrow A\backslash\{q\} \\ (\xi, t) &\longmapsto q + t\xi \end{aligned} \tag{18.18}$$

is a submersion—the first partial differential is a homothety, hence is an isomorphism—and therefore is transverse to $f(X) \subset A\backslash\{q\}$. Now apply Theorem 15.18. ☐

The Jordan-Brouwer Separation Theorem

The setup for this section is (18.2) with $f = i_X : X \to A$ the inclusion of a submanifold.

Theorem 18.19. *Given the data (18.2), assume $X \subset A$ is a compact connected hypersurface. Then $A\backslash X$ has two components D_0, D_1; exactly one component, say D_1, is bounded. The closure $\overline{D_1}$ is a compact manifold with boundary X. For $q \in A\backslash X$, we have $q \in D_j$ iff $W_2(i_X, q) = j$, $j \in \mathbb{Z}/2\mathbb{Z}$.*

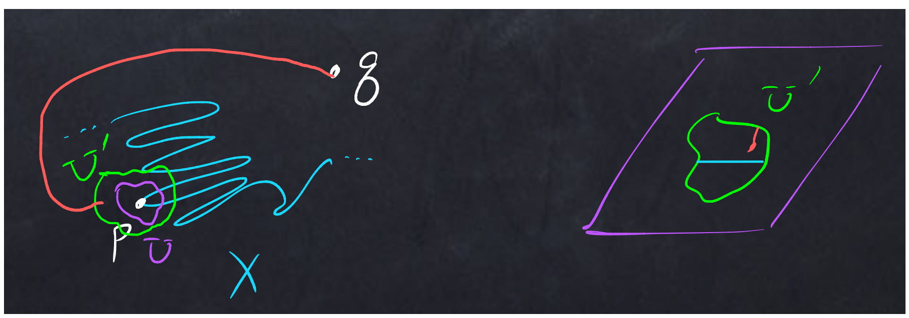


FIGURE 63. Joining $q \in A\backslash X$ to a neighborhood of $p \in X$

Proof. Fix $q \in A\backslash X$. Let $X' \subset X$ be the subset of $p \in X$ such that for any open neighborhood $U \subset A$ of p there exists a smooth motion in $A\backslash X$ which begins at q and terminates at a point of U. We claim X' is nonempty, for if $p_0 \in X$ minimizes the positive distance function $p \mapsto \|p - q\|$ on the compact manifold X, then the constant velocity motion from q to p enters any neighborhood U of p before intersecting X. Also, X' is closed: if $p_n \to p$ is a convergent sequence in X with $p_n \in X'$, and U is an open neighborhood of p, then U is an open neighborhood of p_n for n sufficiently large. Finally, X' is open. To prove this, let $p \in X'$ and choose a submanifold chart on a connected open neighborhood U' of p such that $U' \cap X$ is also connected, as in Figure 63. Fix a motion γ in $A\backslash X$ from q to U'. If $p' \in U' \cap X$, and $U \subset A$ is any open neighborhood of p', then $U \cap U'$ intersects both components of $U' \setminus X \cap U'$. Hence we can extend the motion γ to terminate in $U \cap U'$. Thus $p' \in X'$, and so $U' \cap X$ is an open neighborhood of p in X', which proves that X' is open. Since X is connected, the nonempty open and closed subset $X' \subset X$ is equal to X. It follows that $A\backslash X$ has at most two components.

We prove that $A\backslash X$ has exactly two (path) components. Fix $q_0 \in A\backslash X$ and by Proposition 18.17 choose a ray $Z_{q_0}(\xi)$ such that $Z_{q_0}(\xi) \cap X \neq \emptyset$ and $Z_{q_0}(\xi) \pitchfork X$. Let $q_1 \in Z_{q_0}(\xi)\backslash X$ be a point on

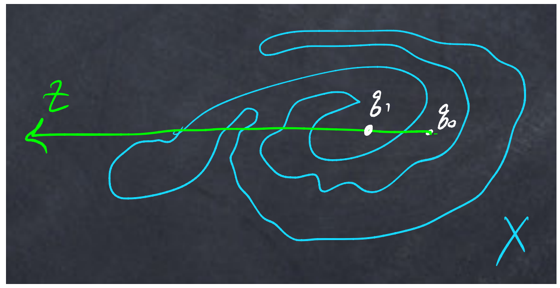


FIGURE 64. Mod 2 winding number as an invariant of $\pi_0(A\backslash X)$

the ray past the first intersection point with X but not past any further intersection points, as in Figure 64. Apply Theorem 18.14 to $Z_{q_0}(\xi)$ and $Z_{q_1}(\xi)$ to conclude that $W_2(i_X,q_0) \neq W_2(i_X,q_1)$. Then the first statement in Proposition 18.8 implies that q_0 and q_1 lie in different path components of $A\backslash X$. Therefore, we conclude that $\#\pi_0(A\backslash X) = 2$, and furthermore the path components are

$$D_j = \{q \in A\backslash X : W_2(i_X,q) = j\}, \qquad j \in \mathbb{Z}/2\mathbb{Z}. \tag{18.20}$$

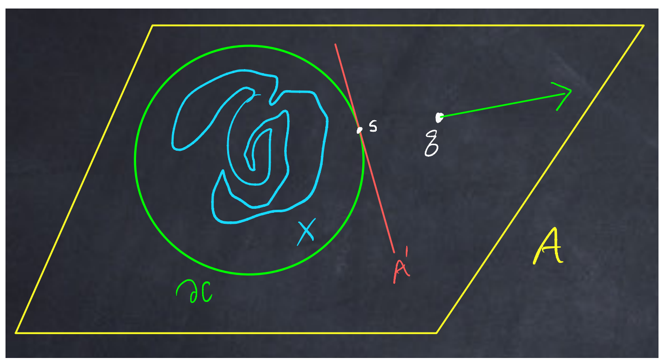


FIGURE 65. The unbounded component is D_0

Since X is compact, there is a closed disk $C \subset A$ such that $X \subset C$, as in Figure 65. Let $q \in A\backslash C$, choose $s \in \partial C$ which minimizes the distance from q to ∂C, and let $A' = s + T_s\partial C$ be the affine hyperplane tangent to ∂C at s. Then X lies in the half space with boundary A' which does not contain q, and so $Z_q(\xi) \cap X = \emptyset$, where $\xi = q - s$. Therefore, $q \in D_0$. Hence $D_1 \subset C$ is bounded and D_0 is unbounded.

Now $\overline{D_1} = D_1 \amalg X$ is compact, and $D_1 \subset A$ is an open submanifold. To prove that $\overline{D_1}$ is a manifold with boundary, for $p \in X$ we must produce a boundary chart. We do so using a submanifold chart for X at p; see Figure 63. □

(18.21) *An application to a nonembedding theorem.* Recall our discussion in Example 11.8 of the minimal embedding dimension of real projective spaces. Here we apply Theorem 18.19 to prove that $\mathbb{RP}^2$ does not embed in $\mathbb{A}^3$. However, we do not quite have[20] all the tools to carry out the proof, so beware that the last step requires more technical foundations.

Recall Definition 13.34 of an orientation of a real line.

[20]We need to integrate a vector field as in Theorem 14.32, now simultaneously for a manifold Z of initial conditions.

Proposition 18.22. *Given the data (18.2), assume $X \subset A$ is a compact connected hypersurface. Then the normal bundle $\nu \to X$ is orientable.*

In fact, the normal bundle carries a canonical orientation, which we choose in the proof.

Proof. Let $p \in X$; then the normal space is $\nu_p = V/T_pX$. A vector $\xi \in V \backslash T_pX$ has a nonzero image in the quotient space ν, and for all sufficiently small $t > 0$ we have either $p + t\xi \in D_0$ or $p + t\xi \in D_1$. (Argue as in the proof of Theorem 18.19 based on Figure 63.) We say $[\xi] \in \nu_p \backslash \{0\}$ is positively oriented if $p + t\xi \in D_0$ for all sufficiently small $t > 0$. □

Now suppose there exists an embedding $\mathbb{RP}^2 \hookrightarrow \mathbb{A}^3$. Let $Z = \mathbb{RP}^1 \subset \mathbb{RP}^2$. Then at $p \in Z$ we have the *full flag*

$$0 \subset T_pZ \subset T_p\mathbb{RP}^2 \subset \mathbb{R}^3 \tag{18.23}$$

in the vector space $\mathbb{R}^3$. Since $\mathbb{RP}^1 \approx S^1$ we can consistently orient T_pZ for all $p \in Z$. By Proposition 18.22 the quotient $\mathbb{R}^3/T_p\mathbb{RP}^2$ is oriented. Choose $e_1 \in T_pZ$ be positively oriented and choose $e_3 \in \mathbb{R}^3 \backslash T_p\mathbb{RP}^2$ so that the image of e_3 in the normal line is positively oriented. Relying on your knowledge of an orientation of $\mathbb{R}^3$ (the "right hand rule"), choose $e_2 \in T_p\mathbb{RP}^2 \backslash T_pZ$ so that e_1, e_2, e_3 is a positively oriented basis of $\mathbb{R}^3$. Rigidify these constructions: choose e_1 to have unit length, e_3 to be a unit vector orthogonal to $T_p\mathbb{RP}^2$, and e_2 to complete to a normal basis. Then e_2 is a nonzero normal vector field to $Z \subset \mathbb{RP}^2$. Use it to "push" Z off of itself to a parallel circle in $\mathbb{RP}^2$. Then $\#_2(Z, Z) = 0$ in $\mathbb{RP}^2$ since by a homotopy we have deformed i_Z to have image disjoint from Z. This contradicts Example 17.24.

The Borsuk-Ulam Theorem

For the standard n-sphere (3.4), the *antipodal map* $\alpha\colon S^n \to S^n$ is the map $x \mapsto -x$. A map $g\colon S^n \to V$ into a vector space V is *odd* if $g\big(\alpha(p)\big) = -g(p)$ for all $p \in S^n$.

Theorem 18.24. *Fix a positive integer n. Let W be a real vector space of dimension n and V a real vector space of dimension $n + 1$.*

1. *If $f\colon S^n \to W$ is a smooth map, then there exists $p \in S^n$ such that $f\big(\alpha(p)\big) = f(p)$.*
2. *If $g\colon S^n \to W$ is an odd map, then there exists $p \in S^n$ such that $g(p) = 0$.*
3. *If $h\colon S^n \to V$ is an odd map, and $0 \notin h(S^n)$, then $W_2(h, 0) = 1$.*

Proof. If (3) is true, and $g\colon S^n \to W$ is odd, set $V = W \oplus \mathbb{R}$ and define

$$\begin{aligned} h\colon S^n &\longrightarrow W \oplus \mathbb{R} \\ p &\longmapsto g(p) \oplus 0 \end{aligned} \tag{18.25}$$

Then h is odd. If g never vanishes, then $h(S^n) \cap \big(\{0\} \oplus \mathbb{R}\big) = \emptyset$, from which Theorem 18.14 implies $W_2(h, 0) = 0$ in contradiction to (3).

If (2) is true, and $f\colon S^n \to W$ is given, set $g(p) = f\big(\alpha(p)\big) - f(p)$ and then deduce (1) from (2).

We prove (3). Fix an inner product on V. Set

$$\varphi = \frac{h}{\|h\|}\colon S^n \longrightarrow S(V) \approx S^n; \tag{18.26}$$

then

$$\varphi\bigl(\alpha(p)\bigr) = \alpha\bigl(\varphi(p)\bigr), \qquad p \in S^n, \tag{18.27}$$

and by (18.6) we have $W_2(h,0) = \deg_2 \varphi$. We proceed by induction on n.

For $n = 1$ write $\varphi\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ and use covering space theory to lift to a smooth function $\widetilde{\varphi}\colon \mathbb{R} \to \mathbb{R}$ which satisfies $\widetilde{\varphi}(x+1) = \widetilde{\varphi}(x) + d$ for some $d \in \mathbb{Z}$; then $\deg_2 \varphi = d \pmod 2$. But by (18.27) we also have $\widetilde{\varphi}(x+1/2) = \widetilde{\varphi}(x) + 1/2 + e$ for some $e \in \mathbb{Z}$. Iterating we deduce $d = 2e+1$.

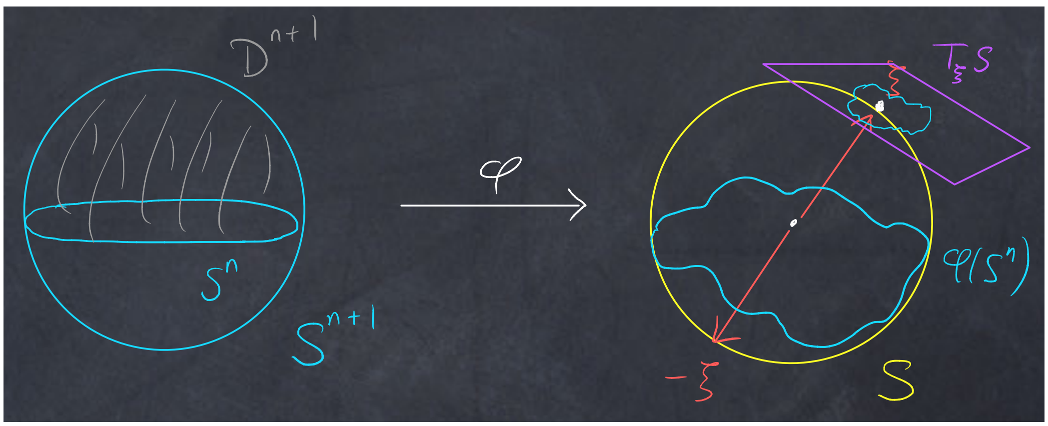


FIGURE 66. The inductive step

It remains to prove the inductive step. Suppose $\varphi\colon S^{n+1} \to S = S(V)$ satisfies (18.27), and assume the theorem holds in dimensions $\leqslant n$. Let $S^n \subset S^{n+1}$ be an equitorial n-sphere. Choose $\xi \in S$ such that ξ is a regular value of φ and $\xi \notin \varphi(S^n)$. Then $\deg_2 \varphi = \#\varphi^{-1}(\xi) \pmod 2$. Compose $\psi = \varphi\big|_{S^n}$ with orthogonal projection $\pi\colon S \subset V \to T_\xi S$; then ψ is odd and $0 \notin \psi(S^n)$. The restriction of $\pi \circ \varphi$ to the upper hemisphere D^{n+1} is an extension of ψ, so Theorem 18.10 implies

$$W_2(\psi,0) = \#\Bigl(\pi\circ\varphi\big|_{D^{n+1}}\Bigr)^{-1}(0) = \#\Bigl(\varphi\big|_{D^{n+1}}\Bigr)^{-1}(\xi) + \#\Bigl(\varphi\big|_{D^{n+1}}\Bigr)^{-1}(-\xi) = \deg_2\varphi. \tag{18.28}$$

At the last stage we use (18.27). By the inductive hypothesis we have $W_2(\psi,0) = 1$, and now $W_2(h,0) = \deg_2\varphi = 1$ follows from (18.28). □

Lectures 19: Motivation for differential forms; universal properties

We begin with a heuristic motivation for the calculus of differential forms and for exterior algebra. Then, as a first step toward the construction of the exterior algebra, we introduce the idea of a *universal property* that characterizes an algebraic construction. Here we illustrate with the free vector space on a set. In the next lecture we use the same fundamental idea to construct the tensor algebra of a vector space and a quotient, the exterior algebra.

Differential forms: motivation

(19.1) *The differential of a real-valued function.* Let M be a smooth manifold. Introduce the notation

$$\begin{aligned} \Omega^0(M) &= \{f\colon M \to \mathbb{R}\} \\ \Omega^1(M) &= \{\alpha\colon M \to T^*M : \alpha \text{ is a section of } T^*M \to M\} \end{aligned} \tag{19.2}$$

In both cases we implicitly require smoothness. Each of $\Omega^0(M)$ and $\Omega^1(M)$ is a real vector space. If M has a component of positive dimension, then each is infinite dimensional.

Definition 19.3. A *differential 1-form* on M, or simply a *1-form*, is an element of $\Omega^1(M)$.

If $\alpha \in \Omega^1(M)$ is a 1-form, then $\alpha_p \in T_p^*M = (T_pM)^*$ is a linear functional $T_pM \to \mathbb{R}$ for all $p \in M$.

Remark 19.4. Let V be a linear space, A an affine space over V, $U \subset A$ an open set. Then the space of 1-forms on U is

$$\Omega^1(U) = \{\alpha\colon U \to V^*\} \tag{19.5}$$

The differential is a linear map

$$\Omega^0(M) \xrightarrow{\;d\;} \Omega^1(M) \tag{19.6}$$

Recall that the differential $df \in \Omega^1(M)$ of $f \in \Omega^0(M)$ is defined in terms of the directional derivative:

$$df_p(\xi) = (\xi f)(p), \qquad p \in M, \quad \xi \in T_pM. \tag{19.7}$$

Remark 19.8. The notation suggests that there are vector spaces $\Omega^k(M)$ for all $k \in \mathbb{Z}^{\geqslant 0}$ and, perhaps, that there is an extension of the differential to a linear map

$$\Omega^k(M) \xrightarrow{\;d\;} \Omega^{k+1}(M) \tag{19.9}$$

for each k. Both are true, as we shall see presently. Our task now is to motivate such an extension.

Example 19.10. Let $A = \mathbb{A}^n$ be affine space with standard coordinates $x^1, \dots, x^n$. Then any 1-form on an open set $U \subset \mathbb{A}^n$ can be written

$$\alpha = \alpha_i(x^1, \dots, x^n)\, dx^i \tag{19.11}$$

for some smooth functions $\alpha_i \in \Omega^0(U)$, $i = 1, \dots, n$. If $f \in \Omega^0(U)$, then

$$df = \frac{\partial f}{\partial x^i}\, dx^i. \tag{19.12}$$

Note that $f, x^1, \dots, x^n$ are functions on U, so their differentials (19.7) are defined. One can view (19.12) as the definition of the partial derivatives, or alternatively check that it follows from a definition of the partial derivatives that you already know.

(19.13) *Prescribing the differential.* In the general situation of **(19.1)** we ask the following: given $\alpha \in \Omega^1(M)$ does there exist $f \in \Omega^0(M)$ such that

$$df = \alpha? \tag{19.14}$$

In other words, can we prescribe the differential of a function arbitrarily? The uniqueness aspect of (19.14) is straightforward. Since this is an affine equation, the difference $g = f_1 - f_0$ between two solutions satisfies the linear equation $dg = 0$, i.e., g is locally constant. Introduce the space of locally constant functions

$$H^0_{\mathrm{dR}}(M) = \{g \in \Omega^0(M) : dg = 0\}. \tag{19.15}$$

Then the the space of solutions to (19.14), *if nonempty*, is an affine space over $H^0_{\mathrm{dR}}(M)$. We remark that (19.14) is a first-order linear *partial* differential equation if $\dim M > 1$. The word 'partial' refers to the partial derivatives which appear when we write (19.14) in coordinates, as in (19.12).

Remark 19.16. We can already observe that the existence and uniqueness theory of (19.14) is tied to the topology of M. For example, if M is an interval in $\mathbb{R}$, then there is a 1-dimensional vector space of locally constant functions, whereas if M is the union of ℓ disjoint open intervals then the space of locally constant functions has dimension ℓ. In other words, (19.15) detects the *connectivity* of M. Notice that solutions to (19.14) exist on open subsets of $\mathbb{R}$: if $M = (a, b) \subset \mathbb{R}$ with coordinate x, and we write $\alpha = g(x)\, dx$, then (19.14) reduces to the equation $f'(x) = g(x)$ which has the solution

$$f(x) = \int_{x_0}^{x} g(t)\, dt \ + C \tag{19.17}$$

for any $x_0 \in (a, b)$ and $C \in \mathbb{R}$.

Remark 19.18. The notation in (19.15) suggests the existence of real vector spaces $H^k_{\mathrm{dR}}(M)$ for all $k \in \mathbb{Z}^{\geqslant 0}$. These are French mathematician Georges de Rham's eponymous vector spaces: *de Rham cohomology*.

(19.19) *A necessary condition.* For concreteness and ease of notation specialize to $M = U \subset \mathbb{A}^2$ an open subset with standard coordinates x, y. Write

$$\alpha = P(x,y)dx + Q(x,y)dy \tag{19.20}$$

for functions $P, Q\colon U \to \mathbb{R}$. Then (19.14) is equivalent to the system of equations

$$\begin{aligned} \frac{\partial f}{\partial x} &= P \\ \frac{\partial f}{\partial y} &= Q \end{aligned} \tag{19.21}$$

We can immediately see an obstruction to existence. For if f is a C^2 function, then by the equality of mixed partials we have

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial P}{\partial y}. \tag{19.22}$$

Equation (19.22) is a necessary condition for existence. It is not satisfied by every 1-form, for example not by $\alpha = xdy - ydx$.

(19.23) *The 2-form obstruction.* Since the necessary condition (19.22) involves first derivatives of the coefficients of α, we are motivated to express it directly in terms of a derivative of α. That is precisely what we contemplated in (19.9), but as of yet we have not defined that operator. Nonetheless, assuming the most basic properties of a first order differential operator—that d be linear and obey a Leibniz rule—we compute from (19.20) by simply juxtaposing 1-forms to indicate some as-of-yet-not-defined multiplication:

$$\begin{aligned} d\alpha &= d(Pdx + Qdy) \\ &= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy\right)dx + P\,d^2x \\ &\qquad \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy\right)dy + Q\,d^2y \end{aligned} \tag{19.24}$$

As desired, we see the relevant derivatives $\partial P/\partial y$ and $\partial Q/\partial x$ appearing, but there are 4 extraneous terms. They will be set to zero if we stipulate the following rules:

$$df \wedge dg = -dg \wedge df \tag{19.25}$$

$$d^2 f = 0 \tag{19.26}$$

for all functions f, g, in particular for the coordinate functions x and y. Here, in view of the odd commutativity rule (19.25) we change notation and write '$\wedge$' for the product of 1-forms. With this understood, (19.24) reduces to

$$d\alpha = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy. \tag{19.27}$$

The necessary condition (19.22) for a solution to (19.21) is now the equation

$$d\alpha = 0. \tag{19.28}$$

Said differently, $d\alpha$ is an obstruction to solving the equation (19.14).

Remark 19.29. Quite generally, the necessary condition (19.28) for a solution to (19.14) follows immediately by applying d and using the rule (19.26).

(19.30) *The road ahead.* This discussion reinforces our desire to define $\Omega^2(M)$ and the extension (19.9) of d, which then gives a sequence of linear maps

$$\Omega^0(M) \xrightarrow{\;d\;} \Omega^1(M) \xrightarrow{\;d\;} \Omega^2(M) \tag{19.31}$$

such that the composition is zero. Furthermore, in view of (19.5) for the case of an open subset U of affine space, we might anticipate constructing a new vector space, ${\textstyle\bigwedge}^2 V^*$ so that

$$\Omega^2(U) = \{\omega \colon U \to {\textstyle\bigwedge}^2 V^*\}. \tag{19.32}$$

We also have to incorporate the *wedge product* into our theory. It should be a pointwise operation on 1-forms, more precisely a bilinear map

$$\wedge \colon V^* \times V^* \longrightarrow {\textstyle\bigwedge}^2 V^*. \tag{19.33}$$

Anticipating higher degrees, we will construct the *exterior algebra* ${\textstyle\bigwedge} V^*$ whose multiplication $\wedge$ satisfies the odd commutativity rule (19.25).

Exterior algebra: motivation

We turn now to some motivation for the algebra we will develop.

(19.34) *Alternating and skew-symmetric bilinear forms.* The bilinear form (19.33) satisfies a special symmetry property.

Definition 19.35. Let W, U be vector spaces and suppose $B\colon W \times W \to U$ is a bilinear form.

(1) B is *alternating* if $B(\xi, \xi) = 0$ for all $\xi \in W$.
(2) B is *skew-symmetric* if $B(\xi_1, \xi_2) = -B(\xi_2, \xi_1)$ for all $\xi_1, \xi_2 \in W$.

If B is alternating, then it is skew-symmetric: expand $B(\xi_1 + \xi_2, \xi_1 + \xi_2) = 0$. The converse is true over any field not of characteristic two, in particular over the real numbers.

Remark 19.36. The definitions extend to a k-linear form $B\colon W \times W \times \cdots \times W \to U$ for all $k \in \mathbb{Z}^{\geqslant 2}$. The form is alternating if it vanishes whenever two inputs agree, and it is skew-symmetric if its value changes sign upon swapping two inputs.

The wedge product (19.33) is alternating, so let us write $B(\xi_1, \xi_2) = \xi_1 \wedge \xi_2$.

(19.37) *Linear dependence and the span.* Observe first that if $\xi_1, \xi_2 \in W$ are linearly dependent, then $\xi_1 \wedge \xi_2 = 0$. For if $\lambda^i \xi_i = 0$ for some $\lambda^1, \lambda^2 \in \mathbb{R}$, then assuming $\lambda^1 \neq 0$,

$$0 = (\lambda^1 \xi_1 + \lambda^2 \xi_2) \wedge \xi_2 = \lambda^1 \xi_1 \wedge \xi_2. \tag{19.38}$$

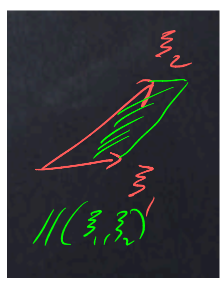


FIGURE 67. The parallelogram spanned by two vectors

Now suppose ξ_1, ξ_2 are linearly independent, and let $W' \subset W$ be their span. Also, define the *parallelogram* they span as

$$/\!/(\xi_1, \xi_2) = \{t^1 \xi_1 + t^2 \xi_2 : 0 \leqslant t^1, t^2 \leqslant 1\} \subset W'. \tag{19.39}$$

Suppose η_1, η_2 is another ordered basis of W', and write the change of ordered basis as $\xi_j = A^i_j \eta_i$ for some matrix $A = \left(A^i_j\right)$. Expanding out we have

$$\begin{aligned} \xi_1 &= A^1_1 \eta_1 + A^2_1 \eta_2 \\ \xi_2 &= A^1_2 \eta_1 + A^2_2 \eta_2 \end{aligned} \tag{19.40}$$

Now use the alternating property to compute

$$\begin{aligned}\xi_1 \wedge \xi_2 &= (A^1_1\eta_1 + A^2_1\eta_2) \wedge (A^1_2\eta_1 + A^2_2\eta_2) \\ &= (A^1_1A^2_2 - A^2_1A^1_2)\,\eta_1 \wedge \eta_2 \\ &= \det(A)\,\eta_1 \wedge \eta_2\end{aligned} \tag{19.41}$$

We draw several conclusions from (19.41).

(1) Although the vector space W' does *not* come with a notion of *area*, it does have an intrinsic notion of *ratios* of areas. Namely, for any "reasonable" notion of area, the ratio of the area of $/\!\!/(\xi_1, \xi_2)$ and $/\!\!/(\eta_1, \eta_2)$ is $|\det(A)|$.

(2) Although the vector space W' does *not* come with a notion of *orientation*, it does have an intrinsic notion of *ratios* of orientations. Namely, for any "reasonable" notion of orientation, the ratio of the orientation given by the ordered basis ξ_1, ξ_2 and the orientation given by the ordered basis η_1, η_2 is $\operatorname{sign}\bigl(\det(A)\bigr) \in \mu_2 = \{\pm 1\}$.

(3) Combine (1) and (2) into a ratio of *signed* areas.

(4) We are motivated, then, to view the wedge product $\xi_1 \wedge \xi_2$ as representing the parallelogram $/\!\!/(\xi_1, \xi_2)$. This representation identifies parallelograms in the span $W' \subset W$ of ξ_1, ξ_2 with the same area and orientation, i.e., the same signed area. (Again: while signed area is not defined, the ratio of signed areas is.)

(5) Equation (19.41) implies that the line spanned by $\xi_1 \wedge \xi_2$ is independent of the choice of basis ξ_1, ξ_2 of $W' \subset W$. This leads to the *Plucker embedding* of the Grassmannian into the projective space of the exterior square:

$$\begin{aligned}\mathrm{Gr}_2(W) &\longrightarrow \mathbb{P}(\textstyle\bigwedge^2 W) \\ W' &\longmapsto \mathbb{R}\cdot(\xi_1 \wedge \xi_2)\end{aligned} \tag{19.42}$$

where ξ_1, ξ_2 is any basis of W'.

Characterization by a universal property

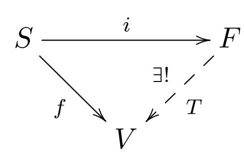
(19.43) *Introduction.* One of the triumphs of mathematics is its conceptualization of structure. (This is only one of many triumphs!) Consider, for example, the definition/theorem about the real numbers: $\mathbb{R}$ is the complete ordered field. The three words 'complete', 'ordered', and 'field' are historically hard-won. Infinities had to be tamed in order to arrive at the first of these, and essential structures had to be extracted to formulate the latter two. Imagine a different, unlikely version of history in which these concepts are known, but the real numbers do not yet exist. Then we can wish them into existence by defining $\mathbb{R}$ as the complete ordered field. We could go on to prove all the properties we need from these three words, and in practice this is what one does in a first course in analysis. But one must prove existence by exhibiting a construction. By contrast, uniqueness, indicated by that innocuous looking 'the' which precedes the three magic words, is automatic. But what uniqueness? After all, there are two common constructions of the real numbers: Dedekind cuts and limits of Cauchy sequences. These construct unequal sets R_1, R_2 that we call $\mathbb{R}$. The uniqueness of which we speak is the strongest possible uniqueness for sets: there are unique inverse maps $R_1 \leftrightarrows R_2$ which preserve the complete ordered field structures.[21]

(19.44) *The vector space generated by a set.* As a warmup to illustrate characterization by a universal property, suppose S is a set. We want to find a vector space F "generated" by S. Intuitively it should contain S and all elements and relations forced by the structure of a vector space, but should not contain more. So, for example, F has a zero vector, and it contains all finite linear combinations of elements of S. We could try to *construct* F along these lines, but instead we *encode* our specifications as follows.

Definition 19.45. Let S be a set. A pair (F, i) consisting of a vector space F and a map of sets $i\colon S \to F$ is a *free vector space generated by* S if for every (W, f) consisting of a vector space W and a map of sets $f\colon S \to W$ there exists a unique linear map $T\colon F \to W$ such that $f = T \circ i$.

The definition is summarized by the diagram

(19.46)



The dashed line indicates that T is output whereas i and f are inputs. The symbol $\exists!$ indicates the existence of a *unique* map T. The two ways of traveling from S to V are assumed equal—the diagram commutes—which is the condition $f = T \circ i$. Notice that in the definition we use the article 'a' in front of 'free vector space generated by S', we do not use the article 'the'; uniqueness is a theorem (Theorem 19.52 below). Also, we do not assume that i is injective; that is also a theorem (an immediate corollary of Theorem 19.58 below). Intuitively, the existence of T ensures that F is big enough and the uniqueness of T ensures that F is not too big. The word 'free' evokes this middle ground. We say that (F, i) is *universal* among all pairs (W, f), and Definition 19.45 spells out the precise universal property.

The power of the universal property is illustrated by the theorems which follow.

Example 19.47. If $S = \emptyset$ is the empty set, then the only choice for F is the zero vector space.

Example 19.48. If $S = \{s\}$ is a singleton, then we can exhibit many free vector spaces generated by S. For example, for $k \in \mathbb{Z}$ define $F = \mathbb{R}$ and define $i_k\colon S \to \mathbb{R}$ by $i_k(s) = k$. The factorization problem (19.46) is solved uniquely by the linear map

$$T_k(x) = \frac{x}{k} f(s), \qquad x \in \mathbb{R}, \tag{19.49}$$

so long as $k \neq 0$. Hence $(\mathbb{R}, i_k)$ is a free vector space generated by S for all $k \in \mathbb{Z}^{\neq 0}$. You can verify that $(\mathbb{R}, i_0)$ does not satisfy the universal property.

[21] 0 and 1 are preserved, from which it follows that the map matches the copies of $\mathbb{Q}$ sitting in R_1 and R_2. The requirement that the maps preserve completeness, say in the form of least upper bounds, determines the rest.

Example 19.50. Similarly, if $S = \{s_1, \dots, s_n\}$ is a finite set, define $F = \mathbb{R}^n$ and $i\colon S \to \mathbb{R}^n$ by $i(s_j) = e_j$, where $e_1, \dots, e_n$ is the standard ordered basis of $\mathbb{R}^n$. The factorization is solved uniquely by

$$T(\xi^1, \dots, \xi^n) = \xi^j f(s_j). \tag{19.51}$$

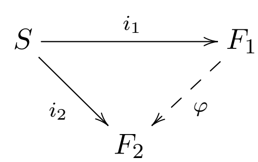
Theorem 19.52. *Let (F_1, i_1), (F_2, i_2) be free vector spaces generated by a set S. Then there is a unique linear isomorphism $\varphi\colon F_1 \to F_2$ such that $i_2 = \varphi \circ i_1$.*

Call a map $\varphi\colon F_1 \to F_2$ which satisfies $i_2 = \varphi \circ i_1$ a *morphism* from (F_1, i_1) to (F_2, i_2). Then Theorem 19.52 asserts that any two solutions to the universal problem (19.46) are *unique up to unique isomorphism*. This is the strongest form of uniqueness for a problem whose answer is a set.[22] In this circumstance we speak of *the* free vector space generated by S and introduce a special notation $F(S)$; the map i is implicit.

Proof. Apply (19.46) four times:

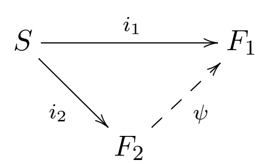
(i) Use existence in the universal property for (F_1, i_1) to construct $\varphi\colon F_1 \to F_2$:

(19.53)



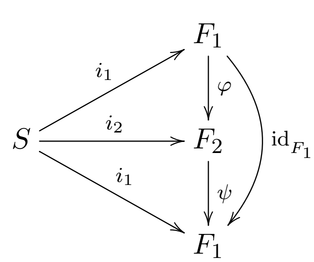
(ii) Use existence in the universal property for (F_1, i_1) to construct $\psi\colon F_2 \to F_1$:

(19.54)



(iii) Use uniqueness in the universal property for (F_1, i_1) to prove $\psi \circ \varphi = \mathrm{id}_{F_1}$:

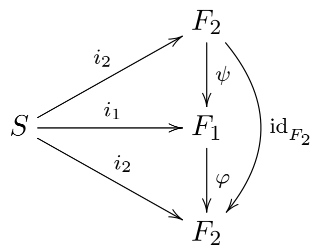
(19.55)



[22] The pairs (F, i) which solve the universal problem (19.46) are the objects of a category in which there is a unique morphism between any two objects. Such a category is termed 'contractible'. This is the technical meaning of 'unique up to unique isomorphism'.

(iv) Use uniqueness in the universal property for (F_2, i_2) to prove $\varphi \circ \psi = \mathrm{id}_{F_2}$:

(19.56)



Hence φ and ψ are inverse isomorphisms. □

A *basis* of a vector space F is a subset $B \subset F$ such that every $\eta \in F$ can be written uniquely as a linear combination

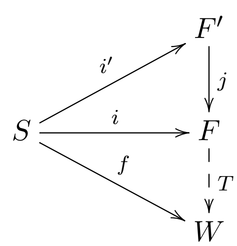
$$\eta = c^1\xi_1 + \cdots + c^n\xi_n \tag{19.57}$$

for a finite subset $\{\xi_1, \ldots, \xi_n\} \subset B$ and scalars $c^1, \ldots, c^n$.

Theorem 19.58. *Let (F, i) be the free vector space generated by a set S. Then $i(S) \subset F$ is a basis.*

Proof. Let $F' \subset F$ be the *span* of $i(S)$ and $i' \colon S \to F'$ the inclusion. Then (F', i') satisfies the universal property, as follows from the existence and uniqueness of T in the diagram

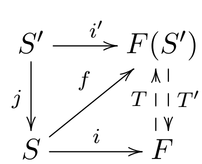
(19.59)



Then the uniqueness Theorem 19.52 implies that the inclusion j is an isomorphism, so $F' = F$. This proves the existence of (19.57) for each η.

Next, we claim that the image of the restriction of i to every finite subset $S' \subset S$ is a linearly independent set in F; this is equivalent to the uniqueness of (19.57) for each η. Suppose $S' \subset S$ has cardinality n. Example 19.50 shows that the map $i' \colon S' \to F(S')$ is injective. Use the universal property of $(F(S'), i')$ and (F, i) to construct T' and T in the diagram

(19.60)



where

$$f(s) = \begin{cases} i'(s'), & s' \in S'; \\ 0, & s \notin S'. \end{cases} \tag{19.61}$$

The diagram commutes. Now an argument similar to (3) in the proof of Theorem 19.52 shows $T \circ T' = \mathrm{id}_{F(S')}$. In particular, T' is injective, from which $T' \circ i' = i \circ j$ is injective, which is the claim. ☐

(19.62) *Existence.* We have still not proved existence of a free vector space generated by an infinite set. I leave that to the problem set.

Lecture 20: Tensor and exterior algebras

(20.1) *The base field.* Throughout this lecture the base field can be arbitrary, though our applications of this algebra in this class only use vector spaces over the real numbers. A few cautions are necessary. When a field has characteristic 2, there is a difference between skew-symmetric and alternating maps; see **(20.43)**. Also, we define the $\mathbb{Z}$-grading on the tensor and exterior algebras using the action by invertible scalars. Over the reals or complexes that argument works directly; a modification (Remark 20.35) works in general.

We begin with the tensor product of two or more vector spaces, which is a preliminary to the tensor and exterior algebras.

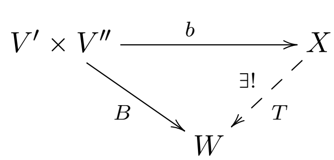
Tensor products of vector spaces

The tensor product is the codomain for the universal bilinear map out of a Cartesian product of vector spaces. We characterize it by a universal property.

Definition 20.2. Let V', V'' be vector spaces. A *tensor product* (X, b) is a vector space X and a bilinear map $b\colon V' \times V'' \to X$ such that for all pairs (W, B) of a vector space W and a bilinear map $B\colon V' \times V'' \to W$, there exists a unique linear map $T\colon X \to W$ such that $B = T \circ b$.

The definition is encoded in the commutative diagram

(20.3)



The argument in Theorem 19.52 proves uniqueness of the tensor product up to unique isomorphism.

Theorem 20.4. *There exists a tensor product* (X, b) *of vector spaces* V' *and* V''*.*

Proof. Let $F(V' \times V'')$ be the free vector space generated by $V' \times V''$. Let $R(V' \times V'')$ be the subspace generated by vectors

$$(c^1\xi'_1 + c^2\xi'_2, \xi'') - c^1(\xi'_1, \xi'') - c^2(\xi'_2, \xi''),$$
$$(\xi', c^1\xi''_1 + c^2\xi''_2) - c^1(\xi', \xi''_1) - c^2(\xi', \xi''_2), \tag{20.5}$$

for all choices of $\xi', \xi'_1, \xi'_2 \in V'$, $\xi'', \xi''_1, \xi''_2 \in V''$, and $c^1, c^2 \in \mathbb{R}$. Define the quotient vector space

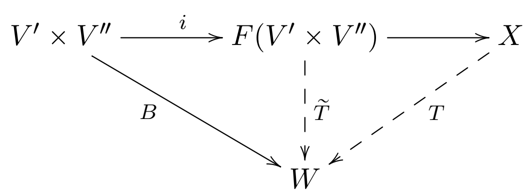
$$X = F(V' \times V'') \big/ R(V' \times V'') \tag{20.6}$$

and the composition

$$b\colon V' \times V'' \xrightarrow{\ i\ } F(V' \times V'') \longrightarrow X, \tag{20.7}$$

where i is the map in Definition 19.45 of the free vector space and the second map is the natural quotient map. (We can define the quotient vector space as satisfying a universal property with respect to that quotient map.) The relations (20.5) imply that b is bilinear. If (W, B) is as in Definition 20.2, then in the diagram

(20.8)



the unique map $\widetilde{T}$ is the one in the universal property of the free vector space, and then the unique map T which completes the right triangle exists because of the universal property of the quotient. ∎

(20.9) *Notation for tensor product.* The standard notation is $X = V' \otimes V''$ and $b(\xi', \xi'') = \xi' \otimes \xi''$ for $\xi' \in V'$ and $\xi'' \in V''$. Since the tensor product is unique up to unique isomorphism, we speak of *the* tensor product.

Theorem 20.10. *Let V', V'' be vector spaces with bases S', S''.*

(1) *Every vector in $V' \otimes V''$ has a unique expression as a finite sum*

$$\sum_i \eta'_i \otimes \xi''_i, \qquad \eta'_i \in V', \quad \xi''_i \in S''. \tag{20.11}$$

(2) *The set*

$$S = \{\xi' \otimes \xi'' : \xi' \in S',\ \xi'' \in S''\} \tag{20.12}$$

is a basis of $V' \otimes V''$.

Proof. Let $X \subset V' \otimes V''$ be the subspace of vectors of the form (20.11). Define the bilinear map $b\colon V' \times V'' \to X$ by $b(\eta', \xi'') = \eta' \otimes \xi''$ for $\eta' \in V'$ and $\xi'' \in S''$. Since S'' is a basis of V'' this suffices to define the bilinear map b. Then (X, b) satisfies the universal property of the tensor product: construct a factorization in (20.3) using $V' \otimes V''$ and then restrict to the subspace X. By the uniqueness of the tensor product, the inclusion map is an isomorphism $X = V' \otimes V''$. This proves existence in (1). If there is not uniqueness, then for some vectors $\eta'_i \in V'$, $\xi''_i \in S''$, $i = 1, \dots, N$, we have

$$\sum_{i=1}^{N} \eta'_i \otimes \xi''_i = 0. \tag{20.13}$$

Let $L\colon V' \to \mathbb{R}$ be a linear functional such that $L(\eta'_1) = 1$. Then

$$\begin{aligned} B\colon V' \times V'' &\longrightarrow V'' \\ \eta' \ , \ \eta'' &\longmapsto L(\eta')\eta'' \end{aligned} \tag{20.14}$$

is a bilinear map which sends equation (20.13) to the nontrivial linear relation $\sum L(\eta'_i)\xi''_i = 0$ among basis elements in S'', which is absurd. This completes the proof of (1).

Assertion (2) is an immediate corollary: expand each $\eta'_i \in V'$ in (20.11) in the basis S' to write any vector in $V' \otimes V''$ uniquely as a linear combination of a finite subset of elements of S. ☐

Corollary 20.15. *If V', V'' are finite dimensional vector spaces, then*

$$\dim(V' \otimes V'') = (\dim V')(\dim V''). \tag{20.16}$$

(20.17) *Commutativity and associativity of tensor product.* The tensor product satisfies commutative and associative "laws", but rather than equalities of elements of a set these are isomorphisms between vector spaces. We write them on decomposable vectors for vector spaces V', V'', V''' as

$$\begin{aligned} V' \otimes V'' &\longrightarrow V'' \otimes V' \\ \xi' \otimes \xi'' &\longmapsto \xi'' \otimes \xi' \end{aligned} \tag{20.18}$$

and

$$\begin{aligned} (V' \otimes V'') \otimes V''' &\longrightarrow V' \otimes (V'' \otimes V''') \\ (\xi' \otimes \xi'') \otimes \xi''' &\longmapsto \xi' \otimes (\xi'' \otimes \xi''') \end{aligned} \tag{20.19}$$

These isomorphism satisfy "equations" known as the pentagon and hexagon diagrams, and tell that vector spaces with tensor product form a *symmetric monoidal category*, which is a "higher" version of an abelian group. We will not pursue this idea here, but will implicitly use the associativity. In particular, we use the notation '$V' \otimes V'' \otimes V'''$' for either[23] of the vector spaces in (20.19).

[23]The correct categorical notion is that of a colimit or limit of the map (20.19).

Remark 20.20. One has to work to prove that tensor products of arbitrary finite collections of vector spaces are unambiguously defined, independent of ordering and of putting in parentheses.

Notation 20.21. For a vector space V write

$$\begin{aligned} V^{\otimes 0} &= \otimes^0 V = \mathbb{R} \\ V^{\otimes 1} &= \otimes^1 V = V \\ V^{\otimes 2} &= \otimes^2 V = V \otimes V \\ V^{\otimes 3} &= \otimes^3 V = V \otimes V \otimes V, \\ &\text{etc.} \end{aligned} \tag{20.22}$$

Tensor algebra

(20.23) *Algebras: basic definitions.* A vector space has two operations: vector addition and scalar multiplication. An algebra has another binary operation called multiplication.

Definition 20.24.

(1) An *algebra* is a nonzero vector space A, a bilinear map $m\colon A \times A \to A$, and a nonzero element $1 \in A$ such that multiplication m is associative and 1 is a unit for m. Write $a_1a_2 = m(a_1, a_2)$ for the product of $a_1, a_2 \in A$.
(2) A *homomorphism* $\varphi\colon A \to B$ of algebras is a linear map which preserves units and multiplication: $\varphi(1) = 1$ and $\varphi(a_1a_2) = \varphi(a_1)\varphi(a_2)$ for all $a_1, a_2 \in A$.
(3) A *subalgebra* of an algebra A is a linear subspace $A' \subset A$ which contains 1 and such that $a_1'a_2' \in A'$ for all $a_1', a_2' \in A'$.
(4) A *2-sided ideal* $I \subset A$ is a linear subspace such that $AI = I$ and $IA = I$.
(5) A $\mathbb{Z}$-*grading* of an algebra A is a direct sum decomposition $A = \bigoplus_{k\in\mathbb{Z}} A^k$ such that $A^{k_1}A^{k_2} \subset A^{k_1+k_2}$ for all $k_1, k_2 \in \mathbb{Z}$.
(6) An algebra A is *commutative* if

$$a_1a_2 = a_2a_1, \qquad a_1, a_2 \in A. \tag{20.25}$$

(7) A $\mathbb{Z}$-graded algebra is (*super*)*commutative* if

$$a_1a_2 = (-1)^{k_1k_2}a_2a_1, \qquad a_1 \in A^{k_1}, \quad a_2 \in A^{k_2}. \tag{20.26}$$

(8) If A is a $\mathbb{Z}$-graded algebra and $a \in A^k$, $k \in \mathbb{Z}^{>0}$, then a is *decomposable* if it is expressible as a product $a = a_1 \cdots a_k$ for $a_1, \dots, a_k \in A^1$. If not, we say a is *indecomposable.*

The associative law and identity law are $(a_1a_2)a_3 = a_1(a_2a_3)$ and $1a = a1 = a$ for all $a, a_1, a_2, a_3 \in A$. If $I \subset A$ is a 2-sided ideal, then the quotient vector space A/I inherits a product, since the bilinear map $m\colon A \times A \to A$ factors to a bilinear map $\overline{m}\colon A/I \times A/I \to A/I$. For a $\mathbb{Z}$-graded algebra we often use the notation $A^\bullet$ to signify the grading. A $\mathbb{Z}$-grading is a structure, not a condition. Elements in the summand A^k are called *homogeneous of degree* k. An element which is not homogeneous is called *inhomogeneous*. The sign in (20.26) is referred to as the *Koszul sign rule*. We motivated it in (19.25) in the context of differential forms.

Example 20.27. The polynomial algebra $\mathbb{R}[x]$ is $\mathbb{Z}$-graded but is not commutative as a $\mathbb{Z}$-graded algebra; it is commutative as an ungraded algebra. The algebra of 2×2 matrices is noncommutative.

(20.28) *The tensor algebra.* The tensor algebra is the free algebra generated by a vector space; that is, there are no relations beyond those in Definition 20.24(1).

Definition 20.29. Let V be a vector space. A *tensor algebra* (A, i) *over* V is an algebra A and a linear map $i\colon V \to A$ such that for all pairs (B, T) consisting of an algebra B and a linear map $T\colon V \to B$ there exists a unique algebra homomorphism $\varphi\colon A \to B$ such that $T = \varphi \circ i$.

A commutative diagram encodes the universal property:

(20.30)
$$\begin{array}{ccc} V & \xrightarrow{\quad i \quad} & A \\ & {\scriptstyle T}\searrow \qquad \swarrow{\scriptstyle \exists!\,\varphi} & \\ & B & \end{array}$$

By the usual argument a tensor algebra is unique up to unique isomorphism, if it exists. It does, but before we construct it we deduce consequences of the universal property.

Theorem 20.31. *Let V be a vector space and (A, i) a tensor algebra of V.*

(1) *The linear map i is injective.*

(2) *If V' is another vector space and (A', i') is a tensor algebra of V', then a linear map $T\colon V' \to V$ induces an algebra homomorphism $\otimes T\colon A' \to A$. Furthermore, the tensor algebra homomorphisms for a composition $V'' \xrightarrow{T'} V' \xrightarrow{T} V$ of linear maps satisfy*

(20.32)
$$\otimes(T \circ T') = \otimes T \circ \otimes T'.$$

(3) *There exists a canonical $\mathbb{Z}$-grading on A.*

The meaning of 'canonical' is that if $T\colon V \to W$ is a linear map of vector spaces, then the induced algebra homomorphism $\varphi_T\colon A \to B$ between choices of tensor algebras for V and W, respectively, preserves the $\mathbb{Z}$-gradings in the sense that $\varphi(A^k) \subset B^k$ for all $k \in \mathbb{Z}$.

Proof. Suppose $\xi \in V$ is nonzero and $i(\xi) = 0$. Define the algebra $\mathbb{R} \oplus \mathbb{R}\xi$ to have $\xi^2 = 0$. Choose a linear map $\pi\colon V \to \mathbb{R}\xi$ which is the identity on $\mathbb{R}\xi$. Define φ by the universal property

(20.33)
$$\begin{array}{ccc} V & \xrightarrow{\quad i \quad} & A \\ & {\scriptstyle \pi}\searrow \qquad \swarrow{\scriptstyle \varphi} & \\ & \mathbb{R} \oplus \mathbb{R}\xi & \end{array}$$

Then $0 \neq \xi = \pi(\xi) = \varphi i(\xi) = 0$. This contradiction proves that i is injective.

The algebra homomorphism in (2) is constructed from the diagram

$$\begin{array}{ccc} V' & \xrightarrow{\;i'\;} & A' \\ {\scriptstyle T}\downarrow & \searrow^{i\circ T} & \downarrow{\scriptstyle \otimes T} \\ V & \xrightarrow{\;i\;} & A \end{array} \tag{20.34}$$

using the universal property (20.30). Use the uniqueness in the universal property to prove (20.32).

For (3), if $\lambda \in \mathbb{R}$ is nonzero, let $T_\lambda\colon V \to V$ denote scalar multiplication by λ. The universal property (20.30) implies the existence of a unique extension to $\varphi_\lambda\colon A \to A$, an algebra homomorphism. Suppose $\lambda \neq 1$ and let $A^k \subset A$ denote the eigenspace of φ_λ with eigenvalue λ^k. Then $A' = \bigoplus_{k\in\mathbb{Z}} A^k \subset A$ is a subalgebra: it contains $1 \in A^0$ and is closed under multiplication. It comes to us as a $\mathbb{Z}$-graded algebra. Also, $i(V) \subset A^1$. Then A' with the factored linear map $i\colon V \to A'$ satisfies the universal property: apply the universal property of A and restrict the resulting map to $A' \subset A$. So the inclusion $A' \hookrightarrow A$ is an isomorphism. We leave the reader to prove that the $\mathbb{Z}$-grading is canonical. ☐

Remark 20.35. We can consider all $\lambda \in \mathbb{R}^\times$ at once and then we are decomposing A under a representation of this multiplicative group. The argument fails over a finite field F. In that, or even the general, case we can extend scalars to the ring $F[x, x^{-1}]$ and consider the linear operator multiplication by x to construct the $\mathbb{Z}$-grading on the tensor algebra. (In this argument we work with modules over a ring rather than vector spaces over a field.)

Remark 20.36. The tensor algebra is not commutative in either sense of Definition 20.24(6).

Theorem 20.37. *Let V be a vector space. Then a tensor algebra over V exists.*

Proof. Define

$$A = \bigoplus_{k=0}^{\infty} \otimes^k V, \tag{20.38}$$

where we use the notation introduced in (20.22), and let $i\colon V \to A$ include into the summand with $k = 1$. Apply Theorem 20.10(2), extended to tensor products of k vector spaces, to see that it suffices to define multiplication on decomposable vectors. Set

$$\begin{aligned} (\xi_1 \otimes \cdots \otimes \xi_{k_1})(\eta_1 \otimes \cdots \otimes \eta_{k_2}) &= \xi_1 \otimes \cdots \otimes \xi_{k_1} \otimes \eta_1 \otimes \cdots \otimes \eta_{k_2}, \\ &\qquad \xi_1, \dots, \xi_{k_1}, \eta_1, \dots, \eta_{k_2} \in V. \end{aligned} \tag{20.39}$$

If B is an algebra and $T\colon V \to B$ a linear map, define $\varphi\colon A \to B$ by

$$\varphi(\xi_1 \otimes \cdots \otimes \xi_k) = T(\xi_1) \cdots T(\xi_k), \qquad \xi_1, \dots, \xi_k \in V, \tag{20.40}$$

and extend to be linear. It follows that φ is an algebra homomorphism. In fact, the requirement that φ be an algebra homomorphism forces (20.40), so φ is unique. This proves the universal property for (A, i). ☐

The explicit construction implies both that the components of the tensor algebra A in negative degree vanish and that for $k \in \mathbb{Z}^{>0}$ the component in degree k is generated by the image of the k-linear map

(20.41)
$$\begin{aligned} V \times \cdots \times V &\longrightarrow A \\ \xi_1, \dots, \xi_k &\longmapsto i(\xi_1)\cdots i(\xi_k). \end{aligned}$$

Notation 20.42. We denote the $\mathbb{Z}$-graded tensor algebra as '$\otimes^\bullet V$'.

Exterior algebra

(20.43) *Alternation and skew-symmetry.* The characteristic property of the exterior algebra is the skew-symmetry of the product on vectors, as we motivated in (19.25). Skew-symmetry of the wedge product on vectors is implied by the alternating property

(20.44)
$$\xi \wedge \xi = 0, \qquad \xi \in V.$$

(Recall Definition 19.35.)

Definition 20.45. Let V be a vector space. An *exterior algebra* (E, j) *over* V is an algebra E and a linear map $j\colon V \to E$ satisfying $j(\xi)^2 = 0$ for all $\xi \in V$ such that for all pairs (B, T) consisting of an algebra B and a linear map $T\colon V \to B$ satisfying $T(\xi)^2 = 0$ for all $\xi \in V$, there exists a unique algebra homomorphism $\varphi\colon E \to B$ such that $T = \varphi \circ j$.

Uniqueness up to unique isomorphism follows from the universal property. We prove existence by constructing the exterior algebra as a quotient of the tensor algebra.

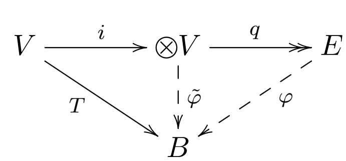
Theorem 20.46. *Let V be a vector space. Then an exterior algebra over V exists.*

Proof. Let $\otimes V$ be the tensor algebra of V. Define $Q(V) \subset \otimes^2 V$ as

(20.47)
$$Q(V) = \{\xi \otimes \xi : \xi \in V\},$$

and let $I(V) \subset \otimes V$ be the 2-sided ideal generated[24] by $Q(V)$. Since $\otimes^{\geqslant 2} V \subset V$ is a 2-sided ideal, then so too is the intersection $I(V) \cap \otimes^{\geqslant 2} V$. But $Q(V) \subset I(V) \cap \otimes^{\geqslant 2} V \subset I(V)$, and since $I(V)$ is generated by $Q(V)$ it follows that $I(V) \subset \otimes^{\geqslant 2} V$.

Define $E = \otimes V / I(V)$ and let $j\colon V \xrightarrow{i} \otimes V \xrightarrow{q} E$ be the composition of the inclusion $\otimes^1 V \hookrightarrow \otimes V$ and the quotient map. We claim (E, j) is an exterior algebra. To prove the universal property, given (B, T) as in Definition 20.45, construct $\tilde{\varphi}$ in

(20.48)


[24] $I(V)$ is the intersection of all 2-sided ideals containing $Q(V)$.

using the universal property for the tensor algebra. It factors through E since $qi(\xi)^2 = 0$ for all $\xi \in V$, which implies $\tilde{\varphi}(Q(V)) = 0$, and then finally $\tilde{\varphi}(I(V)) = 0$. Uniqueness of φ follows immediately from uniqueness of $\tilde{\varphi}$. ☐

Notation 20.49. We denote the exterior algebra as '$\bigwedge V$' and use '$\wedge$' for the product.

Example 20.50. Let $V = 0$ be the zero vector space. Then $\otimes V = \mathbb{R}$ and $\bigwedge V = \mathbb{R}$. We leave the reader to deduce these assertions from the universal properties.

Example 20.51. Let $V = L$ be a line, i.e., a 1-dimensional vector space. The we claim the algebra $E = \mathbb{R} \oplus L$ with $\ell^2 = 0$ for all $\ell \in L$ and the obvious inclusion $j\colon L \hookrightarrow E$ is an exterior algebra over L. This is a straightforward consequence of the universal property. Notice that $\otimes L$ is infinite dimensional.

More about the exterior algebra: $\mathbb{Z}$-grading and commutativity

(20.52) *The tensor algebra maps to the exterior algebra.* Our construction in the proof of Theorem 20.46 expresses the exterior algebra $\bigwedge V$ of a vector space V as a quotient of the tensor algebra $\bigotimes V$ by an ideal $I(V)$. In fact, it follows easily from the universal property of the tensor algebra (Definition 20.29) that there is such a homomorphism.

Proposition 20.53. *Let V be a vector space. Then there is a canonical homomorphism*

$$q\colon \bigotimes V \to \bigwedge V. \tag{20.54}$$

Proof. Factor the linear map $j\colon V \to \bigwedge V$ through the tensor algebra using the universal property (20.30) of the tensor algebra. ☐

(20.55) *Induced maps.* We prove the analog of Theorem 20.31(2) for exterior algebras. Let V, V' be vector spaces and suppose $T\colon V' \to V$ is a linear map. i, i' denote the inclusions of V, V' into their exterior algebras. Then construct the algebra homomorphism $\bigwedge T$ in the diagram

$$\begin{array}{ccc} V' & \xrightarrow{\;i'\;} & \bigwedge V' \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \bigwedge T} \\ V & \xrightarrow{\;i\;} & \bigwedge V \end{array} \tag{20.56}$$

via the universal property of the exterior algebra $\bigwedge V'$ applied to $i \circ T$; see Definition 20.45. We leave the reader to use the uniqueness in the universal property to prove that for a sequence $V'' \xrightarrow{T'} V' \xrightarrow{T} V$ of linear maps, the induced algebra homomorphisms on the exterior algebras satisfy $\bigwedge(T' \circ T) = \bigwedge T' \circ \bigwedge T$, analogous to (20.32).

(20.57) *$\mathbb{Z}$-gradings.* As in Theorem 20.31(2) for the tensor algebra, the exterior algebra admits a canonical $\mathbb{Z}$-grading. The proof is similar.

Theorem 20.58. *Let V be a vector space. Then the exterior algebra $\bigwedge V$ is $\mathbb{Z}$-graded. Furthermore, the homomorphism q in (20.54) preserves the $\mathbb{Z}$-gradings. Also, if $T\colon V' \to V$ is a linear map, then the induced algebra homomorphism $\bigwedge T\colon \bigwedge V' \to \bigwedge V$ preserves the $\mathbb{Z}$-gradings.*

Proof. For $\lambda \in \mathbb{R}$ let $T_\lambda\colon V \to V$ denote scalar multiplication by λ. The $\mathbb{Z}$-grading on $\bigwedge V$ is constructed by choosing $\lambda \neq 0, 1$ and setting $\bigwedge^k V$ to be the eigenspace for $\bigwedge T_\lambda$ with eigenvalue λ^k. The subspace $j(V) \subset \bigwedge V$ is contained in $\bigwedge^1 V$. We leave the reader to check that the inclusion of V into $\bigoplus_{k=0}^\infty \bigwedge^k V$ satisfies the universal property, from which it follows that the inclusion of this direct sum into $\bigwedge V$ is an isomorphism.

The remaining assertions follow from the fact that scalar multiplication commutes with all linear maps. □

(20.59) *The $\mathbb{Z}$-graded ideal.* We can also deduce the $\mathbb{Z}$-grading from the construction in the proof of Theorem 20.46. Recall the subset $Q(V) \subset \bigotimes^2 V$ defined in (20.47) and the 2-sided ideal $I(V) \subset \bigotimes V$ which it generates. Define $I^k(V) = I(V) \cap \bigotimes^k V$. The proof of Theorem 20.46 shows that $I^k(V) = 0$ for $k < 2$.

Theorem 20.60. *The ideal $I(V)$ is $\mathbb{Z}$-graded in the sense that $I(V) = \bigoplus\limits_{k=2}^{\infty} I^k(V)$.*

Proof. Let $\bigotimes T_\lambda\colon \bigotimes V \to \bigotimes V$ denote the homomorphism induced on the tensor algebra by scalar multiplication. We claim $\bigotimes T_\lambda$ maps $I(V)$ into itself. If $\lambda \neq 0$, then $T_{\lambda^{-1}}$ preserves $Q(V)$, and so $(\bigotimes T_{\lambda^{-1}})(I(V))$ is a 2-sided ideal containing $Q(V)$. Since $I(V)$ is the smallest 2-sided ideal containing $Q(V)$, we have $I(V) \subset (\bigotimes T_{\lambda^{-1}})(I(V))$, or equivalently $(\bigotimes T_\lambda)(I(V)) \subset I(V)$. The λ^k-eigenspace of the restriction of $\bigotimes T_\lambda$ to $I(V)$, for $\lambda \neq 1$, is by definition $I^k(V)$. Clearly $\bigoplus\limits_{k=2}^{\infty} I^k(V) \subset I(V)$. We claim that $\bigoplus\limits_{k=2}^{\infty} I^k(V) \subset \bigotimes V$ is a 2-sided ideal. By Theorem 20.31(3) it suffices to show that the product of an element of $I^k(V)$ and an element of $\otimes^\ell(V)$ lies in $I^{k+\ell}(V)$, but this is clear since $I(V)$ is a 2-sided ideal. Now $Q(V) \subset I^2(V)$, and so $\bigoplus\limits_{k=2}^{\infty} I^k(V)$ is a 2-sided ideal containing $Q(V)$ and contained in the smallest 2-sided ideal $I(V)$ containing $Q(V)$. It follows that $I(V) = \bigoplus\limits_{k=2}^{\infty} I^k(V)$. □

Corollary 20.61. *As a vector space, $\bigwedge^k V \cong \bigotimes^k V/I^k(V)$. In particular, we have $\bigwedge^0 V = \mathbb{R}$ and $\bigwedge^1 V = V$.*

The first statement follows from the definition of the $\mathbb{Z}$-grading as an eigenspace decomposition for scalar multiplication. The last assertion follows since $I^k(V) = 0$ for $k = 0, 1$.

Corollary 20.62. *The image of the alternating k-linear map*

$$\begin{aligned} V \times \cdots \times V &\longrightarrow \textstyle\bigwedge^k V \\ \xi_1, \dots, \xi_k &\longmapsto \xi_1 \wedge \cdots \wedge \xi_k \end{aligned} \tag{20.63}$$

generates $\bigwedge^k V$*.*

This follows from the corresponding statement (20.41) for the tensor algebra, which in turn follows from the explicit construction.

Remark 20.64*.* A k-linear map is *alternating* if it vanishes when two arguments are equal. In fact, (20.63) is the universal alternating k-linear map, in the same sense that the tensor product of two vector spaces is the universal bilinear map (Definition 20.2). That follows from the fact that (20.41) (with codomain $\bigotimes^k V$) is the universal k-linear map, which we could prove by developing the ideas in **(20.17)**.

(20.65) *Commutativity.* Recall from Definition 20.24(6) that for a $\mathbb{Z}$-graded algebra the definition of commutativity has a sign, the Koszul sign.

Theorem 20.66. *The $\mathbb{Z}$-graded exterior algebra $\bigwedge^\bullet V$ over a vector space V is commutative.*

Proof. Corollary 20.62 implies that it suffices to check (20.26) for decomposable vectors. Let k, k' be positive integers and let $\xi_1, \dots, \xi_k, \xi'_1, \dots, \xi'_{k'}$ be vectors in V. Set $X = \xi_1 \wedge \cdots \wedge \xi_k$ and $X' = \xi'_1 \wedge \cdots \wedge \xi'_{k'}$. Then

$$\begin{aligned} X \wedge X' &= \xi_1 \wedge \cdots \wedge \xi_k \wedge \xi'_1 \wedge \cdots \wedge \xi'_{k'} \\ &= (-1)^{kk'} \xi'_1 \wedge \cdots \wedge \xi'_{k'} \wedge \xi_1 \wedge \cdots \wedge \xi_k \\ &= (-1)^{kk'} X' \wedge X, \end{aligned} \tag{20.67}$$

since in reordering we move k vectors past k' vectors, for a total of kk' transpositions. Each gives a minus sign according to the defining property of the exterior algebra; see **(20.43)**. ☐

Lecture 21: More on exterior algebras; differential forms on affine space

In the first part of the lecture we treat a few more general properties of exterior algebras. First we identify the exterior algebra of a direct sum of vector spaces as a tensor product of exterior algebras. However, *nota bene* the Koszul sign in the definition of the product (21.7). In other words, this is a $\mathbb{Z}$-*graded* tensor product. We have already determined the exterior algebra of a line (Example 20.51). Since every finite dimensional vector space is a direct sum of lines, we can now inductively determine the exterior algebra of a finite dimensional vector space. If V is a vector space of dimension $n \in \mathbb{Z}^{\geqslant 0}$, then the top[25] exterior power $\bigwedge^n V$ is a line, called the *determinant line* of the vector space V. We explain the relationship to the familiar notion of determinant. Finally, for any vector space V we introduce a duality between $\bigwedge^\bullet V^*$ and $\bigwedge^\bullet V$.

In the second part of the lecture we introduce differential forms on open subsets of affine space. The main theorem at this stage is the extension of the differential on functions to a differential on differential forms. In the next lecture we globalize (patch/paste) to differential forms on a smooth manifold.

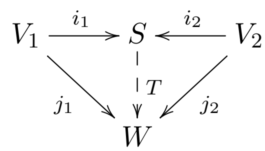
[25]The higher exterior powers $\bigwedge^k V$, $k > n$, are zero.

Direct sums and tensor products

(21.1) *Direct sum via a universal property.* We have used the direct sum of vector spaces many times already in these lectures. Now we make explicit its universal property.

Definition 21.2. Let V_1, V_2 be vector spaces. A *direct sum* (S, i_1, i_2) is a diagram $V_1 \xrightarrow{i_1} S \xleftarrow{i_2} V_2$ of vector spaces and linear maps which is universal in the sense that for any linear maps $j_i\colon V_i \to W$ there exists a unique linear map $T\colon S \to W$ which makes the diagram

(21.3)



commute.

The direct sum exists and is unique up to unique isomorphism. It is denoted $V_1 \oplus V_2$.

(21.4) *Tensor product of algebras.* Let A_1, A_2 be algebras. In particular, they are vector spaces and so we can form the tensor product vector space $A_1 \otimes A_2$, as in Definition 20.2. To endow $A_1 \otimes A_2$ with an algebra structure, as in Definition 20.24(1), we must specify a bilinear map $m\colon A_1 \times A_2 \to A_1 \otimes A_2$ and a unit $1 \in A_1 \otimes A_2$. We can take $m = m_1 \otimes m_2$ and $1 = 1_1 \otimes 1_2$, where $m_i, 1_i$ define the algebra structure of A_i, $i = 1, 2$. This is the correct definition for ungraded algebras, but for $\mathbb{Z}$-graded algebras there is a (Koszul) sign.

Definition 21.5. Let $A_1^\bullet, A_2^\bullet$ be $\mathbb{Z}$-graded algebras with units $1_1, 1_2$. Endow the vector space $A_1 \otimes A_2$ with the $\mathbb{Z}$-grading

$$(A_1 \otimes A_2)^k = \bigoplus_{\substack{k_1, k_2 \in \mathbb{Z} \\ k_1 + k_2 = k}} A_1^{k_1} \otimes A_2^{k_2}, \qquad k \in \mathbb{Z}. \tag{21.6}$$

Endow $A_1 \otimes A_2$ with a $\mathbb{Z}$-graded algebra structure: define the unit $1_1 \otimes 1_2$ and multiplication

$$(a_1 \otimes a_2)(a_1' \otimes a_2') = (-1)^{k_2 k_1'}\, a_1 a_1' \otimes a_2 a_2', \tag{21.7}$$

for $a_1, a_1' \in A_1$, $a_2, a_2' \in A_2$, where a_2 is homogeneous of degree k_2 and a_1' is homogeneous of degree k_1'.

Since multiplication is bilinear, it suffices to define it on decomposable vectors which are tensor products of homogeneous vectors.

Example 21.8. Let L_1, L_2 be lines. Recall the exterior algebras $\bigwedge^\bullet L_i = \mathbb{R} \oplus L_i$, $i = 1, 2$, as in Example 20.51. Let $\ell_i \in L_i$, $i = 1, 2$, be basis elements. The tensor product of the exterior algebras is the $\mathbb{Z}$-graded algebra

$$\big(\textstyle\bigwedge L_1 \otimes \bigwedge L_2\big)^\bullet = \mathbb{R} \,\oplus\, (L_1 \oplus L_2) \,\oplus\, (L_1 \otimes L_2) \tag{21.9}$$

supported in degrees $0, 1, 2$. In this algebra, $\ell_1\ell_2 = -\ell_2\ell_1$ by (21.7). We prove below in Theorem 21.11 that (21.9) is the exterior algebra $\bigwedge^\bullet(L_1 \oplus L_2)$ of the direct sum. In particular, the higher exterior powers $\bigwedge^k(L_1 \oplus L_2)$, $k > 2$, vanish (i.e., equal the zero vector space).

(21.10) *Exterior algebra of a direct sum.* The passage from a vector space to its exterior algebra is an *exponentiation* in the sense that sums go over to products. (The same holds for tensor algebras.)

Theorem 21.11. *Let V_1, V_2 be vector spaces. Then there is a canonical isomorphism*

$$\bigwedge^\bullet(V_1 \oplus V_2) \xrightarrow{\;\cong\;} \big(\bigwedge V_1 \otimes \bigwedge V_2\big)^\bullet \tag{21.12}$$

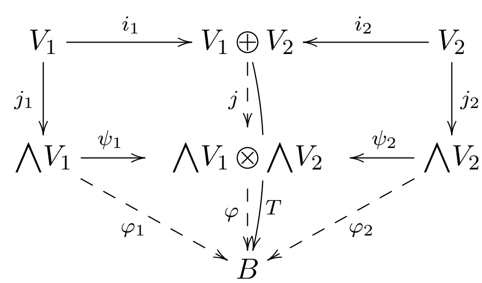
of $\mathbb{Z}$-graded algebras.

Proof. Let $j_i\colon V_i \to \bigwedge V_i$, $i = 1, 2$, be the inclusions into the exterior algebra as the degree one elements. Define algebra homomorphisms $\psi_i\colon \bigwedge V_i \to \bigwedge V_1 \otimes \bigwedge V_2$, $i = 1, 2$, by

$$\begin{aligned} \psi_1(X_1) &= X_1 \otimes 1, \qquad & X_1 \in \bigwedge V_1, \\ \psi_2(X_2) &= 1 \otimes X_2, \qquad & X_2 \in \bigwedge V_2. \end{aligned} \tag{21.13}$$

Consider the diagram

(21.14)



First, use the universal property of the direct sum (Definition 21.2) to construct j. Then we claim that the pair $\big((\bigwedge V_1 \otimes \bigwedge V_2)^\bullet, j\big)$ is an exterior algebra of $V_1 \oplus V_2$, where multiplication is defined on the tensor product with the Koszul sign (21.7). To check the universal property in Definition 20.45 for the pair $(\bigwedge V_1 \otimes \bigwedge V_2, j)$, suppose B is an algebra and $T\colon V_1 \oplus V_2 \to B$ in (21.14) satisfies $T(\xi_1 + \xi_2)^2 = 0$ for all $\xi_1 \in V_1$, $\xi_2 \in V_2$. Apply the universal property of the exterior algebra to $T \circ i_1$ to produce the (unique) algebra homomorphism φ_1 and to $T \circ i_2$ to produce the (unique) algebra homomorphism φ_2. Finally, define the (unique) linear map $\varphi\colon \bigwedge V_1 \otimes \bigwedge V_2 \to B$ by applying the universal property of the tensor product (Definition 20.2) to the bilinear map

$$\begin{aligned} \bigwedge V_1 \times \bigwedge V_2 &\longrightarrow B \\ X_1 \;\;,\;\; X_2 &\longmapsto \varphi_1(X_1)\,\varphi_2(X_2) \end{aligned} \tag{21.15}$$

We leave the reader to check that φ is an algebra homomorphism; it suffices to check on the tensor products of decomposable vectors in $\bigwedge V_1$ and $\bigwedge V_2$. □

Finite dimensional exterior algebras and determinants

(21.16) *Applications of Theorem 21.11.* First, in Example 21.8 we see that (21.9) is isomorphic to $\bigwedge^{\bullet}(L_1 \oplus L_2)$. By induction on the dimension of a finite dimensional vector space, we leave the reader to deduce the following from Theorem 21.11.

Theorem 21.17. *Let n be a positive integer and suppose V is a vector space of dimension n.*

(1) $\bigwedge^k V = 0$ *if* $k < 0$ *or* $k > n$.

(2) *If $e_1, \dots, e_n$ is a basis of V, then for $1 \leqslant k \leqslant n$*

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \leqslant i_1 < \cdots < i_k \leqslant n\} \tag{21.18}$$

is a basis of $\bigwedge^k V$.

(3) $\dim \bigwedge^k V = \binom{n}{k}$, $0 \leqslant k \leqslant n$.

In (2) the expression $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ is the binomial coefficient. In (3) the indices run over all strictly increasing ordered subsets of $\{1, \dots, n\}$ of cardinality k. If we write $V = L_1 \oplus \cdots \oplus L_n$ as a sum of lines, then one proves by induction on $\dim V$ that

$$\bigwedge^k V \cong \bigoplus_{1 \leqslant i_1 < \cdots < i_k \leqslant n} L_{i_1} \otimes \cdots \otimes L_{i_k}, \tag{21.19}$$

which generalizes (21.9).

Corollary 21.20. *Let V be a vector space and $S = \{\xi_1, \dots, \xi_k\} \subset V$. Then S is linearly independent if and only if $\xi_1 \wedge \cdots \wedge \xi_k \in \bigwedge^k V$ is nonzero.*

Proof. Let $V' \subset V$ be the span of S; then $\xi_1 \wedge \cdots \wedge \xi_k \in \bigwedge^k V'$. If S is not linearly independent, then $\dim V' < k$ in which case Theorem 21.17(1) implies $\bigwedge^k V' = 0$. Conversely, if S is a basis of V', then Theorem 21.17(2) implies that $\xi_1 \wedge \cdots \wedge \xi_k$ is a basis of $\bigwedge^k V'$, hence is nonzero. □

(21.21) *The determinant line.* According to Theorem 21.17(3) the top exterior power of a finite dimensional vector space is 1-dimensional. It has a special name and notation.

Definition 21.22. Let n be a positive integer and suppose V is a vector space of dimension n. The *determinant line* of V is the 1-dimensional vector space

$$\operatorname{Det} V = \bigwedge^n V. \tag{21.23}$$

The reason for the name will be apparent shortly. The following is a special case of Corollary 21.20.

Proposition 21.24. *Let V have dimension n. Then $\xi_1, \dots, \xi_n \in V$ is a basis if and only if the wedge product $\xi_1 \wedge \cdots \wedge \xi_n \in \operatorname{Det} V$ is nonzero.*

(21.25) *Determinant of a linear map.* Suppose V, V' are finite dimensional vector spaces and $T\colon V' \to V$ a linear map.

Definition 21.26.

(1) If $\dim V' \neq \dim V$, then define $\det T\colon \operatorname{Det} V' \to \operatorname{Det} V$ to be the zero map.
(2) If $\dim V' = \dim V$, then define $\det T = \bigwedge^n T\colon \operatorname{Det} V' \to \operatorname{Det} V$.

Recall that $\bigwedge^n T$ is the map induced by T on the n^{th} exterior power; see **(20.55)**.

Proposition 21.27. $\det T \neq 0$ *if and only if* T *is invertible.*

Proof. If $\dim V' \neq \dim V$, then T is not invertible and $\det T = 0$. Assume $\dim V' = \dim V$ and $\xi'_1, \dots, \xi'_n \in V'$ is a basis. Proposition 21.24 implies that $\xi'_1 \wedge \cdots \wedge \xi'_n$ is a basis of $\operatorname{Det} V'$. By Definition 21.26(2),

$$(\det T)(\xi'_1 \wedge \cdots \wedge \xi'_n) = T\xi'_1 \wedge \cdots \wedge T\xi'_n. \tag{21.28}$$

This is nonzero iff $T\xi'_1, \dots, T\xi'_n$ is a basis of V iff T is invertible. ∎

We can give another proof using the composition law at the end of **(20.55)**. Namely, if T is invertible, then $\mathrm{id}_{\operatorname{Det} V'} = \det(T^{-1} \circ T) = \det T^{-1} \circ \det T$, which implies $\det T$ is nonzero.

(21.29) *The numerical determinant.* If V is finite dimensional and $T\colon V \to V$ is a linear operator, then $\det T\colon \operatorname{Det} V \to \operatorname{Det} V$ is a linear operator on a line, so it is scalar multiplication by a real number. We identify that scalar with the numerical determinant of the linear operator T. The following computations show that this agrees with the usual determinant of matrices.

Example 21.30 (2-dimensional determinant)**.** Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. This means that for the standard basis e_1, e_2 we have

$$\begin{aligned} Te_1 &= ae_1 + ce_2 \\ Te_2 &= be_1 + de_2 \end{aligned} \tag{21.31}$$

Hence

$$\begin{aligned} (\det T)(e_1 \wedge e_2) &= Te_1 \wedge Te_2 \\ &= (ae_1 + ce_2) \wedge (be_1 + de_2) \\ &= (ad - bc)\, e_1 \wedge e_2. \end{aligned} \tag{21.32}$$

Example 21.33 (n-dimensional determinant)**.** Let n be a positive integer and $T\colon \mathbb{R}^n \to \mathbb{R}^n$ a linear operator. Write

$$Te_j = T^i_j e_i, \qquad j = 1, \dots, n, \tag{21.34}$$

for n^2 numbers $T^i_j \in \mathbb{R}$. Then

$$(21.35)\qquad \begin{aligned}(\det T)(e_1 \wedge \cdots \wedge e_n) &= Te_1 \wedge \cdots \wedge Te_n \\ &= (T^{i_1}_1 e_{i_1}) \wedge \cdots \wedge (T^{i_n}_n e_{i_n}) \\ &= \left\{ \sum_{\sigma \in \mathrm{Sym}_n} \epsilon(\sigma)\, T^{\sigma(1)}_1 \cdots T^{\sigma(n)}_n \right\} e_1 \wedge \cdots \wedge e_n,\end{aligned}$$

where Sym_n is the permutation group of $\{1, \dots, n\}$ and $\epsilon(\sigma) = \pm 1$ is the sign of the permutation. The homogeneous polynomial of degree n in braces is the usual expression for the determinant of a matrix.

Standard properties of the determinant, including the formula (21.35), are easily derived from the definition using the basic properties of exterior algebras.

Duality and exterior algebras

Let n be a positive integer and V an n-dimensional real vector space. The exterior powers of V and of the dual space V^* form an array

$$(21.36)\qquad \begin{array}{ccccc} \mathbb{R} & V^* & \bigwedge^2 V^* & \cdots & \bigwedge^n V^* = \operatorname{Det} V^* \\ \\ \mathbb{R} & V & \bigwedge^2 V & \cdots & \bigwedge^n V = \operatorname{Det} V \end{array}$$

There is a natural duality between the vector spaces in each column.

Proposition 21.37. *For all $k \in \mathbb{Z}^{>0}$ the pairing*

$$(21.38)\qquad \begin{aligned} \textstyle\bigwedge^k V^* \quad \times \quad \bigwedge^k V \quad &\longrightarrow \quad \mathbb{R} \\ \theta^1 \wedge \cdots \wedge \theta^k \;\;,\;\; \xi_1 \wedge \cdots \wedge \xi_k &\longmapsto \det\bigl(\theta^i(\xi_j)\bigr)_{i,j} \end{aligned}$$

is nondegenerate

We have only specified the pairing on decomposable vectors in the exterior powers; it extends to all vectors using bilinearity. This determinant pairing identifies $\bigwedge^k V^*$ as the dual space to $\bigwedge^k V$. Compose with the alternating k-linear map (20.63) to identify $\bigwedge^k V^*$ as the space of k-linear alternating functions $V \times \cdots \times \to \mathbb{R}$. Namely, if $\alpha \in \bigwedge^k V^*$, define

$$(21.39)\qquad \begin{aligned} \hat{\alpha}\colon V \times \cdots \times V &\longrightarrow \mathbb{R} \\ \xi_1, \dots, \xi_k &\longmapsto \langle \alpha, \xi_1 \wedge \cdots \wedge \xi_k \rangle, \end{aligned}$$

using the determinant pairing (21.38). We usually omit the carrot over α in (21.39) and simply identify the k-form α with this alternating k-linear map.

Proof. Let $e_1, \dots, e_n$ be a basis of V and $e^1, \dots, e^n$ the dual basis of V^*. Then by Theorem 21.17 we obtain a bases of $\bigwedge^k V$ and $\bigwedge^k V^*$. Introduce the multi-index notation $I = (i_1 \cdots i_k)$ for an increasing set of indices $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$. Suppose $X = X^I e_I \in \bigwedge^k V$ lies in the kernel of (21.38). Writing the determinant pairing (21.38) as $\langle -, - \rangle$ we have

$$0 = \langle e^J, X \rangle = X^J \tag{21.40}$$

for all multi-indices J. This implies $X = 0$ and proves nondegeneracy. □

Example 21.41. For $k = 2$, if $\theta^1 \wedge \theta^2$ is a decomposable 2-form, the product of 1-forms $\theta^1, \theta^2 \in V^*$, then

$$(\theta^1 \wedge \theta^2)(\xi_1, \xi_2) = \theta^1(\xi_1)\theta^2(\xi_2) - \theta^1(\xi_2)\theta^2(\xi_1), \qquad \xi_1, \xi_2 \in V. \tag{21.42}$$

This is an oft-used formula.

Remark 21.43. Note the absence of numerical factors (such as 1/2) in (21.42) (as derived from (21.38)).

Differential forms on affine space

(21.44) *Definitions.* Let V be a finite dimensional normed real vector space and A an affine space over V. Let $U \subset A$ be an open subset.

Definition 21.45. A *(differential) k-form on U*, $k \in \mathbb{Z}^{\geqslant 0}$, is a function $\alpha \colon U \to \bigwedge^k V^*$. The space of C^∞ k-forms on U is denoted $\Omega^k(U)$ (or as Ω^k_U, depending on the context).

Differential forms of arbitrary degree form a $\mathbb{Z}$-graded vector space

$$\Omega^\bullet(U) = \bigoplus_{k=0}^{\infty} \Omega^k(U). \tag{21.46}$$

The operation of exterior multiplication is defined on differential forms pointwise, and with it $\Omega^\bullet(U)$ is a commutative $\mathbb{Z}$-graded algebra; see Definition 20.24.

Example 21.47. Let $\mathbb{A}^3_{x,y,z}$ be standard affine 3-space with coordinate functions x, y, z. Let $U \subset \mathbb{A}^3$ be open. At each point $p \in U$ the differentials dx_p, dy_p, dz_p form a basis of $(\mathbb{R}^3)^*$. (Since $x, y, z \colon U \to \mathbb{R}$ are affine functions, the differentials are constant, so the basis is independent of p.) Therefore, by Theorem 21.17(2) we can write any element of $\Omega^2(U)$ as a linear combination

$$f(x,y,z)dx \wedge dy \; + \; g(x,y,z)dx \wedge dz \; + \; h(x,y,z)dy \wedge dz \tag{21.48}$$

for functions $f, g, h \colon U \to \mathbb{R}$.

(21.49) *The Cartan d.* The exterior differential is characterized by a few basic properties.

Theorem 21.50. *There exists a unique map $d\colon \Omega^\bullet(U) \to \Omega^\bullet(U)$ of degree $+1$ such that*

(i) *d is linear,*
(ii) $d(\alpha_1 \wedge \alpha_2) = d\alpha_1 \wedge \alpha_2 + (-1)^k \alpha_1 \wedge d\alpha_2, \qquad \alpha_1 \in \Omega^{k_1}(U), \quad \alpha_2 \in \Omega^\bullet(U),$
(iii) $d^2 = 0,$
(iv) *d agrees with the usual differential on $\Omega^0(U)$.*

That d is homogeneous of degree $+1$ means that if $\alpha \in \Omega^k(U)$ is homogeneous of degree k, then $d\alpha \in \Omega^{k+1}(U)$ is homogeneous of degree $k+1$. Note the sign in (ii) is consistent with the Koszul rule: in passing the degree 1 operator d past the degree k differential form α_1 we pick up the sign $(-1)^{1\cdot k} = (-1)^k$. We prove Theorem 21.50 below but first exhibit some explicit computations.

Example 21.51. Consider the differential form

$$\alpha = e^{2x} - x^2y^2\,dy - x\,dx \wedge dy \tag{21.52}$$

on $\mathbb{A}^2_{x,y}$. Then using the rules in Theorem 21.50 we compute

$$\begin{aligned} d\alpha &= d(e^{2x}) - d(x^2y^2\,dy) - d(x\,dx \wedge dy) \\ &= 2e^{2x}\,dx - d(x^2y^2) \wedge dy - dx \wedge dx \wedge dy \\ &= 2e^{2x}\,dx - 2xy^2\,dx \wedge dy. \end{aligned} \tag{21.53}$$

In this example both α and $d\alpha$ are inhomogeneous.

Example 21.54. Consider

$$\alpha = x\,dy \wedge dz + y\,dx \wedge dz + z\,dx \wedge dy \tag{21.55}$$

in $\Omega^2(\mathbb{A}^3_{x,y,z})$. Then

$$d\alpha = 3\,dx \wedge dy \wedge dz. \tag{21.56}$$

Example 21.57. On an open set $U \subset \mathbb{A}^n_{x^1,\dots,x^n}$, consider the general smooth $(n-1)$-form

$$\alpha = f^1 dx^2 \wedge \cdots \wedge dx^n - f^2 dx^1 \wedge dx^3 \wedge \cdots \wedge dx^n + \cdots, \tag{21.58}$$

where $f^i\colon U \to \mathbb{R}$ are smooth functions. Then

$$d\alpha = \left(\sum_i \frac{\partial f^i}{\partial x^i}\right) dx^1 \wedge \cdots \wedge dx^n. \tag{21.59}$$

Proof of Theorem 21.50. Let $x^1, \dots, x^n \colon A \to \mathbb{R}$ be affine coordinates on A, and restrict them to functions on U. It suffices to define d on k-forms for all $k \in \mathbb{Z}^{>0}$ and then use (i) to extend uniquely by linearity, since every differential form is a finite sum of homogeneous forms. By Theorem 21.17(2) we can write $\alpha \in \Omega^k(U)$ uniquely as

$$\alpha = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} f_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \tag{21.60}$$

for functions $f_{i_1 \cdots i_k} \colon U \to \mathbb{R}$. Then if d exists satisfying (i)–(iv) we compute

$$\begin{aligned}
d\alpha &= d\left(\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} f_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right) \\
&\overset{\text{(i)}}{=} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} d(f_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\
&\overset{\text{(ii)}}{=} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} df_{i_1 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\
&\qquad\qquad + f_{i_1 \cdots i_k} d^2 x^{i_1} \wedge \cdots \wedge dx^{i_k} \\
&\qquad\qquad\qquad - f_{i_1 \cdots i_k} dx^{i_1} \wedge d^2 x^{i_2} \wedge dx^{i_3} \cdots \wedge dx^{i_k} \\
&\qquad\qquad\qquad\qquad + \cdots \\
&\overset{\text{(iii)}}{=} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} df_{i_1 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\
&\overset{\text{(iv)}}{=} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \frac{\partial f_{i_1 \cdots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.
\end{aligned} \tag{21.61}$$

Therefore, if d exists and satisfies (i)–(iv) it must be given by the formula (21.61). This proves the uniqueness. To prove existence we *define* d on k-forms by formula (21.61) and check (i)–(iv). Property (i) is easy. As a variation, for (ii) use increasing multi-indices, as in the proof of Proposition 21.37. So as not to have index wars let us call the forms $\alpha \in \Omega^k(U)$ and $\beta \in \Omega^\ell(U)$. (By linearity it suffices to take the second form homogeneous as well.) Write

$$\begin{aligned}
\alpha &= \alpha_I \, dx^I \\
\beta &= \beta_J \, dx^J
\end{aligned} \tag{21.62}$$

for functions $\alpha_I, \beta_J \colon U \to \mathbb{R}$. Then

$$\begin{aligned}
d(\alpha \wedge \beta) &= d(\alpha_I \beta_J \, dx^I \wedge dx^J) \\
&= \beta_J \frac{\partial \alpha_I}{dx^i} dx^i \wedge dx^I \wedge dx^J + \alpha_I \frac{\partial \beta_J}{dx^j} dx^j \wedge dx^I \wedge dx^J \\
&= \left(\frac{\partial \alpha_I}{dx^i} dx^i \wedge dx^I \right) \wedge \left(\beta_J \, dx^J \right) \; + \; (-1)^k \left(\alpha_I \, dx^I \right) \wedge \left(\frac{\partial \beta_J}{dx^j} dx^j \wedge dx^J \right) \\
&= d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.
\end{aligned} \tag{21.63}$$

The sign comes from commuting the 1-form dx^j past the k-form dx^I. For (iii) compute

$$\begin{aligned}
d^2\alpha &= d\left(\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \frac{\partial f_{i_1\cdots i_k}}{\partial x^j}\,dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}\right)\\
&= \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \frac{\partial^2 f_{i_1\cdots i_k}}{\partial x^\ell\partial x^j}\,dx^\ell\wedge dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}\\
&= \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \frac12\left(\frac{\partial^2 f_{i_1\cdots i_k}}{\partial x^\ell\partial x^j}+\frac{\partial^2 f_{i_1\cdots i_k}}{\partial x^j\partial x^\ell}\right)dx^\ell\wedge dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}\\
&= \frac12\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \frac{\partial^2 f_{i_1\cdots i_k}}{\partial x^\ell\partial x^j}\,dx^\ell\wedge dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}\\
&\qquad+\frac12\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \frac{\partial^2 f_{i_1\cdots i_k}}{\partial x^\ell\partial x^j}\,dx^j\wedge dx^\ell\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}\\
&= \frac12\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \frac{\partial^2 f_{i_1\cdots i_k}}{\partial x^\ell\partial x^j}\,dx^\ell\wedge dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}\\
&\qquad-\frac12\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \frac{\partial^2 f_{i_1\cdots i_k}}{\partial x^\ell\partial x^j}\,dx^\ell\wedge dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}\\
&= 0.
\end{aligned} \tag{21.64}$$

In the fourth equality we exchange the dummy indices j and ℓ; in the penultimate equality we use $dx^j\wedge dx^\ell = -dx^\ell\wedge dx^j$. The main point is the third equality, which expresses that second partials are symmetric; this contrasts the skew-symmetry of the wedge product. Finally, the definition (21.61) reduces to (19.12) for $k=0$, which proves (iv). □

(21.65) *Expression for d in terms of directional derivatives.* We continue with $U\subset A$ an open subset of a finite dimensional affine space A over a normed linear space V. Recall that for $f\in\Omega^0(U)$ and $\xi\in V$ we have

$$df(\xi)=\xi f, \tag{21.66}$$

an equality of functions on U. The following generalizes (21.66) for all k. In this theorem we treat a differential k-form as an alternating k-linear function on vectors; see Proposition 21.37.

Theorem 21.67. *Let $k\in\mathbb{Z}^{\geqslant 0}$, $\alpha\in\Omega^k(U)$, and $\xi_1,\dots,\xi_{k+1}\in V$. Then*

$$\begin{aligned}
d\alpha(\xi_1,\dots,\xi_{k+1}) &= \xi_1\,\alpha(\xi_2,\dots,\xi_{k+1})-\xi_2\,\alpha(\xi_1,\xi_3,\dots,\xi_{k+1})\\
&\qquad+\cdots+(-1)^k\,\xi_{k+1}\,\alpha(\xi_1,\dots,\xi_k).
\end{aligned} \tag{21.68}$$

Each term is the directional derivative of the k-form α evaluated on k vectors. Note the oft-used special case $k = 1$ in which (21.68) reduces to[26]

$$d\alpha(\xi_1, \xi_2) = \xi_1\, \alpha(\xi_2) - \xi_2\, \alpha(\xi_1). \tag{21.69}$$

Proof. Choose affine coordinates $x^1, \dots, x^n$ as in the proof of Theorem 21.50. By multilinearity suffices to verify (21.68) when each

$$\xi_j = \partial/\partial x^{i_j} \tag{21.70}$$

is a basis vector for some $1 \leqslant i_1, \dots, i_{k+1} \leqslant n$. If any two indices are equal, then it is straightforward to check that both sides of (21.68) vanish using the alternating property of differential forms. Furthermore, by the skew-symmetry property of differential forms it suffices to assume that $1 \leqslant i_1 < \cdots < i_{k+1} \leqslant n$. Write $\alpha = \alpha_I \, dx^I$ as a sum over increasing indices of length k. Then with (21.70) the right hand side of (21.68) is

$$\sum_j (-1)^j \frac{\partial}{\partial x^{i_j}} \alpha_{i_1 \cdots \not{i_j} \cdots i_{k+1}}. \tag{21.71}$$

Using the pairing (21.38) we compute the left hand side as

$$\Big\langle \sum_{j,I} \frac{\partial \alpha_I}{\partial x^j} dx^j \wedge dx^I \,,\, \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_{k+1}}} \Big\rangle = \sum_j (-1)^j \frac{\partial}{\partial x^{i_j}} \alpha_{i_1 \cdots \not{i_j} \cdots i_{k+1}}. \tag{21.72}$$

The terms in the first sum which contribute are those in which the multi-index jI of length $k+1$ is a permutation of $i_1, \dots, i_{k+1}$; the sign is the determinant of the permutation matrix. □

(21.73) *The differential is local.* The next proposition asserts that the differential does not increase supports.

Proposition 21.74. *Let $U \subset \mathbb{A}^n$ be an open set. If for some open $W \subset X$ we have* $\operatorname{supp} \alpha \subset W$*, then* $\operatorname{supp} d\alpha \subset W$*.*

Proof. This locality follows immediately from the explicit formula (21.61) for the differential. □

Lecture 22: Pullbacks, differential forms on smooth manifolds

There remains one more item to consider about differential forms on affine space: pullbacks. We do so in the first part of this lecture. Then we turn the local-to-global process which takes us

[26] There is a generalization on smooth manifolds in which ξ_1, ξ_2 are vector fields. In that case there is an additional term in (21.69) which involves the *Lie bracket* of the vector fields.

from differential forms on open sets in affine space to differential forms on smooth manifolds. As you might guess, partitions of unity are a key tool. We characterize of differential forms among alternating multilinear maps from vector fields to functions. This introduces the key idea of *linearity over functions*, which is the hallmark of *tensoriality.* Finally, we transport the Cartan d operator from affine space to a smooth manifold and so complete the construction of the de Rham complex.

Pullbacks of differential forms

(22.1) *Pullbacks of 0-forms and 1-forms.* Let V, V' be normed linear spaces; A, A' affine spaces over V, V', respectively; $U \subset A$, $U' \subset A'$ open sets; and $\varphi\colon U' \to U$ a C^1 map. Suppose first that $f\colon U \to \mathbb{R}$ is a smooth function. Then the *pullback* function $\varphi^* f\colon U' \to \mathbb{R}$ is the composition

$$\varphi^* f = f \circ \varphi. \tag{22.2}$$

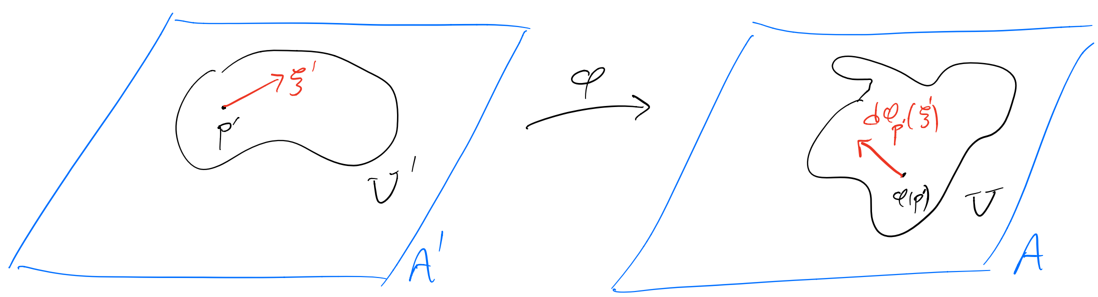


Figure 68. Pullback of a 1-form

The pullback of a 1-form is a bit more complicated. Namely, at each $p' \in U'$ the differential of φ is a linear map

$$d\varphi_{p'}\colon V' \longrightarrow V. \tag{22.3}$$

Its dual is a continuous linear map

$$d\varphi^*_{p'}\colon V^* \longrightarrow (V')^* \tag{22.4}$$

defined by

$$d\varphi^*_{p'}(\theta)(\xi') = \theta(d\varphi_{p'}(\xi')), \qquad \theta \in V^*, \quad \xi' \in V'. \tag{22.5}$$

Then if $\alpha\colon U \to V^*$ is a smooth 1-form on U, its pullback $\varphi^*\alpha\colon U' \to (V')^*$ is the smooth 1-form on U' defined by (see Figure 68)

$$\varphi^*(\alpha)_{p'}(\xi') = \alpha_{\varphi(p')}(d\varphi_{p'}(\xi')), \qquad p' \in U', \quad \xi' \in V'. \tag{22.6}$$

Write 'φ_*' for the differential $d\varphi$. Equation (22.6) is more transparent in the form

$$\varphi^*\alpha(\xi') = \alpha(\varphi_* \xi'). \tag{22.7}$$

Example 22.8. In practice, pullback is computed by blind substitution. Consider the 1-form

$$\alpha = y\,dx - x\,dy \tag{22.9}$$

on $\mathbb{A}^2_{x,y}$, and let $\gamma\colon \mathbb{R}_t \to \mathbb{A}^2_{x,y}$ be the map defined by

$$\begin{aligned} x &= t \\ y &= t^2 \end{aligned} \tag{22.10}$$

The left hand sides of (22.10) are shorthand for the pullbacks γ^*x, γ^*y of the coordinate functions. In a similar vein, blindly applying d we obtain the pullbacks $\gamma^*(dx), \gamma^*(dy)$ of their differentials:

$$\begin{aligned} dx &= dt \\ dy &= 2t\,dt \end{aligned} \tag{22.11}$$

Then the pullback $\gamma^*\alpha$ is computed by plugging (22.10) and (22.11) into (22.9):

$$\alpha = t^2dt - t(2t\,dt) = -t^2dt. \tag{22.12}$$

We use implicitly that γ^* and d commute, as we prove below in **(22.26)**.

(22.13) *Pullbacks in arbitrary degrees.* Let V, V' be real vector spaces. Let A, A' be affine over V, V' and $U \subset A$, $U' \subset A'$ open subsets. Finally, let $\varphi\colon U' \to U$ be a C^1 map and $p' \in U'$. Then the differential and its dual give linear maps

$$\begin{gathered} V' \xrightarrow{d\varphi_{p'}} V \\ (V')^* \xleftarrow{d\varphi^*_{p'}} V^* \\ {\textstyle\bigwedge}^\bullet (V')^* \xleftarrow{\bigwedge d\varphi^*_{p'}} {\textstyle\bigwedge}^\bullet V^* \end{gathered} \tag{22.14}$$

where the last map is the induced map **(20.55)** on the exterior algebra.

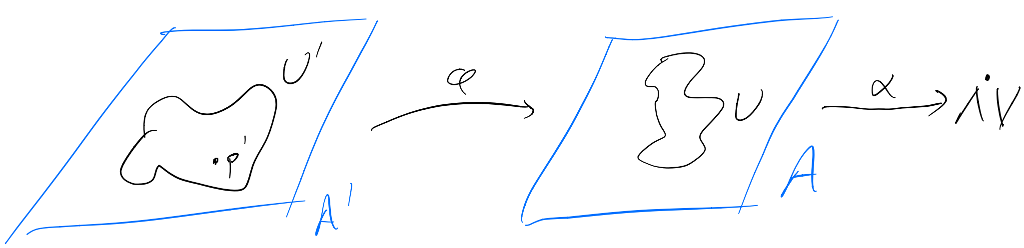


FIGURE 69. Pullback of differential forms

Definition 22.15. Let $\alpha \in \Omega^\bullet(U)$ be a differential form. The *pullback* $\varphi^*\alpha \in \Omega^\bullet(U')$ is

$$(\varphi^*\alpha)_{p'} = {\textstyle\bigwedge} d\varphi^*_{p'}(\alpha_{\varphi(p')}), \qquad p' \in U'. \tag{22.16}$$

This definition is a generalization of (22.6).

(22.17) *Pullbacks and products.* The following is an immediate consequence of the fact that $\bigwedge d\varphi^*_{p'}$ is an algebra homomorphism.

Proposition 22.18. *If $\alpha, \beta \in \Omega^\bullet(U)$, then*

$$\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta. \tag{22.19}$$

(22.20) *Computation.* Definition 22.15 is not applied directly in computations; rather one plugs and chugs as illustrated in Example 22.8. Here is another illustration.

Example 22.21. Recall the 2-form α in (21.55). Consider the map

$$\begin{aligned} \varphi\colon (0,\pi) \times (0,2\pi) &\longrightarrow \mathbb{A}^3_{x,y,z} \\ \phi \quad , \quad \theta \quad &\longmapsto \sin\phi\cos\theta\,,\ \sin\phi\sin\theta\,,\ \cos\phi \end{aligned} \tag{22.22}$$

This embeds an open rectangle in $\mathbb{A}^2_{\phi,\theta}$ into the unit sphere in $\mathbb{A}^3$. To compute we write

$$\begin{aligned} x &= \sin\phi\cos\theta \\ y &= \sin\phi\sin\theta \\ z &= \cos\phi \end{aligned} \tag{22.23}$$

and then apply d:

$$\begin{aligned} dx &= \cos\phi\cos\theta\,d\phi - \sin\phi\sin\theta\,d\theta \\ dy &= \cos\phi\sin\theta\,d\phi + \sin\phi\cos\theta\,d\theta \\ dz &= -\sin\phi\,d\phi. \end{aligned} \tag{22.24}$$

Substitute (22.23) and (22.24) into (21.55) and use the rules of exterior algebra to deduce

$$\varphi^*\alpha = \sin\phi\,d\phi \wedge d\theta. \tag{22.25}$$

In computations it is customary to omit ‘ φ^* ’ in (22.25), as we did in (22.12).

(22.26) *Pullback and d.* Just as pullback commutes with products (Proposition 22.18), pullback also commutes with exterior d. In the following proof we use the fact that any differential form on a finite dimensional affine space A is a finite sum of expressions

$$f\,dg_1 \wedge \cdots \wedge dg_k \tag{22.27}$$

for functions $f, g_1, \dots, g_k$. For example, choose affine coordinates on A and then exhibit a representation in which $g_1, \dots, g_k$ are the affine coordinate functions; see Theorem 21.17(2) as applied in (21.60). This leads to an important proof technique for statements about differential forms, which we employ to prove the following.

Proposition 22.28. *Assume A, A' are finite dimensional affine spaces. Let $\alpha \in \Omega^\bullet(U)$. Then*

$$d\varphi^*\alpha = \varphi^* d\alpha. \tag{22.29}$$

Proof. Since pullback commutes with products (Proposition 22.18), by the remark preceding the proof it suffices to prove (22.29) for 0-forms f and for exact 1-forms dg, where $f, g \in \Omega^0(U)$. For $\xi' \in V'$ we have

$$\varphi^* df(\xi') = df(\varphi_*\xi') = (df \circ d\varphi)(\xi') = d(f \circ \varphi)(\xi') = d(\varphi^* f)(\xi'). \tag{22.30}$$

For an exact 1-form $\alpha = dg$ we compute

$$\begin{aligned} d(\varphi^*\alpha) &= d(\varphi^* dg) = d^2\varphi^* g = 0 \\ \varphi^*(d\alpha) &= \varphi^*(d^2 g) = 0 \end{aligned} \tag{22.31}$$

so (22.29) is satisfied. ∎

Differential forms on manifolds

(22.32) *0-forms and 1-forms.* Let X be a smooth manifold. The real vector spaces

$$\begin{aligned} \Omega^0(X) &= \{\text{functions } X \to \mathbb{R}\} \\ \Omega^1(X) &= \{\text{sections of } T^*X \longrightarrow X\} \end{aligned} \tag{22.33}$$

of smooth functions and smooth 1-forms have been defined, and the differential is a linear map

$$\Omega^0(X) \xrightarrow{\;d\;} \Omega^1(X) \tag{22.34}$$

which satisfies the Leibniz rule

$$d(f_1 f_2) = d(f_1)f_2 + f_1 d(f_2). \tag{22.35}$$

Our goal is to extend these definitions to higher degree forms.

(22.36) *Bundles of exterior algebras.* Let $E \to X$ be a real vector bundle. Recall from Lecture 9 that we can construct new vector bundles from a given vector bundle from a functorial construction on vector spaces. The dual bundle **(9.51)** is an example; the general procedure is discussed in **(9.44)**. Now for each $k \in \mathbb{Z}^{>0}$ we construct a vector bundle $\bigwedge^k T^*X \to X$ from the cotangent bundle $T^*X \to X$ using Theorem 9.45. The same construction gives a bundle $\bigwedge^\bullet T^*X \to X$ whose fibers are exterior algebras. Define the vector space of *differential k-forms*

$$\Omega^k(X) = \bigl\{\text{sections of } \textstyle\bigwedge^k T^*X \longrightarrow X\bigr\}. \tag{22.37}$$

The algebra of differential forms is the direct sum

$$\Omega^\bullet(X) = \bigoplus_{k=0}^{\infty} \Omega^k(X) \tag{22.38}$$

with the wedge product defined pointwise; it may be identified as the algebra of sections of $\bigwedge^\bullet T^*X \to X$. It is important to observe the *pointwise* nature of the algebraic structure, namely of the wedge product of differential forms.

(22.39) *Vector fields.* A vector field[27] on a manifold is a smooth choice of tangent vector at each point. The vector space of vector fields on X is

$$\mathfrak{X}(X) = \bigl\{\text{sections of } TX \longrightarrow X\bigr\}. \tag{22.40}$$

(22.41) *Differential forms as functionals of vector fields.* Recall that if V is a real vector space, then there is a duality pairing (Proposition 21.37)

$$\textstyle\bigwedge^k V^* \times \bigwedge^k V \longrightarrow \mathbb{R} \tag{22.42}$$

Combine with the multiplication map $V \times \cdots \times V \to \bigwedge^k V$ to construct an isomorphism between $\bigwedge^k V^*$ and the vector space of k-linear alternating maps $V \times \cdots \times V \to \mathbb{R}$.

If X is a smooth manifold and $\alpha \in \Omega^k(X)$ is a differential k-form, then pointwise evaluation determines an alternating k-linear map

$$\widehat{\alpha}\colon \mathfrak{X}(X) \times \cdots \times \mathfrak{X}(X) \longrightarrow \Omega^0(X) \tag{22.43}$$

where for $\xi_1, \dots, \xi_k \in \mathfrak{X}(X)$ we have

$$\widehat{\alpha}(\xi_1, \dots, \xi_k)(p) = \alpha_p\bigl(\xi_1(p), \dots, \xi_k(p)\bigr), \qquad p \in X. \tag{22.44}$$

[27]In general an *x-field* on a manifold is a smooth choice of x at each point, where x can be 'vector', 'covector', 'tensor', 'scalar', 'spinor', ...

There is an additional important property beyond skew-symmetry and multilinearity: $\widehat{\alpha}$ is *linear over functions*. Namely, $\widehat{\alpha}$ is a multilinear function over the ground ring $\Omega^0(X)$, not just over the ground field $\mathbb{R}$. Thus if $f_1, \dots, f_k \in \Omega^0(X)$ are functions, and $\xi_1, \dots, \xi_k \in \mathfrak{X}(X)$ are vector fields,

$$\widehat{\alpha}(f_1\xi_1, \dots, f_k\xi_k) = f_1 \dots f_k\, \widehat{\alpha}(\xi_1, \dots, \xi_k). \tag{22.45}$$

The following proposition is a converse: it constructs a differential form from a map (22.43).

Proposition 22.46. *Let*

$$\widehat{\alpha}\colon \mathfrak{X}(X) \times \cdots \times \mathfrak{X}(X) \longrightarrow \Omega^0(X) \tag{22.47}$$

be a k-linear alternating map which is linear over functions in the sense that (22.45) holds. Then there is a unique differential k-form $\alpha \in \Omega^k(X)$ such that (22.44) holds.

A function (22.47) which is linear over functions is said to be *tensorial*. Proposition 22.46 holds for other kinds of tensor fields on a smooth manifold. For example, a Riemannian metric is (determined by) a tensorial positive definite symmetric bilinear form on vector fields.

Example 22.48. Let $U \subset A$ be an open subset of an affine space A with tangent space V, and fix a smooth function $f \in \Omega^0(U)$ which is not locally constant. Define

$$\widehat{\alpha}_f(\xi_1, \xi_2) = \xi_1\xi_2 f \;-\; \xi_2\xi_1 f, \qquad \xi_1, \xi_2 \in \mathfrak{X}(U). \tag{22.49}$$

Each term in (22.49) is an iterated directional derivative. Then $\widehat{\alpha}_f$ is bilinear and alternating, but it is not linear over functions so does not define a 2-form on U.

Proof of Proposition 22.46. For ease of notion we take $k = 1$. For $p \in X$ and $\xi_p \in T_pX$, we would like to define

$$\alpha_p(\xi_p) = \widehat{\alpha}(\xi)(p), \tag{22.50}$$

where $\xi \in \mathfrak{X}(X)$ is any vector field such that $\xi(p) = \xi_p$, i.e., ξ is an extension of the vector $\xi_p \in T_pX$ to a vector field. We must prove that (22.50) is independent of the extension ξ. Equivalently, if $\eta \in \mathfrak{X}(X)$ satisfies $\eta(p) = 0$, then we claim $\widehat{\alpha}(\eta)(p) = 0$.

To prove the claim, let $(U; x^1, \dots, x^n)$ be a standard coordinate chart about p and let $\rho\colon X \to \mathbb{R}$ be a function with $\operatorname{supp}(\rho) \subset U$ and $\rho(p) = 1$. Write $\eta\big|_U = f^i\, \partial/\partial x^i$ for functions $f^i \in \Omega^0(U)$. Since $\eta(p) = 0$, we have $f^i(p) = 0$ for all i. Now write

$$\eta = (\rho f^i)\left(\rho\frac{\partial}{\partial x^i}\right) \;+\; (1-\rho^2)\eta. \tag{22.51}$$

Notice that each factor in each term is a global function or a global vector field on X (after extension by zero on the complement of U). By the hypothesis that $\widehat{\alpha}$ is linear over functions, we deduce

$$\begin{aligned}\widehat{\alpha}(\eta)(p) &= \rho(p) f^i(p)\, \widehat{\alpha}\left(\rho\frac{\partial}{\partial x^i}\right)(p) \;+\; \big(1-\rho^2(p)\big)\widehat{\alpha}(\eta)(p) \\ &= 0\end{aligned} \tag{22.52}$$

□

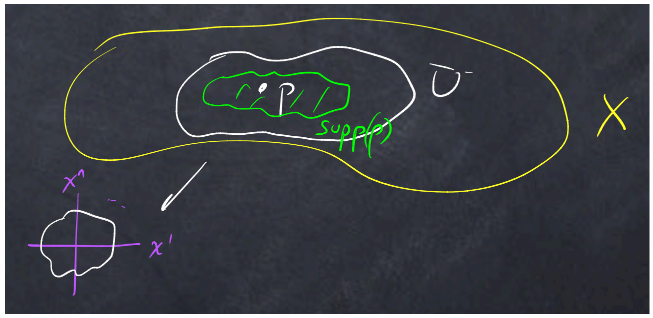


FIGURE 70. A cutoff function ρ near p

The Cartan d operator

(22.53) *The de Rham complex.* The goal is to construct the *de Rham complex*

$$0 \longrightarrow \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(X) \longrightarrow 0, \tag{22.54}$$

assuming $\dim X = n$. We have already constructed the vector spaces $\Omega^k(X)$ in (22.37), so it remains to construct the Cartan d operator. To do so we build on the affine space differential constructed in Theorem 21.50. Recall that differential forms and d on affine space satisfy properties spelled out in Proposition 21.74, Proposition 22.18 and Proposition 22.28.

(22.55) *The de Rham differential d.* The locality of d and the fact that d commutes with diffeomorphisms are key ingredients in the proof of the following.

Theorem 22.56. *Let X be a smooth manifold. Then there exists a unique map $d\colon \Omega^\bullet(X) \to \Omega^\bullet(X)$ of degree $+1$ such that*

(i) *d is linear,*
(ii) *$d(\alpha_1 \wedge \alpha_2) = d\alpha_1 \wedge \alpha_2 + (-1)^k \alpha_1 \wedge d\alpha_2, \qquad \alpha_1 \in \Omega^{k_1}(X), \quad \alpha_2 \in \Omega^\bullet(X),$*
(iii) *$d^2 = 0$,*
(iv) *d agrees with the usual differential on $\Omega^0(X)$.*

Furthermore, for all $\alpha \in \Omega^\bullet(X)$ we have $\operatorname{supp}(d\alpha) \subset \operatorname{supp}(\alpha)$. Also, if $\varphi\colon X' \to X$ is a smooth map of manifolds, and $\alpha \in \Omega^\bullet(X)$, then

$$\varphi^* d\alpha = d\varphi^*\alpha. \tag{22.57}$$

The $\mathbb{Z}$-graded algebra $\Omega^\bullet(X)$ in (22.38) is upgraded to a *differential graded algebra* (DGA) $\big(\Omega^\bullet(X), d\big)$ when equipped with the Cartan d operator.

Proof. Let $\{(U_i, x_i)\}_{i\in I}$ be a cover of X by coordinate charts, and choose a partition of unity $\{\rho_i\}_{i\in I}$ subordinate to the cover. For any $\alpha \in \Omega^\bullet(X)$ we have $\operatorname{supp}(\rho_i\alpha) \subset U_i$ for all i. Define

$$d\alpha = \sum_{i\in I} d\big((x_i^{-1})^*(\rho_i\alpha)\big), \tag{22.58}$$

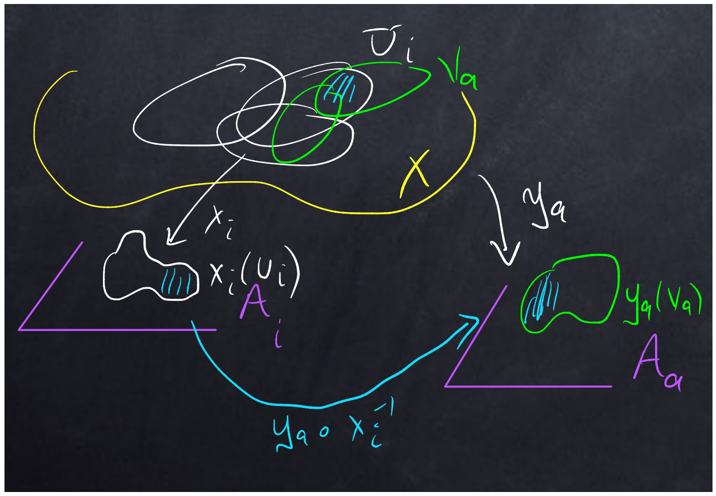


FIGURE 71. The overlap map between two charts

where the differential of $(x_i^{-1})^*(\rho_i\alpha) \in \Omega^\bullet\big(x_i(U_i)\big)$ is the one determined uniquely on affine space in Theorem 21.50. Since $\alpha = \sum_{i\in I}\rho_i\alpha$ and since d is assumed linear in (i), the definition (22.58) is uniquely determined by the conditions in the theorem. But it remains to prove that the definition (22.58) is independent of the choice of cover and partition of unity.

Suppose $\{(V_a, y_a)\}_{a\in A}$ is another cover by coordinate charts, and $\{\sigma_a\}_{a\in A}$ is a subordinate partition of unity. Then

$$\begin{aligned}\sum_{i\in I} d\big((x_i^{-1})^*(\rho_i\alpha)\big) &= \sum_{i\in I}\sum_{a\in A} d\big((x_i^{-1})^*(\rho_i\sigma_a\alpha)\big)\\ &= \sum_{a\in A}\sum_{i\in I} d\big((y_a^{-1})^*(\sigma_a\rho_i\alpha)\big)\\ &= \sum_{a} d\big((y_a^{-1})^*(\sigma_a\alpha)\big).\end{aligned} \tag{22.59}$$

Notice that the form $\rho_i\sigma_a\alpha = \sigma_a\rho_i\alpha$ has support in $U_i \cap V_a$. The passage from the first to the second line is Proposition 22.28 applied to $\varphi = y_a \circ x_i^{-1}\colon x_i(U_i\cap V_a) \to y_a(U_i \cap V_a)$. ☐

In the proof we implicitly use the fact that any two covers have a common refinement. So if a quantity defined in terms of a cover, and we want to compare the computation in two different covers, then we compare each to the computation in a common refinement. Here we use this idea to prove that the differential is well-defined. We will use the same technique shortly to prove that the integral is well-defined.

Lecture 23: Orientations and volumes

The problem of computing lengths, areas, and volumes dates from the beginnings of geometry in ancient Babylonia around 5000 years ago. Our treatment is more modern, but does not start from first principles. Rather, we tell the data needed to introduce a notion of n-dimensional volume in an n-dimensional real vector space. We do so in terms of the exterior algebra, and what naturally emerges is a notion of *signed* volume. We begin with the sign, which is the structure of an *orientation*.

Orientations

(23.1) *Intuition: dimension 1.* The notion of an orientation is familiar in low dimensions. There are two directions to traverse a curve, and an orientation is a choice between them. The linear version is a sense of direction on a 1-dimensional vector space L. This can be expressed as a choice of nonzero vector $e \in L$, but then if we multiply e by a *positive* scalar the resulting vector points in the same direction. So an orientation is a choice of nonzero vector up to positive multiple. If we consider the space $L\backslash\{0\}$ of nonzero vectors, then there are two components in the usual (norm) topology. An orientation $\mathfrak{o}$ is a choice of one component; see Figure 72. (We already gave the formal definition in Definition 13.34.)

FIGURE 72. Orientation of a line

(23.2) *Intuition: dimension 2.* Let V be a 2-dimensional vector space. Then an orientation is a sense of direction of rotation, a choice of clockwise vs. counterclockwise. A basis e_1, e_2 of V determines an orientation: the direction of rotation from e_1 to e_2; see Figure 73. If e_1', e_2' is another basis, and $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ the change-of-basis matrix defined by

$$e_j' = T^i_j e_i, \tag{23.3}$$

then e_1', e_2' and e_1, e_2 determine the same orientation if and only if $\det T > 0$. Recall the map

$$\begin{aligned} \mathcal{B}(V) &\longrightarrow \operatorname{Det} V\backslash\{0\} \\ e_1, e_2 &\longmapsto e_1 \wedge e_2 \end{aligned} \tag{23.4}$$

where $\mathcal{B}(V)$ is the set of bases of V. The foregoing tells that two bases have the same orientation map if and only if they map to the same component of $\operatorname{Det} V\backslash\{0\}$. For example, imagine fixing e_1 and rotating e_2. Compute what happens to $e_1 \wedge e_2$ in the determinant line. We are led to define an orientation of V as an orientation of $\operatorname{Det} V$: a choice of component of $\operatorname{Det} V\backslash\{0\}$, as in (23.1).

FIGURE 73. Orientation of a plane

(23.5) *Intuition: dimension 3.* An orientation of a 3-dimensional vector space is a choice of right-hand rule vs. a left-hand rule. So if e_1, e_2, e_3 is a basis of V, and the right-hand rule orientation is chosen, then the basis is positively oriented if when we take our right hand and point the fingers towards e_1 and rotate so that they curl in the direction of e_2, then our thumb and e_3 should be on the same side of the e_1-e_2 plane.

Here is the formal definition in any dimension.

Definition 23.6.

(1) Let V be a finite dimensional real vector space. An *orientation* of V is a choice $\mathfrak{o}$ of component of $\operatorname{Det} V \setminus \{0\}$.

(2) Suppose V, V' are finite dimensional, $\dim V = \dim V'$, and $\mathfrak{o}, \mathfrak{o}'$ are orientations of V, V'. An invertible linear map $T\colon V' \to V$ *preserves orientation* if $(\det T)(\mathfrak{o}') = \mathfrak{o}$. If instead $(\det T)(\mathfrak{o}') = -\mathfrak{o}$, then we say that T *reverses orientation.*

Remark 23.7. If $V = V'$, then $\det T \in \mathbb{R}^{\neq 0}$, and T preserves orientation if and only if $\det T > 0$. Notice that we do not need to choose an orientation on V to determine whether an automorphism $T\colon V \to V$ preserves or reverses orientation.

(23.8) *Bases and orientation.* If $\dim V = n$, then as in (23.4) there is a map

$$\begin{aligned} \mathcal{B}(V) &\longrightarrow \operatorname{Det} V \setminus \{0\} \\ e_1, \dots, e_n &\longmapsto e_1 \wedge \cdots \wedge e_n \end{aligned} \tag{23.9}$$

which takes a basis to a nonzero point of the determinant line. In this way a basis determines an orientation. Moreover, the inverse images of the two components of $\operatorname{Det} V \setminus \{0\}$ partition the bases into two equivalence classes, the orbits of the action of the group $\mathrm{GL}_n^+(\mathbb{R})$ of invertible $n \times n$ matrices with positive determinant.

(23.10) *Dual orientation.* Let V be a finite dimensional real vector space with an orientation $\mathfrak{o}$. There is an induced orientation $\mathfrak{o}^*$ of the dual space V^*. Namely, we say that a nonzero vector $\omega \in \operatorname{Det} V^*$ lies in $\mathfrak{o}^*$ iff

$$\langle \omega, \Xi \rangle > 0 \qquad \text{for all } \Xi \in \mathfrak{o} \subset \operatorname{Det} V, \tag{23.11}$$

where $\langle -, - \rangle$ is the duality pairing $\operatorname{Det} V^* \times \operatorname{Det} V \to \mathbb{R}$ defined in Proposition 21.37; see Figure 74.

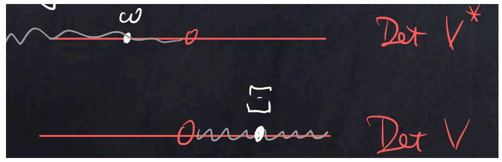


FIGURE 74. The dual orientation

Signed volume

Definition 23.12. Let V be a finite dimensional real vector space. A *volume form* is a nonzero vector $\omega \in \operatorname{Det} V^*$.

A volume form ω determines an orientation $\mathfrak{o}_\omega$ of V^*, namely the component of $\operatorname{Det} V^* \backslash \{0\}$ in which ω lies; see Figure 76. By duality it determines an orientation of V as well.

If A is affine over V, then a volume form on V determines a translation-invariant volume form on A. In linear geometry a volume form gives a notion of signed volume to parallelepipeds.

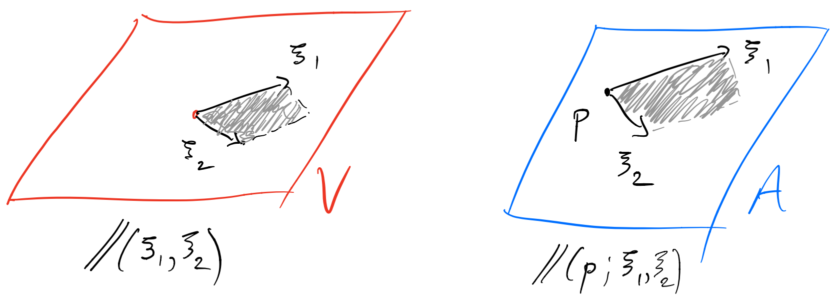


FIGURE 75. Parallelepipeds in a vector space and an affine space

Definition 23.13. Let V be a real vector space and A an affine space over V. A *k-dimensional parallelepiped* in V is the set of vectors

$$/\!\!/(\xi_1, \dots, \xi_k) = \{t^i \xi_i : 0 \leqslant t^i \leqslant 1\} \subset V \tag{23.14}$$

for vectors $\xi_1, \dots, \xi_k \in V$. A *$k$-dimensional parallelepiped* in A is the set of points

$$/\!\!/(p; \xi_1, \dots, \xi_k) = \{p + t^i \xi_i : 0 \leqslant t^i \leqslant 1\} \subset A \tag{23.15}$$

for a point $p \in A$ and vectors $\xi_1, \dots, \xi_k \in V$.

The parallelepiped is *nondegenerate* if $\xi_1, \dots, \xi_k$ are linearly independent; otherwise it is *degenerate*. There are many expressions for the same parallelepiped. In the vector case we can permute the vectors, and in the affine case we can also change the choice of vertex.

(23.16) *Oriented parallelepipeds.* If the parallelepiped is nondegenerate, then a choice of presentation which orders the vectors, as in (23.14) and (23.15), induces an orientation on the subspace spanned by the vectors, which we consider to be an orientation on the parallelepiped. Hence we speak of the *oriented parallelepiped* $/\!\!/(\xi_1, \dots, \xi_k)$.

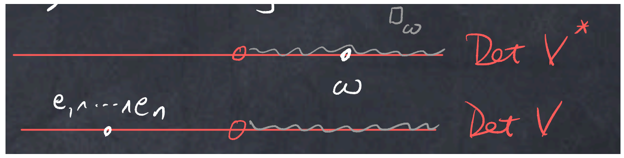


FIGURE 76. Computation of signed volume

(23.17) *Signed volume and volume.* Let $\omega \in \bigwedge^k V^*$. Then for all k-dimensional subspaces $W \subset V$, the restriction of ω to W is either zero or is a volume form on W. If $\xi_1, \dots, \xi_k$ is a linearly independent set in V, define the *signed volume* of the oriented parallelepiped spanned as

$$\omega(\xi_1 \wedge \cdots \wedge \xi_k); \tag{23.18}$$

see Figure 76. Define the volume as

$$\mathrm{Vol}\big(/\!\!/(\xi_1, \dots, \xi_k)\big) = |\omega(\xi_1 \wedge \cdots \wedge \xi_k)|\,. \tag{23.19}$$

Note the special case $\dim V = n$ and $\omega \in \operatorname{Det} V^*$, which gives a notion of (signed) n-dimensional volume.

Remark 23.20. We can also give meaning to $|\omega|$, and in fact to a line $|\operatorname{Det} V^*|$ of *densities* on V which give a notion of volume without defining signed volume.

(23.21) *Standard choices.* The vector space $\mathbb{R}^n$ has a standard orientation in which the standard basis $e_1, \dots, e_n$ is positively oriented. It has a standard volume form $e^1 \wedge \cdots \wedge e^n$, where $e^1, \dots, e^n$ is the dual basis of $(\mathbb{R}^n)^*$. The reader should check for $n = 1, 2, 3$ that (23.18) and (23.19) reproduce standard formulas for length, area, and volume.

Orientations on manifolds

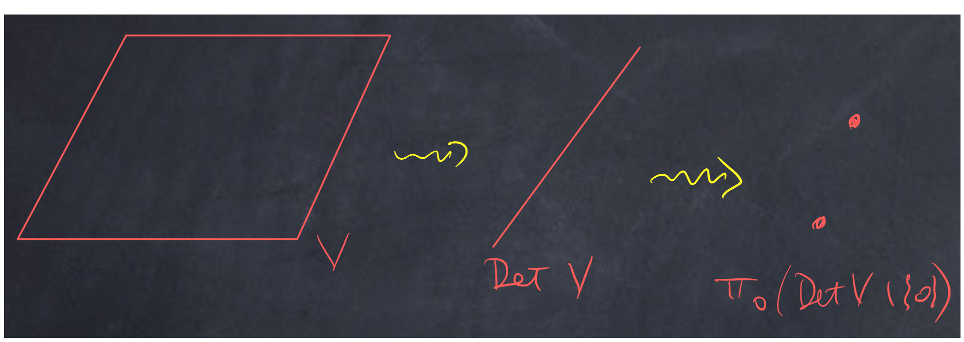


FIGURE 77. The two orientations of a finite dimensional real vector space V

(23.22) *Orientation of a smooth manifold.* Let X be a smooth manifold. Carry out the construction indicated in Figure 77 on each tangent space of X, and use local trivializations of the tangent bundle $TX \to X$ to construct a fiber bundle

$$\widehat{X} \longrightarrow X \tag{23.23}$$

whose fiber at $p \in X$ is $\pi_0\big(\operatorname{Det} T_pX \backslash \{0\}\big)$. Then (23.23) is a double cover, called the *orientation double cover* of X.

Definition 23.24. Let X be a smooth manifold of dimension n.

(1) An *orientation* $\mathfrak{o}$ of X is a section of the orientation double cover (14.11).
(2) X is *orientable* if a section of the orientation double cover exists.
(3) A *volume form* on X is a nowhere vanishing n-form $\omega \in \Omega^n(X)$.

An orientation is data; orientability is a condition. A volume form induces an orientation.

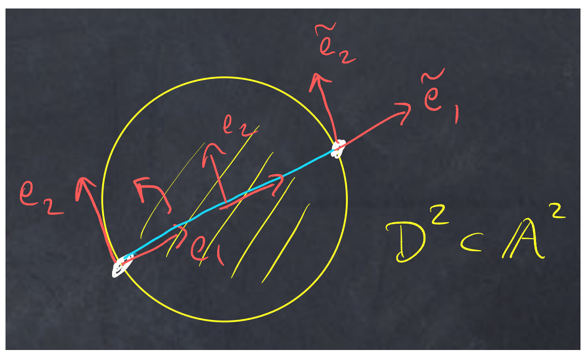


FIGURE 78. Nonorientability of the real projective plane

Example 23.25 ($\mathbb{RP}^2$ is not orientable)**.** The argument which proves this assertion is indicated in Figure 78. Represent $\mathbb{RP}^2$ as the unit disk $D^2 \subset \mathbb{A}^2$ with antipodal points of the boundary $\partial D^2 = S^1$ identified. The blue line segment pictured is an embedded $\mathbb{RP}^1 \subset \mathbb{RP}^2$, which is geometrically a circle. Suppose the pictured basis e_1, e_2 at the left endpoint of $\mathbb{RP}^1$ is positively oriented. Then transporting it along $\mathbb{RP}^1$ we must get positively oriented bases. By the time we end up at the right endpoint we deduce that the pictured basis $\tilde{e}_1, \tilde{e}_2$ is also positively oriented. However, under the differential of the antipodal map we have $\tilde{e}_1 = e_1$ and $\tilde{e}_2 = -e_2$, from which $\tilde{e}_1 \wedge \tilde{e}_2 = -e_1 \wedge e_2$, and hence the two bases are oppositely oriented. This contradiction shows that $\mathbb{RP}^2$ is not orientable.

(23.26) *Oriented charts.* An orientation of a manifold distinguishes a subset of standard charts, that is, charts with codomain $\mathbb{A}^n$ for some n.

Definition 23.27. Let X be an oriented manifold. A standard chart $(U; x^1, \dots, x^n)$ with values in $\mathbb{A}^n$ is *oriented* if for each $p \in U$ the basis $\partial/\partial x^1\big|_p, \dots, \partial/\partial x^n\big|_p$ is a positively oriented basis of T_pX.

If U is connected and $n \geqslant 1$, and if the standard chart $(U; x^1, \dots, x^n)$ is not oriented, then the standard chart $(U; -x^1, x^2, \dots, x^n)$ is oriented.

Proposition 23.28. *Let X be an oriented manifold and suppose $(U; x^1, \dots, x^n)$, $(V; y^1, \dots, y^n)$ are oriented charts. Then*

(23.29) $$\det \left(\frac{\partial x^i}{\partial y^a} \right)_{1 \leqslant i, a \leqslant n} > 0$$

on $U \cap V$.

Here we write $x^i = x^i(y^1, \dots, y^n)$; the matrix in (23.29) is the differential of the coordinate change.

Proof. Let $J \colon U \cap V \to \mathbb{R}$ be the function on the left hand side of (23.29). Then

(23.30) $$dx^1 \wedge \cdots \wedge dx^n = J\, dy^1 \wedge \cdots \wedge dy^n,$$

and since both $dx^1(p), \dots, dx^n(p)$ and $dy^1(p), \dots, dy^n(p)$ are positively oriented bases of T_p^*X, it follows that $J(p) > 0$. □

Lecture 24: Integration on manifolds

In previous lectures we passed from differential calculus on affine space to differential calculus on smooth manifolds: that is, to the differential calculus of differential forms. In this lecture we do the same for integral calculus. The natural objects to integrate on manifolds are *densities*, which are twisted differential forms. In the presence of an orientation on a manifold, densities are canonically equivalent to differential forms, and it is the integration of differential forms over *oriented* manifolds that we treat here. (It is a small variation to treat densities.) We begin with some heuristic motivation for why 1-forms and 2-forms are the natural geometric objects to integrate along curves and surfaces, respectively.

The key is the change of variables formula for the integral in affine space (Theorem 24.15). The globalization of the integral is parallel to the globalization of the differential (Theorem 22.56). In the next lecture we prove Stokes' theorem, which is a generalization of the fundamental theorem of calculus that relates integration and differentiation.

Motivation

(24.1) *A Riemann sum for 1-forms.* Let V be a normed linear space, A an affine space over V, and $U \subset A$ an open set. Suppose $C \subset U$ is a connected compact 1-manifold with boundary, which we write as the image of an injective immersion $\gamma \colon [a, b] \to U$, where $a < b$ are real numbers. Our goal is to define an integral which does not depend on the parametrization, so we will not use γ

in any essential way. Let $\alpha\colon U \to V^*$ be a 1-form[28] on U. We claim that 1-forms are the natural objects to integrate over curves. To see why, approximate C by a piecewise affine curve as follows: choose points $p_1, p_2, \dots, p_{n+1}$ "in order"[29] along C. Define $\xi_i = p_{i+1} - p_i \in V$ to be the displacement vector from p_i to p_{i+1}; see Figure 79. Then define the "Riemann sum"

$$\sum_{i=1}^{n} \alpha_{p_i}(\xi_i). \tag{24.2}$$

In essence this approximates C by a union of affine line segments p_ip_{i+1} and replaces α on each line segment by the constant 1-form with value α_{p_i}. The formula is obtained from the stipulation that the integral of a constant 1-form along an affine line segment is the pairing of the 1-form with the displacement vector. In other words, to define $\int_C \alpha$ we use the pairing

$$V^* \times V \longrightarrow \mathbb{R} \tag{24.3}$$

on a piecewise constant approximation of the integrand and a piecewise constant approximation of the region of integration.

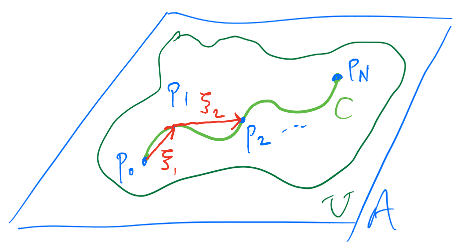


FIGURE 79. Integration of a 1-form along a curve

Remark 24.4. Most definitions of integration begin with the notion of an integral of a piecewise constant quantity and then introduce a limiting process.

(24.5) *Orientation.* Notice that (24.2) changes sign if we traverse the curve in the opposite direction. For then we sum in the opposite order—this is irrelevant since addition in $\mathbb{R}$ is commutative—but also each ξ_i is replaced by $-\xi_i$, and this does change the sign of the sum. Therefore, we need one more piece of data for the integral to be well-defined: an *orientation* of C.

(24.6) *2-forms and surfaces.* Now let $\Sigma \subset U$ be a surface. We illustrate that, modulo the question of orienting Σ, the natural object to integrate over Σ is a differential 2-form. We gave a very different motivation for introducing 2-forms in **(19.23)**.

Following the strategy of **(24.1)** choose a grid of points p_{ij} in Σ $(1 \leqslant i, j \leqslant n+1)$ as depicted in Figure 80. For example, we can suppose that we have a global parametrization by an injective

[28]It need only be continuous, not necessarily smooth, in order to define the integral, and we can also relax continuity with a more sophisticated theory of integration.

[29]Use γ as a crutch: choose $t_1 = a < t_2 < \cdots < t_{n+1} = b$ in $[a, b]$ and set $p_i = \gamma(t_i)$.

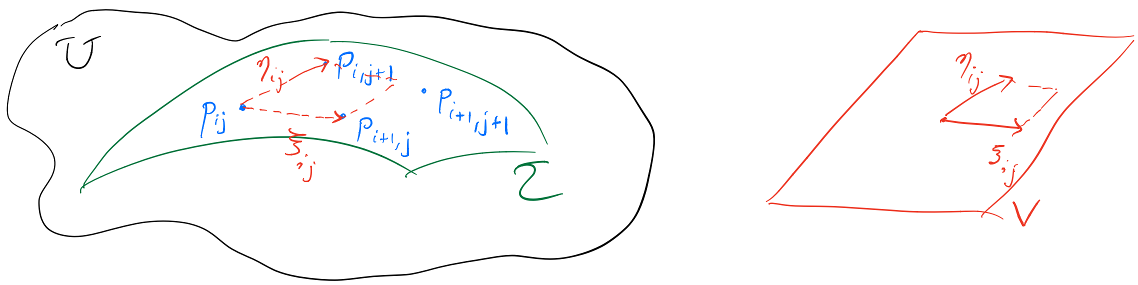


Figure 80. Piecewise parallelogram approximation of a surface

immersion from a square, and we take the p_{ij} to be the image of a lattice in the square. Then for each $i, j \in \{1, \dots, n\}$ approximate a piece of Σ by the affine parallelogram with vertices

$$p_{ij}, \quad p_{ij} + \xi_{ij}, \quad p_{ij} + \eta_{ij}, \quad p_{ij} + \xi_{ij} + \eta_{ij}, \tag{24.7}$$

where

$$\begin{aligned} \xi_{ij} &= p_{i+1,j} - p_{ij} \\ \eta_{ij} &= p_{i,j+1} - p_{ij} \end{aligned} \tag{24.8}$$

The 2-dimensional version of the Riemann sum (24.2) is

$$\sum_{i,j=1}^{n} \omega_{p_{ij}}(\xi_{ij} \wedge \eta_{ij}), \tag{24.9}$$

where the wedge product $\xi_{ij} \wedge \eta_{ij}$ of vectors represents the linear parallelogram in V spanned by ξ_{ij} and η_{ij}; see **(19.37)**. The pairing in (24.9) is

$$\textstyle\bigwedge^2 V^* \times \bigwedge^2 V \longrightarrow \mathbb{R}, \tag{24.10}$$

as defined in Proposition 21.37.

Remark 24.11. Just as the integral of a 1-form along a curve requires an orientation of the curve, so too does the integral of a 2-form over a surface. An orientation in two dimensions is heuristically a coherent sense of rotation, which here determines the ordering of the sides ξ_{ij}, η_{ij} of each parallelogram.

Integration and the change of variables formula in flat space

(24.12) *Integration of compactly supported functions.* Let $U\subset \mathbb{A}^n$ be an open set. There is a generalization of Riemann's integral in one dimension to compactly supported functions in n dimensions. It is superceded by the theory of the Lebesgue integral. In either case we obtain a linear function

$$\int_U : \Omega^0_c(U) \longrightarrow \mathbb{R} \tag{24.13}$$

It is well-define on *continuous* functions, though the notation refers to *smooth* functions. In this class we assume that this integration map on affine space is given.

(24.14) *Behavior of the integral under a diffeomorphism.* After linearity, the main property of the integral (24.13) is the following.

Theorem 24.15 (change of variables)**.** *Let $U,U'\subset\mathbb{A}^n$ be open sets and $f\colon U\to\mathbb{R}$ a bounded continuous function of compact support. Suppose $\varphi\colon U'\to U$ is a C^1 diffeomorphism. Then*

$$\int_U f = \int_{U'} \varphi^*f\,|\det d\varphi|. \tag{24.16}$$

The determinant factor is the continuous function which is the composition

$$U' \xrightarrow{\;d\varphi\;} \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^n) \xrightarrow{\;\det\;} \mathbb{R} \xrightarrow{\;|-|\;} \mathbb{R} \tag{24.17}$$

It tells the instantaneous stretching factor on volumes of the diffeomorphism φ. We have stated Theorem 24.15 for continuous functions and C^1 diffeomorphisms; in this class we will only use smooth functions and diffeomorphisms.

Example 24.18. The special case $n=1$ of (24.16) is not the usual change of variables formula you first learned in calculus, which is for *definite* integrals. Thus if into an integral you substitute

$$\begin{aligned} x &= -2y \\ dx &= -2dy \end{aligned} \tag{24.19}$$

you obtain formulas such as

$$\int_2^4 x^2\,dx = \int_{-1}^{-2} (4y^2)(-2dy). \tag{24.20}$$

On the other hand, Theorem 24.15 addresses the integral over a subset, which here we apply to a closed subset, to obtain instead

$$\int_{[2,4]} x^2\,|dx| = \int_{[-2,-1]} 4y^2\,2|dy|. \tag{24.21}$$

In (24.16) we did not write the standard measure on $\mathbb{A}^n$, which here we render in standard affine coordinates $x^1,\dots,x^n$ as $|dx^1\cdots dx^n|$. The absolute value is consonant with the absolute value in the change of variables formula (24.16).

Remark 24.22. The change of variables (24.20) treats $x^2\,dx$ as a differential 1-form on the closed interval $[2,4]$, whereas change of variables (24.21) is for the integral of the *density* $x^2\,|dx|$.

(24.23) *Interlude: densities.* Let V be an n-dimensional real vector space and $\mathcal{B}(V)$ the set of bases, i.e., isomorphisms $\mathbb{R}^n \to V$.

Definition 24.24. The *line of densities* of V is

$$|\operatorname{Det} V^*| = \{\mu\colon \mathcal{B}(V) \to \mathbb{R} : \mu(b \cdot g) = |\det g|\mu(b) \text{ for all } b \in \mathcal{B}(V),\ g \in \mathrm{GL}_n\mathbb{R}\}. \tag{24.25}$$

Let $|\operatorname{Det} V^*|_+ \subset |\operatorname{Det} V^*|$ be the ray of positive functions. A *(positive) density* is an element of $|\operatorname{Det} V^*|_+$.

Recall that $\mathcal{B}(V)$ is a right $\mathrm{GL}_n\,\mathbb{R}$-torsor, that is, the group of invertible $n \times n$ matrices acts simply transitively on $\mathcal{B}(V)$ by right composition. A density μ is a volume function on parallelepipeds in V. If A is an affine space over V, then μ defines a translation-invariant density on A, in particular a translation-invariant volume function on parallelepipeds in A.

More generally, we can consider variable densities

$$\mu\colon U \longrightarrow |\operatorname{Det} V^*| \tag{24.26}$$

defined on an open set $U \subset A$. The product of a function and a density is a density, so for example in (24.21), the integrand $x^2|dx|$ is a variable density on the closed interval $[2, 4]$.

On standard affine space $\mathbb{A}^n$ we have the standard constant density $|dx^1 \cdots dx^n|$. The integral on functions in (24.13) is obtained as the composition of multiplication $f \mapsto f\,|dx^1 \cdots dx^n|$ by the standard density with integration of densities.

(24.27) *The integral on n-forms in $\mathbb{A}^n$.* On standard flat space we use the isomorphism

$$\begin{aligned} \Omega^0_c(U) &\xrightarrow{\ \cong\ } \Omega^n_c(U) \\ f &\longmapsto \omega_f = f\,dx^1 \wedge \cdots \wedge dx^n \end{aligned} \tag{24.28}$$

to port the integral (24.13) of functions to an integral

$$\begin{aligned} \int_U : \Omega^n_c(U) &\longrightarrow \mathbb{R} \\ \omega_f &\longmapsto \int_U f \end{aligned} \tag{24.29}$$

of n-forms compactly supported in an open set $U \subset \mathbb{A}^n$. Note that under the diffeomorphism $\varphi\colon U' \to U$ in Theorem 24.15 we have

$$\varphi^*\omega_f = \det d\varphi \cdot \omega_{\varphi^* f}. \tag{24.30}$$

Therefore, the integral (24.29) (i) is linear, and (ii) satisfies the change of variables formula

$$\int_U \omega = \int_{U'} \varphi^*\omega \tag{24.31}$$

if φ is orientation-preserving so that $\det d\varphi_{p'} > 0$ for all $p' \in U'$.

(24.32) *The integral of n-forms on an arbitrary affine space.* Now let A be an affine space over an n-dimensional normed linear space V. Assume V is oriented. Then A is oriented (as an n-dimensional manifold). Let $U \subset A$ be an open subset. Then there is an integral

$$\int_U : \Omega_c^n(U) \longrightarrow \mathbb{R} \tag{24.33}$$

which can be defined by pullback from (24.29) using an *oriented* affine coordinate system $A \xrightarrow{\cong} \mathbb{A}^n$. The change of variables formula (24.31) shows that the resulting integral (24.33) is independent of the choice of oriented affine coordinate system.

Integration of differential forms on oriented manifolds

The next step is to deploy a partition of unity to globalize the integral (24.29) to a smooth manifold equipped with an orientation.

Theorem 24.34. *Let X be an oriented manifold. Then there exists a unique linear map*

$$\int_X : \Omega_c^n(X) \longrightarrow \mathbb{R} \tag{24.35}$$

such that if (U, x) is an oriented chart and $\omega \in \Omega_c^n(U)$, then

$$\int_X \omega = \int_{x(U)} (x^{-1})^*\omega. \tag{24.36}$$

We illustrate (24.36) in Figure 81.

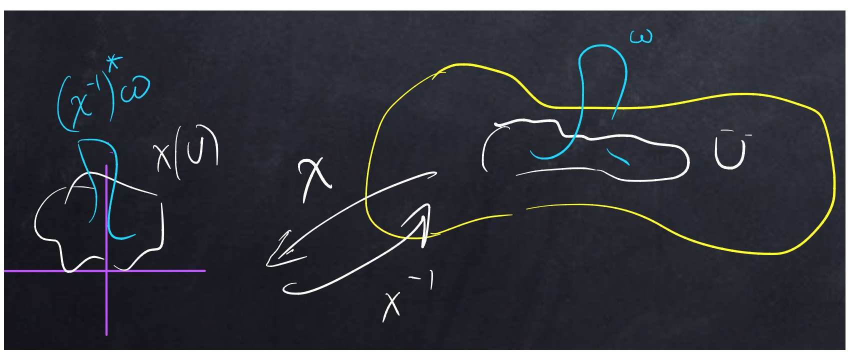


FIGURE 81. Reducing the integral on a manifold to the integral on affine space

Proof. Let $\{(U_i, x_i)\}_{i\in I}$ be a cover of X by oriented charts, and let $\{\rho_i\}$ be a subordinate partition of unity. Then $\omega = \sum_{i\in I}(\rho_i\omega)$, and $\operatorname{supp}(\rho_i\omega) \subset \operatorname{supp}(\rho_i) \cap \operatorname{supp}(\omega)$ is a compact subset of U_i. The integral of $\rho_i\omega$ is determined by (24.36), so we must have

$$\int_X \omega = \sum_i \int_{x_i(U_i)} (x_i^{-1})^*(\rho_i\omega). \tag{24.37}$$

This proves uniqueness. For existence we define the integral by (24.35) and check it is independent of choices. Thus let $\{(V_a, y_a)\}_{a\in A}$ be another cover of X by oriented charts and $\{\sigma_a\}_{a\in A}$ a subordinate partition of unity. Then

$$\begin{aligned}\int_X \omega &= \sum_i \int_{x_i(U_i)} (x_i^{-1})^*(\rho_i\omega)\\ &= \sum_i\sum_a \int_{x_i(U_i\cap V_a)} (x_i^{-1})^*(\rho_i\sigma_a\omega)\\ &= \sum_a\sum_i \int_{y_a(U_i\cap V_a)} (x_i\circ y_a^{-1})^*(x_i^{-1})^*(\rho_i\sigma_a\omega)\\ &= \sum_a\sum_i \int_{y_a(U_i\cap V_a)} (y_a^{-1})^*(\rho_i\sigma_a\omega)\\ &= \sum_a \int_{y_a(V_a)} (y_a^{-1})^*(\sigma_a\omega)\end{aligned} \tag{24.38}$$

as desired. The passage to the third line is the change of variables formula (24.31), which applies since all charts are positively oriented. ☐

We leave the proof of the following to the reader.

Proposition 24.39. *The integral (24.35) satisfies the following properties.*

(1) *Let $-X$ denote the oppositely oriented manifold to X. Then*

$$\int_{-X}\omega = -\int_X \omega,\qquad \omega\in\Omega_c^n(X). \tag{24.40}$$

(2) *Let X' be an oriented manifold and $\varphi\colon X'\to X$ an orientation-preserving diffeomorphism. Then*

$$\int_{X'}\varphi^*\omega = \int_X \omega,\qquad \omega\in\Omega_c^n(X). \tag{24.41}$$

Example 24.42. Let $S^2\subset \mathbb{A}^3_{x,y,z}$ be the standard unit sphere. For now we assume an orientation; below we discuss a canonical orientation of the boundary of an oriented manifold with boundary, and so S^2 inherits an orientation by virtue of being the boundary of the closed unit ball D^3. Define

$$\omega = x\,dy\wedge dz + y\,dx\wedge dz + z\,dx\wedge dy, \tag{24.43}$$

as in Example 22.21. Then $\omega\in\Omega^2(\mathbb{A}^3)$, and it restricts to an element of $\Omega^2(S^2)$, which necessarily has compact support since S^2 is compact. To integrate it we may omit a set of measure zero from S^2 and parametrize the complement. That we do via the parametrization

$$\begin{aligned}\varphi\colon (0,\pi)\times(0,2\pi) &\longrightarrow \mathbb{A}^3\\ \phi\quad,\quad\theta\quad &\longmapsto (\sin\phi\cos\theta\,,\,\sin\phi\sin\theta\,,\,\cos\phi)\end{aligned} \tag{24.44}$$

This embeds an open rectangle in $\mathbb{A}^2_{\phi,\theta}$ into the unit sphere in $\mathbb{A}^3$; the complement of the image is a closed half great circle, which is the union of the open half great circle and two points. It follows from Corollary 8.4 (see also Proposition 8.10(3)) that this complement has measure zero. To compute we write

$$\begin{aligned} x &= \sin\phi\cos\theta \\ y &= \sin\phi\sin\theta \\ z &= \cos\phi \end{aligned} \tag{24.45}$$

and then apply d:

$$\begin{aligned} dx &= \cos\phi\cos\theta\, d\phi - \sin\phi\sin\theta\, d\theta \\ dy &= \cos\phi\sin\theta\, d\phi + \sin\phi\cos\theta\, d\theta \\ dz &= -\sin\phi\, d\phi. \end{aligned} \tag{24.46}$$

These are identities among functions and 1-forms on S^2. Substitute (24.45) and (24.46) into (24.43) and use the rules of exterior algebra to deduce

$$\varphi^*\alpha = \sin\phi\, d\phi \wedge d\theta. \tag{24.47}$$

(In computations it is customary to omit 'φ^*' in (24.47).) Now apply (24.36) and (24.29) to compute

$$\begin{aligned} \int_{S^2}\omega &= \int_{(0,\pi)\times(0,2\pi)} \sin\phi\, d\phi\wedge d\theta \\ &= \int_0^\pi d\phi \int_0^{2\pi} d\theta\ \sin\phi \\ &= 4\pi, \end{aligned} \tag{24.48}$$

where in the second line we use Fubini's theorem to convert the integral over the rectangle into an iteration of integrals over an interval.

The boundary orientation

(24.49) *Quotient Before Sub.* Suppose

$$0 \longrightarrow V' \xrightarrow{\ i\ } V \xrightarrow{\ j\ } V'' \longrightarrow 0 \tag{24.50}$$

is a short exact sequence of finite dimensional vector spaces. It is elementary to prove

$$\dim V'' + \dim V' = \dim V. \tag{24.51}$$

The corresponding statement for determinant lines is a canonical isomorphism[30]

$$\mathrm{Det}\, V'' \otimes \mathrm{Det}\, V' \longrightarrow \mathrm{Det}\, V. \tag{24.52}$$

We define (24.52) in terms of three choices:

$$\begin{array}{ll} e'_1, \dots, e'_k & \text{basis of } V' \\ e''_1, \dots, e''_\ell & \text{basis of } V'' \\ \tilde{e}''_1, \dots, \tilde{e}''_\ell & \text{lifts of the } e''_j \text{ to } V \end{array} \tag{24.53}$$

Then $\tilde{e}''_1, \dots, \tilde{e}''_\ell, e'_1, \dots, e'_k$ is a basis of V, and we define (24.52) on basis elements:

$$(e''_1 \wedge \cdots \wedge e''_\ell) \otimes (e'_1 \wedge \cdots \wedge e'_k) \longmapsto \tilde{e}''_1 \wedge \cdots \wedge \tilde{e}''_\ell \wedge i(e'_1) \wedge \cdots \wedge i(e'_k) \tag{24.54}$$

The mnemonic "Quotient Before Sub" comes from the ordering in (24.54), which by experience is a convention that gives nice formulas in many situations.

(24.55) *2 out of 3.* Recall that an orientation of a finite dimensional real vector space is an orientation of its determinant line. Suppose two out of the three vector spaces V', V'', V are equipped with an orientation. Then the isomorphism (24.52) can be used to orient the remaining vector space.

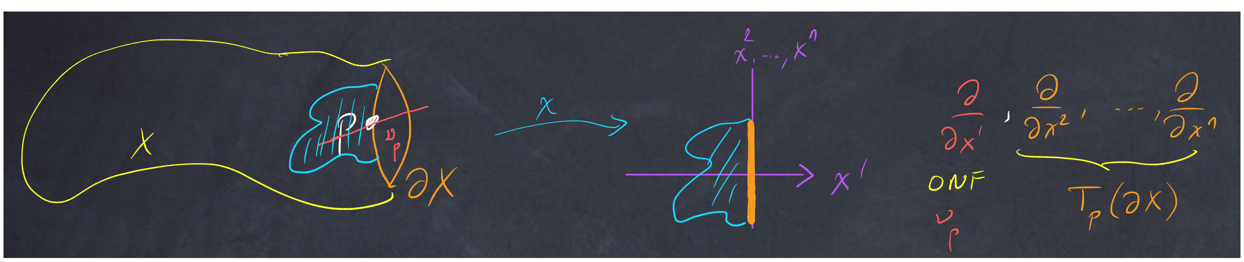


FIGURE 82. Boundary orientation in a standard boundary chart

(24.56) *The induced boundary orientation.* Suppose X is an oriented manifold with boundary. Then at $p \in \partial X$ we have the short exact sequence

$$0 \longrightarrow T_p(\partial X) \longrightarrow T_pX \longrightarrow \nu_p \longrightarrow 0 \tag{24.57}$$

where ν_p is the normal line to the boundary. In this exact sequence ν_p is oriented by the outward normal; see **(13.32)**. Since T_pX is oriented, by the 2-out-of-3 rule there is an induced orientation

[30]One can combine (24.51) and (24.52) into an isomorphism of $\mathbb{Z}$-graded lines. Also, there is a generalization for not-short but finite length exact sequences: the alternating product of the ($\mathbb{Z}$-graded) determinant lines has a canonical nonzero element, which is the determinant of the exact sequence.

of $T_p(\partial X)$. This construction proves that the boundary of an orientable manifold is orientable, and furthermore gives a canonical orientation of the boundary of an oriented manifold. Figure 82 illustrates the boundary orientation in a standard boundary chart, which is arranged so that the Quotient Before Sub convention is compatible with the standard orientation of $\mathbb{A}^n$. See (13.36) as well as Remarks 13.37 and 13.38.

Lecture 25: Stokes' theorem; oriented degree and applications

In this lecture we begin with Stokes' Theorem (Theorem 25.1), which is a generalization of the fundamental theorem of calculus (Example 25.12) and a major theorem in differential topology, despite the relative simplicity of its proof (which uses the fundamental theorem of calculus). We treat the special cases of 0- and 1-manifolds separately, since they are important in the sequel.

We then turn back to topology. Our first move is to generalize the mod 2 degree (Lectures 15 and 17) to an integer degree for maps between equidimensional *oriented* manifolds. This oriented degree is a basic invariant, and we use it to prove the Hairy Ball Theorem (Corollary 25.47) and to prove that $\mathbb{RP}^{2m}$ is not orientable for all $m \in \mathbb{Z}^{\geqslant 1}$ (Corollary 25.50).

Stokes' Theorem

The integral in Theorem 24.34 extends to manifolds with boundary in a straightforward manner. In the following Stokes' theorem we use the boundary orientation induced on an oriented manifold with boundary; see (**24.56**).

Theorem 25.1. *Let X be an oriented n-dimensional manifold with boundary, and suppose $\omega \in \Omega_c^{n-1}(X)$. Denote the inclusion of the boundary as $i\colon \partial X \to X$. Then*

$$\int_X d\omega = \int_{\partial X} i^*\omega. \tag{25.2}$$

Remark 25.3. Write the integral as a pairing between a manifold and a compactly supported differential form of top degree. Then (25.2) becomes an *adjunction*

$$\langle d\omega, X\rangle = \langle \omega, \partial X\rangle \tag{25.4}$$

between the Cartan d-operator and the boundary operator ∂.

Proof. Since both sides of (25.2) are linear in ω, in view of the definition (24.37) of the integral we reduce to the case in which $\operatorname{supp}(\omega)$ is contained in the domain U of a single standard chart (U, x). As illustrated in Figure 83 there are two types of charts: Type I contained in $\operatorname{Int}(X)$ and Type II

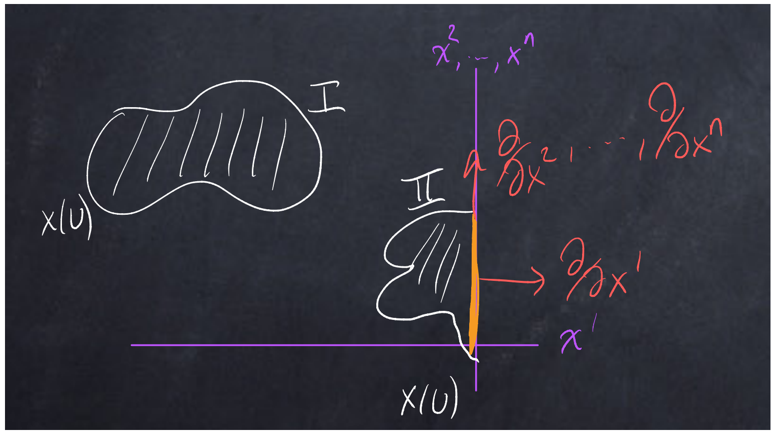


FIGURE 83. Type I and Type II standard charts on a manifold with boundary

if the intersection with ∂X is nonempty; see Lecture 13. We assume ω has been transported to $x(U) \subset \mathbb{A}^n$, and write

$$\omega = f_i \, dx^1 \wedge \cdots \wedge \cancel{dx^i} \wedge \cdots \wedge dx^n, \tag{25.5}$$

where f_i, $i = 1, \ldots, n$, is a smooth function with compact support in $x(U)$ and the expression is summed over i. Then

$$d\omega = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x^i} \, dx^1 \wedge \cdots \wedge dx^n \tag{25.6}$$

and, substituting $x^1 = 0$ into (25.5),

$$i^*\omega = f_1(0, x^2, \ldots, x^n) \, dx^2 \wedge \cdots \wedge dx^n. \tag{25.7}$$

If (U, x) is of Type I, then the left hand side of (25.2) reduces to

$$\sum_{i=1}^{n} \int_{\mathbb{A}^n} (-1)^{i-1} \frac{\partial f_i}{\partial x^i} \, |dx^1 \cdots dx^n|. \tag{25.8}$$

By Fubini we can first integrate the i^{th} term over x^i, and that definite integral vanishes by the fundamental theorem of calculus, since f_i has compact support. The right hand side of (25.2) vanishes since $\operatorname{supp}(\omega) \cap \partial X = \emptyset$.

If (U, x) is of Type II, then the left hand side reduces to (25.8), and the argument for $i = 2, \ldots, n$ is as before: those terms vanish. Only the term with $i = 1$ contributes to

$$\begin{aligned} \int_{x(U)} d\omega &= \int_{\mathbb{A}^{n-1}} |dx^2 \cdots dx^n| \int_{-\infty}^{0} dx^1 \, \frac{\partial f_1}{\partial x^1} \\ &= \int_{\mathbb{A}^{n-1}} |dx^2 \cdots dx^n| \, f_1(0, x^2, \ldots, x^n) \\ &= \int_{\partial x(U)} i^*\omega. \end{aligned} \tag{25.9}$$

At the last stage we use the fact that the oriented chart $(U; x^1, \dots, x^n)$ of X restricts to an oriented chart $(U \cap \partial X; x^2, \dots, x^n)$ of ∂X; see Figure 82. $\square$

Example 25.10. We continue Example 24.42. From (24.43) we compute $d\omega = 3\,dx \wedge dy \wedge dz$. Stokes' theorem implies

$$\int_{S^2} \omega = \int_{D^3} d\omega = 3\,\mathrm{vol}(D^3) = 3(\frac{4}{3}\pi) = 4\pi, \tag{25.11}$$

which agrees with (24.48), as it must. We leave the reader to check that with the standard orientation of D^3, which we use to identify $dx \wedge dy \wedge dz$ with the standard density on $\mathbb{A}^3$, the parametrization (24.44) is positively oriented.

Example 25.12. Theorem 25.1 generalizes the fundamental theorem of calculus, though one cannot derive the latter from the former since the proof reduces the former to the latter. Nonetheless, it is instructive to observe that if $X = [a, b] \subset \mathbb{A}^1 = \mathbb{R}$ with the standard orientation, and $f\colon [a, b] \to \mathbb{R}$ is a smooth function, then

$$\int_{[a,b]} df = \int_a^b f'(x)\,dx \tag{25.13}$$

and

$$\int_{\partial[a,b]} f = f(b) - f(a); \tag{25.14}$$

the equality of (25.13) and (25.14) is the fundamental theorem of calculus. Equation (25.14) requires additional discussion, which we provide in the next section.

The general Stokes' Theorem 25.1 specializes to Green's theorem in the plane, Stokes' theorem for surfaces, Gauss' theorem in 3-space, etc.

Orientations and integrals in zero and one dimensions

(25.15) *The zero-dimensional vector space.* For any finite dimensional real vector space V, the determinant line is the highest exterior power $\mathrm{Det}\, V = \bigwedge^n V$, $n = \dim V$. So, if $V = 0$ we have $\mathrm{Det}\, V = \bigwedge^0 V = \mathbb{R}$. The line $\mathbb{R}$ has a canonical orientation given by the positive real numbers $\mathbb{R}^{>0} \subset \mathbb{R}^{\neq 0}$. Hence there is a canonical orientation of $V = 0$, as pictured in Figure 84. We call this canonical orientation '+' and the opposite orientation '−'.

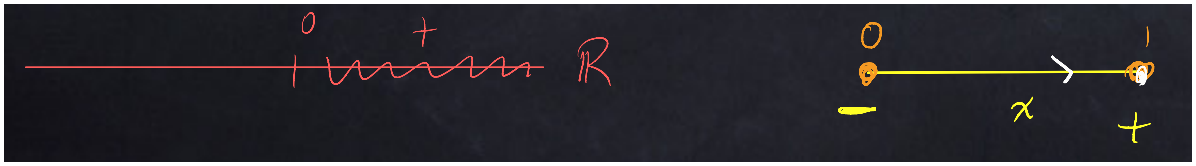


FIGURE 84. The canonical orientation of $\mathbb{R} = \operatorname{Det} 0$, and the boundary orientation of $[0, 1]$

FIGURE 85. The signed count of a compact oriented 0-manifold

(25.16) *Orientations on a zero-dimensional manifold.* A 0-manifold S is a finite or countable set of points, and by **(25.15)** an orientation of S is a function

$$\mathfrak{o}\colon S \longrightarrow \{+, -\}. \tag{25.17}$$

Definition 25.18. Let S be a compact 0-manifold with orientation $\mathfrak{o}$. Then the *signed count* of $S = \{p_1, \dots, p_N\}$ is

$$\#_s S = \sum_{i=1}^{N} \mathfrak{o}(p_i). \tag{25.19}$$

In the sum we interpret $\mathfrak{o}(p_i)$ as $+1$ or -1.

(25.20) *Boundaries of oriented 1-manifolds.* Let $X = [0, 1]$ be equipped with the standard orientation in which the vector field $\partial/\partial x$ is positively oriented. We compute the induced orientation **(24.56)** on the boundary $\partial X = \{0, 1\}$. At the point $p \in \partial X$ the short exact sequence (24.57) reduces to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_p(\partial X) & \longrightarrow & T_pX & \longrightarrow & \nu_p & \longrightarrow & 0 \\ & & \parallel & & \parallel & & \parallel & & \\ & & 0 & & \mathbb{R} & & \mathbb{R} & & \end{array} \tag{25.21}$$

The induced orientation of the zero vector space subspace is $+$ or $-$ according as the isomorphism $T_pX \to \nu_p$ preserves or reverses orientation. The vector $\partial/\partial x \in T_pX$ is positively oriented for $p \in \{0, 1\}$. At $p = 1$ the outward normal is $\partial/\partial x$, and so the induced orientation on the boundary is $+$, whereas at $p = 0$ the outward normal is $-\partial/\partial x$, and so the induced orientation on the boundary is $-$; see the second drawing in Figure 84. In other words, as oriented manifolds we have

$$\partial[0, 1] = \{1\} - \{0\}. \tag{25.22}$$

Theorem 25.23. *Let X be a compact oriented 1-manifold with boundary. Then $\#_s \partial X = 0$.*

Proof. The Classification Theorem 14.1 of 1-manifolds implies that X is a finite union of circles and closed intervals. The result now follows since $\#_s\big(\partial[0, 1]\big) = 0$ by **(25.20)**. □

(25.24) *Stokes' theorem on a closed interval.* We revisit Example 25.12. Let $a < b$ be real numbers and let $f\colon [a,b] \to \mathbb{R}$ be a smooth function. Stokes' Theorem 25.1 implies

$$\int_{[a,b]} df = \int_{\partial[a,b]} f. \tag{25.25}$$

The right hand side is the integral of a function over an oriented 0-manifold, and for that we need a special definition (or give a slightly modified exposition of Theorem 24.34; the existing account does not do well in dimension zero). Namely, if S is a 0-manifold (a finite or countable set), $\mathfrak{o}\colon S \to \{+1, -1\}$ is an orientation, and $f\colon S \to \mathbb{R}$ is a compactly supported function, then define

$$\int_S f = \sum_{p \in S} \mathfrak{o}(p) f(p). \tag{25.26}$$

With that understood, and using $df = f'(x)\,dx$, (25.25) reduces to the fundamental theorem of calculus

$$\int_a^b f'(x)\,dx = f(b) - f(a). \tag{25.27}$$

Orientation of a Cartesian product

(25.28) *Direct sum of vector spaces.* Let V', V'' be finite dimensional oriented real vector spaces. There is an induced direct sum orientation[31] on $V = V' \oplus V''$. Namely, suppose given

$$\begin{aligned} &e'_1, \dots, e'_k &&\text{basis of } V' \\ &e''_1, \dots, e''_\ell &&\text{basis of } V'' \end{aligned} \tag{25.29}$$

Then define an isomorphism

$$\operatorname{Det} V' \otimes \operatorname{Det} V'' \longrightarrow \operatorname{Det} V \tag{25.30}$$

by

$$(e'_1 \wedge \dots \wedge e'_k) \otimes (e''_1 \wedge \dots \wedge e''_\ell) \longmapsto e'_1 \wedge \dots \wedge e'_k \wedge e''_1 \wedge \dots \wedge e''_\ell \tag{25.31}$$

We use this isomorphism to induce an orientation of V from orientations of V' and V''. (In fact, if two out of three of V, V', V'' are oriented, then this isomorphism induces an orientation of the remaining vector space.)

[31] We might be tempted to apply **(24.49)** by writing the direct sum as a short exact sequence. But there are two ways to do so—we can put either V' or V'' as the sub—and if $\dim V', \dim V''$ are both odd we get opposite orientations of V. So we use the order of the direct sum to define the orientation.

(25.32) *Cartesian products.* Let X', X'' be smooth manifolds. Then the Cartesian product is naturally a smooth manifold (Example 3.8), and at $x' \in X'$, $x'' \in X''$ we have

$$T_{(x',x'')}(X' \times X'') = T_{x'}X' \oplus T_{x''}X''. \tag{25.33}$$

Hence if X', X'' are oriented, there is an induced orientation on $X' \times X''$.

In particular, if X is any oriented manifold the Cartesian product $[0,1] \times X$ is an oriented manifold with boundary, and by the discussion in **(25.20)** we deduce a canonical isomorphism

$$\partial\big([0,1] \times X\big) = \{1\} \times X \;-\; \{0\} \times X \tag{25.34}$$

of oriented manifolds; compare (25.22).

Oriented degree

(25.35) *Setup.* We resume **(15.9)** with the addition of orientations. Thus let X be an oriented compact manifold, Y an oriented connected manifold, $f\colon X \to Y$ a smooth map, and assume $\dim X = \dim Y = n$.

(25.36) *The (oriented) degree.* At a regular point $p \in X$ of f, the differential $df_p\colon T_pX \to T_{f(p)}Y$ is an isomorphism.

Definition 25.37. The *local degree* of f at a regular point p is $+1$ if df_p is orientation-preserving and is -1 if df_p is orientation-reversing.

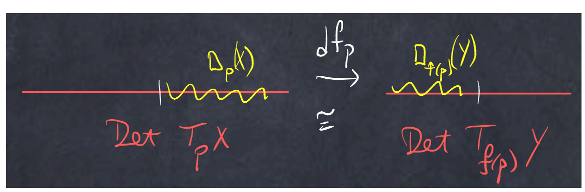


FIGURE 86. The local degree $\deg_p f$ of f at p

If $q \in Y$ is a regular value, define

$$\deg f = \sum_{p \in f^{-1}(q)} \deg_p f. \tag{25.38}$$

The right hand side depends on q, whereas the left hand side purports not to. The following summarizes the basic properties of the degree.

Proposition 25.39.

(1) *The right hand side of* (25.38) *is independent of the regular value* q.
(2) *If* $f\colon [0,1]\times X\to Y$ *is a smooth homotopy, then* $\deg f_0 = \deg f_1$.
(3) *If* W *is a compact oriented manifold with boundary and* $F\colon W\to Y$ *is a smooth map, then* $\deg \partial F = 0$.
(4) *If* $X\xrightarrow{f} Y\xrightarrow{g} Z$ *is a sequence of smooth maps, where* X,Y *are compact,* Y,Z *are connected, and* $\dim X=\dim Y=\dim Z$, *then* $\deg(g\circ f) = (\deg g)(\deg f)$.

The proof depends on yet more constructions with orientations, which we defer to the next lecture. Namely, if W,Y are oriented manifold, $Z\subset Y$ an oriented submanifold, and $F\colon W\to Y$ a smooth map transverse to Z, then the submanifold $F^{-1}(Z)\subset W$ is orientable and in fact has a canonical orientation. Furthermore, if $\dim W=\dim Y$ and $Z=\{q\}\subset Y$ is a single point, then the induced orientation at $p\in F^{-1}(q)$ agrees with the local degree $\deg_p F$. We also implicitly use a comparison of the boundary orientation of a transverse inverse image and the orientation of the transverse inverse image of a boundary, another fact we treat in the next lecture. We suggest the reader to refer back to **(15.12)** and **(17.7)** to review the proofs in the absence of orientations.

Proof. Let $t\mapsto q_t$ be a smooth path in Y between regular values q_0 and q_1. As in Figure 49 form the map $\mathrm{id}_{[0,1]}\times f\colon [0,1]\times X\to[0,1]\times Y$ and let $Z\subset[0,1]\times Y$ be the graph of $t\mapsto q_t$. Use Theorem 17.2 to perturb $\mathrm{id}_{[0,1]}\times f$ to a map F which is transverse to Z and which agrees with f on the boundary. Then $S := F^{-1}(Z)$ is a 1-dimensional submanifold of $[0,1]\times X$ and $\partial S = \{0\}\times f^{-1}(q_0)\ \amalg\ \{1\}\times f^{-1}(q_1)$. As stated before the proof, there is a canonical orientation on S, and the induced boundary orientation agrees with the local degree of f, up to sign. Now

$$\sum_{p\in f^{-1}(q_1)}\deg_p f = \sum_{p\in f^{-1}(q_1)}\deg_p f \tag{25.40}$$

follows from Theorem 25.23.

The proof of (2) is the same as that of Theorem 15.14(2), except that we include orientations and use Theorem 25.23 in place of Corollary 14.3.

The argument for (3) follows that for Proposition 17.9. Let $q\in Y$ be a simultaneous regular value of $F,\partial F$. Then $F^{-1}(q)\subset W$ is a compact 1-dimensional submanifold with $\partial F^{-1}(q)=F^{-1}(q)\cap\partial W$. It inherits an orientation. Now apply Theorem 25.23.

For (4), let $r\in Z$ be a simultaneous regular value of g and $g\circ f$. Then all points of $g^{-1}(r)$ are regular values of f. Now for $p\in(g\circ f)^{-1}(q)$ we have $d(g\circ f)_p = dg_{f(p)}\circ df_p$, and so $\deg_p(g\circ f) = \deg_{f(p)}(g)\cdot\deg_p(f)$. Hence

$$\begin{aligned}\deg(g\circ f) &= \sum_{p\in(g\circ f)^{-1}(r)}\deg_p(g\circ f)\\ &= \sum_{p\in(g\circ f)^{-1}(r)}\deg_{f(p)}(g)\cdot\deg_p(f)\\ &= \sum_{q\in g^{-1}(r)}\deg_q(g)\sum_{p\in f^{-1}(q)}\deg_p(f)\\ &= (\deg g)(\deg f).\end{aligned}\tag{25.41}$$

∎

Applications

(25.42) *The degree of the antipodal map.* Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere and let $\alpha\colon S^n \to S^n$ be the antipodal map. Note that α is the restriction of the map $-1\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$.

Theorem 25.43. *α is not homotopic to the identity map if n is even.*

Proof. We have $\deg(\mathrm{id}_{S^n}) = 1$, as it is for any identity map. We claim $\deg(\alpha) = (-1)^{n+1}$; then the theorem follows immediately from Proposition 25.39(2). For the claim, take $q = (-1, 0, \dots, 0)$, so that $f^{-1}(q)$ consists of the single point $p = (+1, 0, \dots, 0)$. Orient S^n as the boundary of the closed unit ball D^{n+1}. Then both T_pS^n and T_qS^n are identified with the subspace $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ of vectors with first coordinate zero. However, the natural isomorphism $\mathbb{R}^n \to T_pS^n$ is orientation-preserving, whereas the natural isomorphism $\mathbb{R}^n \to T_pS^n$ is orientation-reversing. The differential $d\alpha_p\colon T_pS^n \to T_qS^n$ is $-1\colon \mathbb{R}^n \to \mathbb{R}^n$ under these isomorphisms, which is orientation-preserving iff n is even. The degree computation follows. ☐

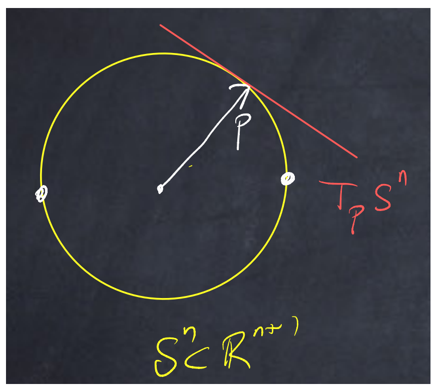


FIGURE 87. The tangent space to the sphere

(25.44) *The hairy ball theorem.* If $p \in S^n$, then

$$T_pS^n = \{\xi \in \mathbb{R}^{n+1} : \langle p, \xi \rangle = 0\}, \tag{25.45}$$

which is obtained by differentiating the defining equation $\langle p, p \rangle = 1$ of $S^n \subset \mathbb{R}^{n+1}$. If n is odd, then

$$\xi_p = (x^2, -x^1, x^4, -x^3, \dots), \qquad p = (x^1, x^2, \dots, x^{n+1}) \in S^n, \tag{25.46}$$

is a nowhere vanishing vector field on S^n.

Corollary 25.47. *If n is even, then S^n does not admit a nowhere vanishing vector field.*

Proof. If ξ is a nowhere vanishing vector field on S^n, define

$$f_t(p) = \cos(t)p + \sin(t)\xi_p, \qquad t \in [0, 1]. \tag{25.48}$$

This is a smooth homotopy from $f_0 = \mathrm{id}_{S^n}$ to $f_1 = \alpha$, which contradicts Theorem 25.43. ☐

(25.49) *Real projective space.* If n is odd, then the antipodal map $\alpha\colon S^n \to S^n$ is orientation-preserving. It is the deck transformation of the double cover $\pi\colon S^n \to \mathbb{RP}^n$, and so an orientation on S^n induces an orientation of the quotient $\mathbb{RP}^n$.

Corollary 25.50. *If n is even, then $\mathbb{RP}^n$ is not orientable.*

Proof. Assume $\mathbb{RP}^n$ is oriented. Then from the commutative diagram

(25.51)
$$\begin{array}{ccc} S^n & \xrightarrow{\ \alpha\ } & S^n \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathbb{RP}^n & \xrightarrow{\ \mathrm{id}_{\mathbb{RP}^n}\ } & \mathbb{RP}^n \end{array}$$

and Proposition 25.39(4) we deduce

(25.52)
$$\deg(\pi)\deg(\alpha) = \deg(\mathrm{id})\deg(\pi),$$

which contradicts $\deg(\alpha) = -1$. $\square$

Lecture 26: Preimage orientation; oriented degree and differential forms

We begin in this lecture by making good on a promise we made following the statement of Proposition 25.39. Namely, we orient the transverse preimage S in an oriented manifold X of an oriented submanifold $Z \subset Y$ of an oriented manifold Y. If X has nonempty boundary, then ∂S has two natural orientations: (i) the boundary orientation as the boundary of the oriented manifold S and (ii) the orientation as the preimage of Z by the boundary map $\partial X \to Y$. The comparison (26.10) of these orientations is a computation with determinant lines, and it illustrates the power of expressing orientations in terms of exterior algebra. Quite generally, algebra is used to express geometric ideas in mathematics. For smooth manifolds it is linear algebra in its many forms which appears in local computations, since a linear space—the tangent space—is attached to each point of a smooth manifold. In algebraic geometry one uses commutative algebra for both local and global computations. In topology one uses homological algebra and category theory, and these spill over into noncommutative geometry and related subjects.

In the second half of this lecture we prove a formula (Theorem 26.18) which relates the integral of differential forms to the oriented degree of a map. It generalizes the change of variables formula for the integral. It is also a first example of a formula which expresses a global quantity—the degree of a map—as an amalgamation (here an integral) of local quantities. We use this formula to define the winding number of a curve in the plane, and then introduce complex differential forms to bring out a new perspective on formulas you've seen in complex analysis.

Orientation of a transverse preimage

(26.1) *Setup.* Let X, Y be oriented manifolds, $Z \subset Y$ an oriented submanifold, and $f\colon X \to Y$ a map such that $f \pitchfork Z$. Set $S := f^{-1}(Z) \subset X$.

(26.2) *Induced orientation.* In this situation the manifold S is orientable, and in fact it carries a canonical orientation. We use the "2 out of 3" rule **(24.55)**, following "Quotient Before Sub" **(24.49)**, applied twice to the short exact sequence (7.9) of tangent spaces. Namely, for $p \in S$ we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_pS & \longrightarrow & T_pX & \longrightarrow & \nu_p(S \subset X) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle df_p} & & \big\downarrow{\scriptstyle df_p} & & \cong\big\downarrow{\scriptstyle df_p} & & \\ 0 & \longrightarrow & T_{f(p)}Z & \longrightarrow & T_{f(p)}Y & \longrightarrow & \nu_{f(p)}(Z \subset Y) & \longrightarrow & 0 \end{array} \tag{26.3}$$

in which the rows are short exact sequences. The differential $df_p\colon T_pX \to T_{f(p)}Y$ induces the left vertical map on subspaces and therefore the right vertical map on the quotients. The latter is an isomorphism since $f \pitchfork_p Z$.

The induced orientation on T_pS is obtained by a 3-step procedure:

(1) Use the orientations of Z and Y to induce an orientation of $\nu_{f(p)}(Z \subset Y)$.
(2) Use the right vertical isomorphism to transport that orientation to $\nu_p(S \subset X)$.
(3) Combine with the orientation of X to induce an orientation of T_pS.

In steps (1) and (3) we deploy the isomorphism (24.52) to induce an orientation.

Remark 26.4. The same procedure works if X is a manifold with boundary and $\partial f \pitchfork Z$ (as well as $f \pitchfork Z$). In this case there are two natural orientations of ∂S, which we compare below.

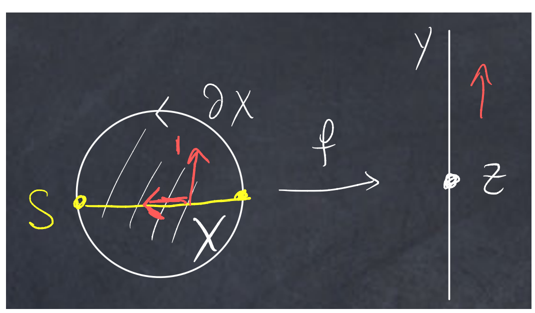


FIGURE 88. Orientation of a transverse preimage

Example 26.5. Let $X \subset \mathbb{A}^2_{x,y}$ be the closed unit disk, $Y = \mathbb{R}$, $Z = \{0\} \subset \mathbb{R}$, and $f\colon X \to \mathbb{R}$ the function $f(x,y) = y$; see Figure 88. Then $S := f^{-1}(0)$ is a diameter of the disk. Use the standard orientation of $\mathbb{A}^2$ in which $\partial/\partial x, \partial/\partial y$ is an oriented basis. The real line $\mathbb{R}$ is oriented as usual, and we take the $+$ orientation of the point $Z = \{0\} \subset \mathbb{R}$. In this case the bottom row of (26.3) degenerates: the sub is the zero vector space and the quotient map is an isomorphism.

Hence the quotient is canonically isomorphic to $T_{f(p)}\mathbb{R}$, and since we use the + orientation on the sub the induced orientation (1) of the quotient agrees with that of the ambient space: it is the usual orientation of $\mathbb{R}$. In step (2) the normal to S in X has positive orientation pointing up in the figure. In step (3) a vector pointing up is the first vector of an oriented basis of X, and so we see from Outward Normal First that $-\partial/\partial x$ is positively oriented on S.

This orientation of S induces a boundary orientation (**24.56**) of ∂S which is + at the left endpoint $(-1,0)$ and $-$ at the right endpoint $(+1,0)$. Alternatively, we can orient ∂S as the transverse inverse image of $Z \subset Y$ via the map $\partial f\colon \partial X \to Y$. In this context consider the 3-step procedure at $p = (-1,0)$. Each sub in (26.3) is the zero vector space. Steps (1) and (2) are the same as in the previous paragraph. Then at p the vector $\partial/\partial y$ is positively oriented in the normal space to p in ∂X but is negatively oriented in the tangent space to ∂X. Hence the induced orientation on ∂S at p is $-$. We leave the reader to check that the induced orientation of ∂S at $(+1,0)$ is +. Observe that the two natural induced orientations of ∂S are opposite in this case, which agrees with Proposition 26.7 below.

(26.6) *Comparison of boundary orientations.* In some applications it is only important to know that the two orientations of ∂S are equal or opposite; the precise sign does not matter. Nonetheless, we compute it.

Proposition 26.7. *Let X be an oriented manifold with boundary, Y an oriented manifold, $Z \subset Y$ an oriented submanifold, and $f\colon X \to Y$ a map such that $f, \partial f \pitchfork Z$. Then as oriented manifolds,*

$$\partial\big[f^{-1}(Z)\big] = (-1)^{\operatorname{codim}(Z\subset Y)}(\partial f)^{-1}(Z). \tag{26.8}$$

The left hand side uses the boundary orientation whereas the right hand side uses the inverse image orientation.

Proof. Set $S = f^{-1}(Z)$. We compute at $p \in \partial S$, but for convenience leave off the point in the notation. We have the exact sequences

$$\begin{array}{ccccccccc} 0 \longrightarrow & T(\partial S) & \longrightarrow & TS & \longrightarrow & \nu(\partial S \subset S) & \longrightarrow 0 \\ 0 \longrightarrow & T(\partial X) & \longrightarrow & TX & \longrightarrow & \nu(\partial X \subset X) & \longrightarrow 0 \\ 0 \longrightarrow & TS & \longrightarrow & TX & \longrightarrow & \nu(S \subset X) & \longrightarrow 0 \\ 0 \longrightarrow & T(\partial S) & \longrightarrow & T(\partial X) & \longrightarrow & \nu(\partial S \subset \partial X) & \longrightarrow 0 \end{array} \tag{26.9}$$

Let **b** denote the boundary orientation and **ii** the inverse image orientation. The dual space to a line L is denoted L^{-1}. Repeatedly apply (24.52) with care about the order, which encodes orientations, and use the convention that $-L$ is the oppositely oriented line to an oriented line:

$$
\begin{aligned}
\operatorname{Det} T(\partial S)^{(\mathbf{b})} &\cong \nu(\partial S \subset S)^{-1} \otimes \operatorname{Det} TS \\
&\cong \nu(\partial X \subset X)^{-1} \otimes \operatorname{Det} TS \\
&\cong \nu(\partial X \subset X)^{-1} \otimes \operatorname{Det} \nu(S \subset X)^{-1} \otimes \operatorname{Det} TX \\
&\cong \nu(\partial X \subset X)^{-1} \otimes \operatorname{Det} \nu(Z \subset Y)^{-1} \otimes \operatorname{Det} TX \\
&\cong \nu(\partial X \subset X)^{-1} \otimes \operatorname{Det} \nu(Z \subset Y)^{-1} \otimes \nu(\partial X \subset X) \otimes \operatorname{Det} T(\partial X) \\
&\cong (-1)^{\operatorname{codim}(Z \subset Y)} \operatorname{Det} \nu(Z \subset Y)^{-1} \otimes \operatorname{Det} T(\partial X) \\
&\cong (-1)^{\operatorname{codim}(Z \subset Y)} \operatorname{Det} \nu(\partial S \subset \partial X)^{-1} \otimes \operatorname{Det} T(\partial X) \\
&\cong (-1)^{\operatorname{codim}(Z \subset Y)} \operatorname{Det} T(\partial S)^{(\mathbf{ii})} .
\end{aligned}
\tag{26.10}
$$

Each isomorphism preserves orientation. The transversality $f \pitchfork Z$ is used to pass from line 3 to line 4, and the transversality $\partial f \pitchfork Z$ is used to pass from line 6 to line 7. □

Transitivity of diffeomorphisms on a connected manifold

As preparation for proving a relationship between degrees and integration of differential forms, we prove that on a connected manifold we can move any point to any other via a diffeomorphism. In fact, the statement is stronger: the diffeomorphism can be chosen to be smoothly homotopic to the identity map and to be the identity outside a compact set.

Definition 26.11. Let Y be a smooth manifold.

(1) The *support* of a diffeomorphism $\psi\colon Y \to Y$ is

$$\operatorname{supp}\psi = \overline{\{y \in Y : \psi(y) \neq y\}}. \tag{26.12}$$

(2) A smooth homotopy $\varphi\colon [0,1] \times Y \to Y$ is an *isotopy* if each $\varphi_t\colon Y \to Y$ is a diffeomorphism.
(3) An isotopy $\varphi\colon [0,1] \times Y \to Y$ has *compact support* if there exists a compact subset $K \subset Y$ such that $\operatorname{supp}\varphi_t \subset K$ for all $t \in [0,1]$.

Theorem 26.13. *Let Y be a smooth connected manifold and $q_0, q_1 \in Y$. Then there exists a compactly supported isotopy $\varphi\colon [0,1] \times Y \to Y$ such that $\varphi_0 = \mathrm{id}_Y$ and $\varphi_1(q_0) = q_1$.*

Proof. Define a relation $\sim$ on Y by letting $q_0 \sim q_1$ if there exists a compactly supported isotopy from the identity to a diffeomorphism which maps q_0 to q_1. Then $\sim$ is an equivalence relation.[32] We claim that each equivalence class is open. If so, then its complement is a union of open sets, so each equivalence class is also closed. It follows that there is a single equivalence class, since Y is connected, and that proves the theorem.

[32] Reflexivity and symmetry are immediate. For transitivity one needs to glue isotopies $\varphi\colon [0,1] \times Y \to Y$ and $\psi\colon [1,2] \times Y \to Y$ along $\{1\} \times Y$. While continuity of the glued function is assured, smoothness is not. To execute such gluings one can employ smooth cutoff functions on the time intervals to reparametrize φ and ψ so that they are constant on $(0.9, 1]$ and $[1, 1.1)$, respectively. Then the glued isotopy is smooth. The same device is used to glue smooth homotopies in general, and so prove that smooth homotopy is an equivalence relation.

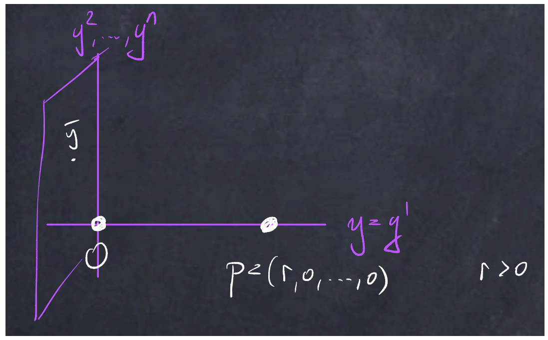


FIGURE 89. A local isotopy

For the claim it suffices to work locally in a coordinate chart, so in standard affine space $\mathbb{A}^n$; see Figure 89. We must prove that there exists $\epsilon > 0$ so that for all $q \in B_\epsilon(0)$ there exists a compactly supported isotopy from $\mathrm{id}_{\mathbb{A}^n}$ to a diffeomorphism which maps the origin to q. Fix smooth cutoff functions $\rho\colon \mathbb{A}^1 \to \mathbb{R}^{\geqslant 0}$ and $\sigma\colon \mathbb{A}^{n-1} \to \mathbb{R}^{\geqslant 0}$ such that $\rho(0) = 1$, $\operatorname{supp}\rho \subset (-1,1)$, $\sigma(0) = 1$, and $\operatorname{supp}\sigma \subset B_1(0)$. It suffices to take $q = (r; 0, \dots, 0)$ for some $r > 0$, since we can always compose with a rotation. Define the map $\varphi\colon [0,1] \times \mathbb{A}^n \to \mathbb{A}^n$ by

$$\varphi_t(y;\bar y) = (y + t\rho(y)\sigma(\bar y)r\,;\,\bar y), \qquad y \in \mathbb{A}^1, \quad \bar y \in \mathbb{A}^{n-1}. \tag{26.14}$$

Then $\varphi_0 = \mathrm{id}_{\mathbb{A}^n}$, $\varphi_1(0;0) = q$, the map φ has compact support, and $\varphi_t(-;\bar y)$ is a monotonic nondecreasing function $\mathbb{R} \to \mathbb{R}$, so φ_t is bijective for all $t \in [0,1]$. It remains to prove that ϕ_t^{-1} is smooth. That follows from the inverse function theorem if we can show that the differential is bijective. Letting I_{n-1} denote the $(n-1)\times(n-1)$ identity matrix, we have

$$d(\varphi_t)_{(y;\bar y)} = \begin{pmatrix} 1 + t\rho'(y)\sigma(\bar y)r & * \\ 0 & I_{n-1}\end{pmatrix} \tag{26.15}$$

as a block $\bigl(1 + (n-1)\bigr) \times \bigl(1 + (n-1)\bigr)$ matrix. Choose $\epsilon = 1/\bigl(2\max|\rho'(y)|\bigr)$. Then the upper left entry is positive for all $t, y, \bar y$, from which $d(\varphi_t)$ is invertible. Apply the inverse function theorem to complete the proof. ∎

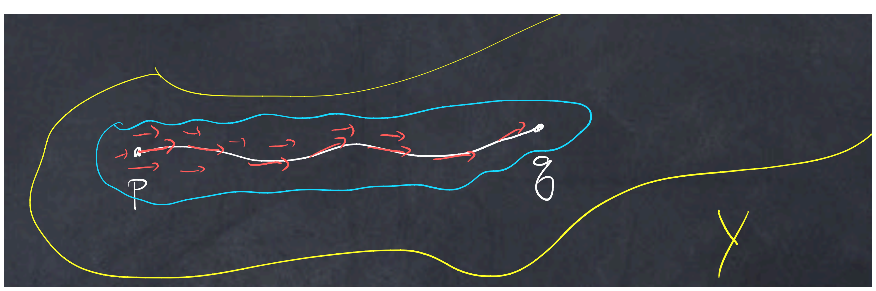


FIGURE 90. Moving p to q by integrating a vector field

Remark 26.16. Here is a sketch of an alternative proof. Construct an embedding $\gamma\colon (-0.1, 1.1) \to Y$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Extend the image of the vector field $\partial/\partial t$ to a vector field on Y with compact support in a neighborhood of the image of γ. The next step beyond the integral curves we discussed in **(14.27)** are the isotopies one constructs from suitable vector fields on manifolds, in particular from compactly supported vector fields. (Imagine the flow of a river; the vector field is the field of instantaneous velocities. One solves ODEs with a family of initial conditions to construct the isotopy/flow from the vector field.)

Oriented degree and integration

(26.17) *Main result.* In this section we generalize to arbitrary maps the change of variables formula Proposition 24.39(2), which applies only to orientation-preserving diffeomorphisms.

Theorem 26.18. *Let X be a compact oriented manifold, let Y be a connected oriented manifold, let $f\colon X \to Y$ be a smooth map, and assume $n = \dim X = \dim Y$. Then for $\omega \in \Omega^n_c(Y)$, we have*

$$\int_X f^*\omega = \deg(f) \int_Y \omega. \tag{26.19}$$

Note the support of $f^*\omega$ is compact, since X is assumed compact. If we normalize ω so that $\int_Y \omega = 1$, then (26.19) is an integral formula for the degree. We treat some preliminaries before proving Theorem 26.18.

Remark 26.20. This formula for degree expresses a global topological invariant—the degree—in terms of a local quantity—the differential form ω. Formulæ which relate local and global permeate differential topology and differential geometry.

(26.21) *Oriented bordism.* I have used the term 'bordism' a few times, but have neglected to give a formal definition. The one here is not perfect, since to glue bordisms we need to impose product structures near the boundary (see footnote [32]), which I do not do explicitly here. I only define *oriented* bordism, whereas the notion of bordism is much more general.

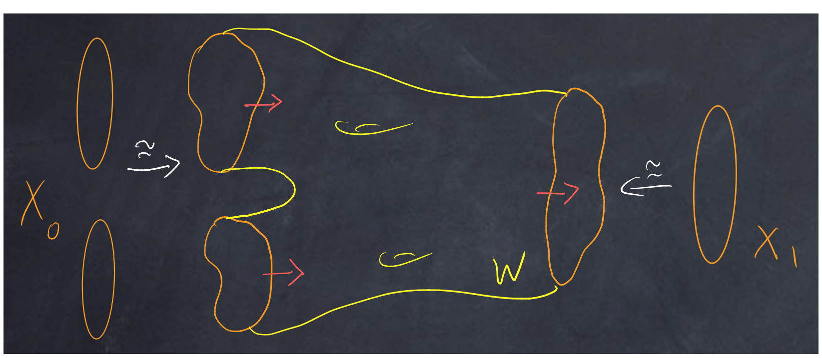


FIGURE 91. A bordism $W\colon X_0 \to X_1$. The red "arrows of time" indicate the partition of ∂W into "incoming" and "outgoing" components.

Definition 26.22. Let X_0, X_1 be closed[33] oriented manifolds of the same dimension.

(1) An *oriented bordism from* X_0 *to* X_1, denoted $W\colon X_0 \to X_1$, is a compact oriented manifold W equipped with a partition $\partial W = \partial W_0 \amalg \partial W_1$ of its boundary and orientation-preserving diffeomorphisms $X_i \xrightarrow{\approx} \partial W_i$, $i = 0, 1$.

(2) We say X_0 is *oriented bordant* to X_1 if there exists an oriented bordism $W\colon X_0 \to X_1$.

(3) Let Y be a smooth manifold and let $f_i\colon X_i \to Y$ be smooth maps. An *oriented bordism from* f_0 *to* f_1 is the data in (1) and a smooth map $f\colon W \to Y$ such that $\partial f = f_0 \amalg f_1$.

Oriented bordism is an equivalence relation on diffeomorphism classes of closed oriented manifolds of a fixed dimension. The equivalence classes form a finitely generated abelian group. These abelian groups were introduced in a sense by Poincaré, further studied by Pontrjagin and especially Thom, and they arise in many interesting geometric contexts, as well as in theoretical physics.

Remark 26.23. A *product bordism* is one diffeomorphic to $[0,1] \times X_0$. A smooth homotopy is an example of a bordism between smooth maps.

(26.24) *An oriented bordism invariant.* The following simple application of Stokes' theorem shows how an integral can produce a topological invariant. It is crucial for the proof of Theorem 26.18.

Proposition 26.25. *Let* $W\colon X_0 \to X_1$ *be an oriented bordism of* n*-manifolds* X_0 *and* X_1*. Suppose* Y *is a smooth manifold,* $f\colon W \to Y$ *a smooth map, and* $\omega \in \Omega^n(Y)$*. Denote* $\partial f = f_0 \amalg f_1$*. Then*

$$\int_{X_0} f_0^*\omega = \int_{X_1} f_1^*\omega. \tag{26.26}$$

We need not assume ω has compact support since X_0, X_1 are compact, so the integrals in (26.26) make sense. Proposition 26.25 applies in particular to a smooth homotopy.

Proof. Apply Stokes Theorem 25.1 to the compactly supported form $f^*\omega$ on W, and observe that $df^*\omega = f^*d\omega = 0$ since every $(n+1)$-form on an n-manifold is identically zero. □

(26.27) *Proof.* Fortified with Proposition 26.25 we are ready.

Proof of Theorem 26.18. Let $q \in Y$ be a regular value of f, set $f^{-1}(q) = \{p_1, \dots, p_N\}$, and define $\epsilon_i = \pm 1$, $i = 1, \dots, N$, where the sign tells if df_{p_i} is orientation-preserving or orientation-reversing. Apply the inverse function theorem to find open neighborhoods $U_i \subset X$ of p_i and $V \subset Y$ of q such that $f^{-1}(V) = U_1 \amalg \cdots \amalg U_N$ and $f\big|_{U_i}\colon U_i \to V$ is a diffeomorphism. Then Proposition 24.39 implies that (26.19) holds if $\operatorname{supp}(\omega) \subset Y$:

$$\begin{aligned}\int_X f^*\omega &= \sum_{i=1}^N \int_{U_i} f^*\omega \\ &= \sum_{i=1}^N \epsilon_i \int_Y \omega \\ &= (\deg f) \int_V \omega.\end{aligned} \tag{26.28}$$

[33]Recall that a *closed manifold* is a compact manifold without boundary.

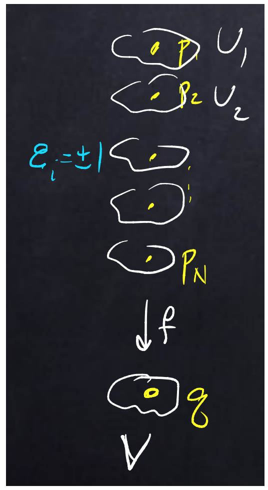


FIGURE 92. An evenly covered open subset of Y

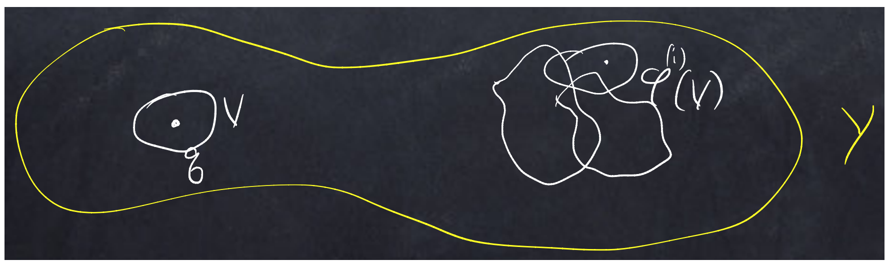


FIGURE 93. A cover of Y by diffeomorphic images of V

Next, apply Theorem 26.13 to construct a cover $\{\varphi^{(i)}(V)\}_{i \in I}$ of Y by images of V under diffeomorphisms isotopic to the identity; in particular, these diffeomorphisms are orientation-preserving. Choose a partition of unity subordinate to this cover and use it to write any $\omega \in \Omega^n(Y)$ as a sum of forms with support in an open set of the cover. It then suffices to prove (26.19) for one of these forms, since each side is linear in ω. If $\mathrm{supp}(\omega) \subset \varphi^{(i)}(V)$, then

$$\begin{aligned} \int_X f^*\omega &= \int_X (\varphi^{(i)} \circ f)^*\omega \\ &= \int_X f^*\big((\varphi^{(i)})^*\omega\big) \\ &= \int_Y (\varphi^{(i)})^*\omega \\ &= \int_Y \omega \end{aligned} \tag{26.29}$$

In the first and last line we use the homotopy invariance of the integral (Proposition 26.25), and in the third line we use the special case proved in (26.28). □

Example: the winding number as an integral

In this section we illustrate Theorem 26.18 via an extended example. Take $Y = \mathbb{A}^2 \backslash \{(0,0)\}$ and $X = S^1$, both equipped with the standard orientation.[34]

(26.30) *The differential form.* In standard real coordinates x, y on $\mathbb{A}^2$, set

$$\omega = \frac{1}{2\pi}\frac{x\,dy - y\,dx}{x^2+y^2}. \tag{26.31}$$

It is useful to write (26.31) in polar coordinates r, θ. Note that whereas $r\colon Y \to \mathbb{R}$ is a global function, a maximal domain for θ is Y with an (open) ray emanating from the origin deleted. The following computation is valid no matter which ray is chosen. Differentiate[35]

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} \tag{26.32}$$

to deduce

$$\omega = \frac{1}{2\pi}d\theta. \tag{26.33}$$

From this we conclude

$$d\omega = 0. \tag{26.34}$$

(26.35) *Some complex differential forms.* Let $\mathbb{C}^\times$ denote the space of nonzero complex numbers; it is identified with Y by writing $\lambda \in \mathbb{C}^\times$ as $\lambda = x + yi$ for $i = \sqrt{-1}$. In terms of polar coordinates we have[36] $\lambda = re^{i\theta}$. Thus

$$d\lambda = dr\,e^{i\theta} + re^{i\theta}i\,d\theta, \tag{26.36}$$

from which

$$\frac{1}{2\pi i}\frac{d\lambda}{\lambda} = \frac{dr}{2\pi i r} + \frac{d\theta}{2\pi} = dh + \frac{d\theta}{2\pi} \tag{26.37}$$

where $h = \log(r)/2\pi i\colon \mathbb{C}^\times \to \mathbb{C}$. Hence from (26.33) we deduce

$$\frac{1}{2\pi i}\frac{d\lambda}{\lambda} = \omega + dh \qquad \text{on } \mathbb{C}^\times. \tag{26.38}$$

[34] Well, which orientation is "standard" on S^1 depends on which S^1 you take. One possibility for the n-sphere: view $S^n \subset \mathbb{A}^{n+1}$ as the boundary of the unit disk D^{n+1}, give $\mathbb{A}^{n+1}$ its standard orientation, restrict to an orientation of D^{n+1}, and use the boundary orientation on S^n.

[35] Equation (26.32) is only valid where $x \neq 0$, but the result holds everywhere θ is defined.

[36] This and the equation $\lambda = x + yi$ are equalities between complex-valued functions on subsets of $Y = \mathbb{C}^\times$.

Let $\mathbb{T} \subset \mathbb{C}^\times$ be the unit circle, defined by $r = |\lambda| = 1$. Then h vanishes on $\mathbb{T}$, and so

(26.39) $$\frac{1}{2\pi i}\frac{d\lambda}{\lambda} = \omega \qquad \text{on } \mathbb{T}.$$

Remove the subset of measure zero $\{1\} \subset \mathbb{T}$ and parametrize the remainder by $\lambda = e^{i\theta}$, $\theta \in (0, 2\pi)$, to carry out the computation[37]

(26.40) $$\int_{\mathbb{T}} \omega = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{d\lambda}{\lambda} = \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1.$$

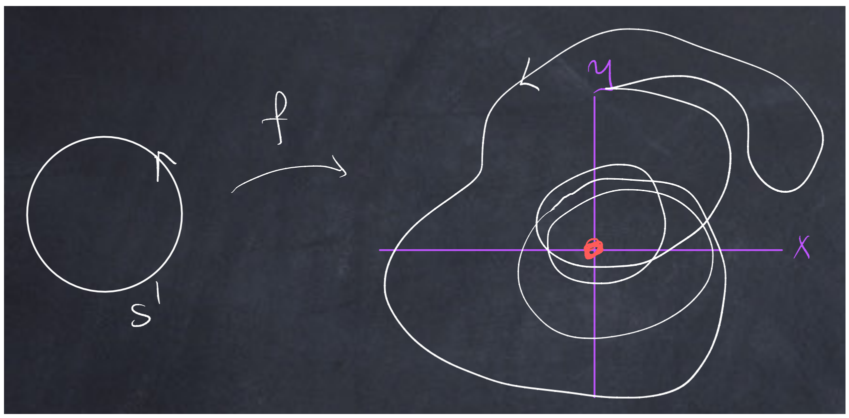


FIGURE 94. The winding number of f about $(0, 0)$

(26.41) *Application to winding number.* Let $w\colon S^1 \to \mathbb{T}$ be a smooth function. Then by Theorem 26.18 we have

(26.42) $$\deg(w) = \int_{S^1} w^*\omega.$$

Suppose $f\colon S^1 \to Y$ is a smooth function. Normalizing as in (18.4) (with $q = (0, 0)$) we obtain a function $w_f\colon S^1 \to \mathbb{T}$. The integer-valued *winding number* about the origin is defined as

(26.43) $$W(f) = \deg(w_f).$$

We claim the following integral formula for the winding number

(26.44) $$W(f) = \int_{S^1} f^*\omega.$$

Observe that the normalized function w_f is smoothly homotopic to f as functions $S^1 \to Y$. Then by Proposition 26.25 we have

(26.45) $$\int_{S^1} f^*\omega \;=\; \int_{S^1} w_f^*\,\omega,$$

from which the claim follows.

[37]We use the orientation of $\mathbb{T}$ discussed in footnote [34].

(26.46) *Counting zeros of complex functions.* Let $W \subset \mathbb{C}$ be a compact manifold with boundary, and assume for simplicity that $\partial W \approx S^1$. Suppose $f\colon W \to \mathbb{C}$ is a smooth function with isolated zeros on a finite subset $\mathrm{Zero}_W(f) = \{p_1, \dots, p_N\} \subset \mathrm{Int}(W)$. We would like to count the number of zeros of f in W. From our experience with oriented degree and oriented intersection number, and also the experience before in the mod 2 case, we know the answer which is stable under perturbations is not simply 'N' in general. We must worry about transversality, signs, etc. In the case of counting zeros, this is encoded by a 'multiplicity' of the zero. In other words, we would like to attach an integer n_i to each zero p_i which represents its multiplicity. This is familiar in the real case too: the function $x \mapsto x^2$ on the real line has a zero of multiplicity two at the origin. You know how to define the multiplicity for real polynomials, and so too for complex functions f which are polynomials. The question is how to define the multiplicity more generally.

An inspired observation is the following. Let us suppose f extends to a complex polynomial

$$P(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0, \qquad a_0, \dots, a_{n-1} \in \mathbb{C} \tag{26.47}$$

on the entire complex line, and let $p \in \mathbb{C}$ be an isolated zero of multiplicity m. Then

$$P(z) = (z-p)^m g(z) \tag{26.48}$$

for some polynomial g with $g(p) \neq 0$. Choose $\delta > 0$ so that g does not vanish on the closed disk D_δ of radius δ about p. Compute on $D_\delta \backslash \{p\}$:

$$P^*\omega = \frac{1}{2\pi i}\frac{dP}{P} = \frac{1}{2\pi i}\left[\frac{m\,dz}{z-p} + \frac{dg}{g}\right]. \tag{26.49}$$

Integrate (26.49) over ∂D_δ. Since g does not vanish on D_δ, the closed 1-form dg/g extends over D_δ, and by Stokes' theorem its integral over ∂D_δ vanishes. To integrate the first term, parametrize ∂D_δ minus a point by $z = p + \delta e^{i\theta}$, $\theta \in (0, 2\pi)$. Then $dz = \delta i e^{i\theta} d\theta$, and

$$\int_{\partial D_\delta} P^*\omega = \frac{1}{2\pi i}\int_0^{2\pi} \frac{m\delta i e^{i\theta}}{\delta e^{i\theta}}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} m\,d\theta = m. \tag{26.50}$$

This is an integral formula for the local multiplicity of the zero of a polynomial.

Recalling that $\mathrm{Zero}_W(P) = \{p_1, \dots, p_N\} \subset \mathrm{Int}(W)$, apply Stokes' theorem to the closed 1-form ω on $W' = W \backslash \amalg_i B_{\delta_i}(p_i)$, where $\delta_i > 0$ is chosen sufficiently small that f does not vanish on $B_{\delta_i}(p_i)\backslash\{p_i\}$. Orient W with the standard orientation on $\mathbb{A}^2$; the induced boundary orientations are indicated in Figure 95. Then

$$\int_{\partial W} \omega = \sum_{i=1}^N \int_{\partial D_{\delta_i}(p_i)} \omega, \tag{26.51}$$

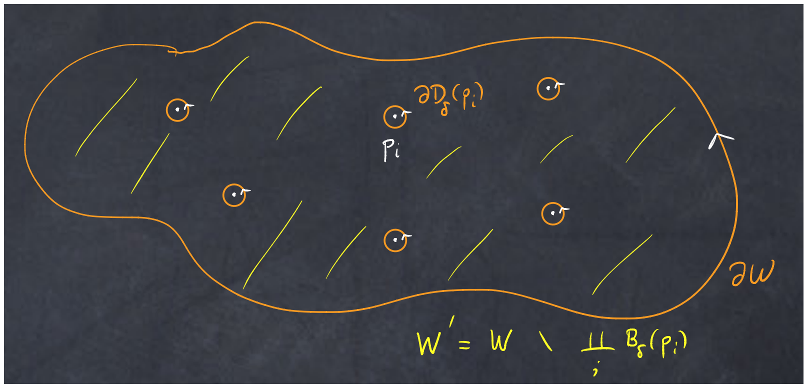


FIGURE 95. Counting zeros by an integral formula

and the right hand side is the sum of the multiplicities of the zeros of the polynomial P which are contained in W. Therefore,

$$\#_m \operatorname{Zero}_W(P) = \frac{1}{2\pi i}\int_{\partial W}\frac{dP}{P}, \tag{26.52}$$

where '$\#_m$' indicates the count with multiplicity.

Remark 26.53. We can use (26.52) to prove the fundamental theorem of algebra. Choose W to be a disk of radius R with center $0 \in \mathbb{C}$, where R is chosen so that

$$|a_{n-1}|R^{n-1} + \cdots + |a_0| < R^n. \tag{26.54}$$

Then P does not have any zeros on ∂W. Define the homotopy

$$P_t(z) = z^n + t\big(a_{n-1}z^{n-1} + \cdots + a_0\big), \qquad t \in [0,1], \tag{26.55}$$

and apply Proposition 26.25 and (26.52) to conclude that the polynomial $P_0(z) = z^n$ has the same number of zeros in W as does $P_1(z) = P(z)$. The latter has n zeros. We give a different, but closely related, proof in the next lecture.

(26.56) *Local multiplicity of holomorphic functions.* Inspired by (26.50) we might define the multiplicity of an isolated zero of a complex function $f\colon W \to \mathbb{C}$ to be

$$\int_{\partial D_\delta} f^*\omega = \frac{1}{2\pi i}\int_{\partial D_\delta}\frac{df}{f}. \tag{26.57}$$

For a *holomorphic* function f this is justified by the theorem which states that we can write f as in (26.48). The manipulations in this section with Stokes' theorem and differential forms, and the relationship to topological quantities such as the winding number, degree, and number of zeros are all ideas you saw in a first course on complex variables.

Lecture 27: Oriented intersection number; linking number

We begin by reproving the fundamental theorem of algebra. (We sketched a proof in Remark 26.53.) Then we turn to the oriented intersection number, developing some basic theory and examples. We apply this to define the linking number in the first nontrivial case, and use it to detect that the Hopf map $S^3 \to S^2$ is not homotopically trivial.

Fundamental theorem of algebra

Theorem 27.1. *Fix $n \in \mathbb{Z}^{>0}$ and suppose $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ is a polynomial with complex coefficients $a_0, \dots, a_{n-1} \in \mathbb{C}$. Then there exists $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.*

In fact, P has n zeros counted with multiplicity; see **(26.46)**.

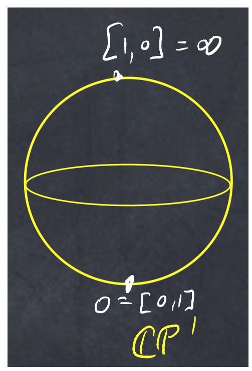


FIGURE 96. The complex projective line

Proof. Let $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ be the complex projective line. Represent points in $\mathbb{CP}^1$ as equivalence classes of ordered pairs $[z, w]$ of complex numbers, not both zero. Identify $z \in \mathbb{C}$ with $[z, 1] \in \mathbb{CP}^1$. Then we claim $f\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ defined by

$$\begin{aligned} f\big([z,1]\big) &= [P(z), 1] \\ f\big([1,0]\big) &= [1,0] \end{aligned} \tag{27.2}$$

is smooth. Namely,

$$f\big([1,w]\big) = \left[1, \frac{1}{P(\frac{1}{w})}\right], \tag{27.3}$$

and

$$\frac{1}{P(\frac{1}{w})} = \frac{w^n}{1 + a_{n-1}w + \cdots + a_0 w^n} \tag{27.4}$$

is smooth near $w = 0$ and converges to 0 as $w \to 0$.

The homotopy (26.55) induces a homotopy $f_t\colon \mathbb{CP}^1 \to \mathbb{CP}^1$, and $\deg(f) = \deg(f_1) = \deg(f_0)$ by the homotopy invariance of degree. Now $f_0(z) = z^n$ has $1 \in \mathbb{C}$ as a regular value, and $f_0^{-1}(1) = \{1, \omega, \dots, \omega^{n-1}\}$, where $\omega = e^{2\pi i/n}$ is a primitive n^{th} root of unity. The differential df_{z_0} at ω^i is the complex linear map $\mathbb{C} \to \mathbb{C}$ which is multiplication by $n\omega^{i(n-1)}$. This is the composition of a homothety and a rotation, each of which is orientation-preserving. Hence the local degree is $+1$ at each inverse image point, and so $\deg(f_0) = n$.

If P has no zeros, then $[0,1] \in \mathbb{CP}^1$ is not in the image of f, from which $\deg(f) = 0$. This contradiction proves the theorem. ∎

Oriented intersection number

(27.5) *Setup.* The setup is **(17.13)** with the addition of orientations. Namely, we have

$$\begin{array}{ll} X & \text{compact oriented manifold} \\ Y & \text{oriented manifold} \\ Z \subset Y & \textit{closed} \text{ oriented submanifold} \\ f\colon X \longrightarrow Y & \text{smooth map} \\ \dim X + \dim Z = \dim Y & \end{array} \tag{27.6}$$

(27.7) *Definition and basic properties.* If $f \pitchfork Z$, then $S := f^{-1}(Z) \subset X$ is a 0-dimensional submanifold, and it inherits an orientation by the 3-step procedure in **(26.2)**. We work out the induced orientation of S in **(27.13)** below.

Definition 27.8. The *oriented intersection number* is

$$\#^Y(f, Z) = \#(f, Z) = \#_s S. \tag{27.9}$$

The oriented intersection number satisfies the following.

Proposition 27.10. *Assume the setup of* (27.6)*.*

(1) *If $f\colon [0,1] \times X \to Y$ is a smooth homotopy, then $\#(f_0, Z) = \#(f_1, Z)$.*
(2) *If W is a compact oriented manifold with boundary and $F\colon W \to Y$, then $\#(\partial F, Z) = 0$.*

Observe that (1) is a special case of (2).

Remark 27.11. Recall the logic of the theory. We first define the oriented intersection number by (27.9), assuming $f \pitchfork Z$. Then we prove Proposition 27.10(1) assuming that $f_0, f_1 \pitchfork Z$. Corollary 16.11 is then used to define the intersection number in general, and the properties in Proposition 27.10 hold.

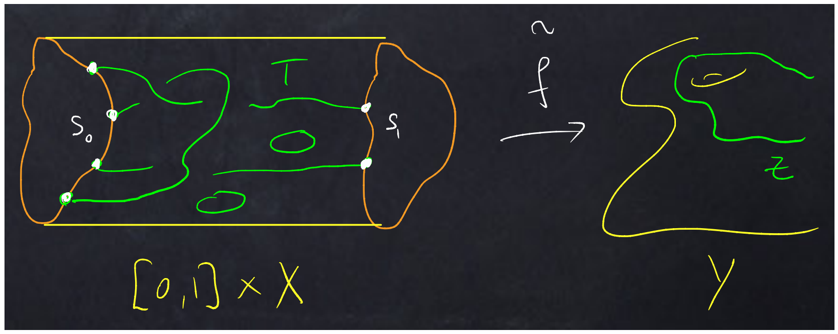


FIGURE 97. Homotopy invariance of the oriented intersection number

(27.12) *A bit of proof.* We give the proof of Proposition 27.10(1) assuming that $f_0, f_1 \pitchfork Z$. The first step is to apply a controlled perturbation of f which leaves it as is on $\{0\} \times X \amalg \{1\} \times X$ and gives a homotopic map $\tilde{f}\colon [0,1] \times X \to Y$ such that $\tilde{f} \pitchfork Z$. We already have $\partial\tilde{f} \pitchfork Z$, and so $T := \tilde{f}^{-1}(Z) \subset [0,1] \times X$ is a 1-dimensional submanifold, oriented by **(26.2)**. Now apply Theorem 25.23 to deduce that the signed count $\#_s\partial T = 0$, where ∂T has the boundary orientation. Define the 0-manifolds $S_0 = f_0^{-1}(Z)$ and $S_1 = f_1^{-1}(Z)$ with the induced orientation defined in **(26.2)**. Then Proposition 26.7 implies $\partial T = S_0 \amalg -S_1$ as oriented 0-manifolds. (This uses (25.34).) Therefore, $0 = \#_s\partial T = \#_s S_0 - \#_s S_1$, which proves the desired smooth homotopy invariance.

(27.13) *The inverse image orientation.* In the situation of (27.6) suppose that $f \pitchfork Z$ so that $S := f^{-1}(Z) \subset X$ is a 0-dimensional submanifold. Then for $p \in S$ the diagram (26.3) simplifies:

$$\begin{array}{ccccccccc} & & 0 & & & & T_pX & & \\ & & \| & & & & \| & & \\ 0 & \longrightarrow & T_pS & \longrightarrow & T_pX & \xrightarrow{\cong} & \nu_p(S \subset X) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle df_p} & & \big\downarrow{\scriptstyle df_p} & & \cong\big\downarrow{\scriptstyle df_p} & & \\ 0 & \longrightarrow & T_{f(p)}Z & \longrightarrow & T_{f(p)}Y & \longrightarrow & \nu_{f(p)}(Z \subset Y) & \longrightarrow & 0 \end{array} \tag{27.14}$$

Hence the orientation ± 1 of T_pS tells if the isomorphism in the first line preserves or reverses the orientation of T_pX. From the last line this, in turn, tells whether the natural isomorphism

$$\operatorname{Det} df_p(T_pX) \otimes \operatorname{Det} T_{f(p)}Z \xrightarrow{\cong} \operatorname{Det} T_{f(p)}Y \tag{27.15}$$

preserves or reverses orientation. It is this latter that we use in examples. The order 'X before Z' follows the order in '$\#(f, Z)$'.

Example 27.16. Consider Figure 98 in which Y is an affine 2-plane in affine 3-space with a handle attached, $Z \subset Y$ is a circle wrapping around the handle, X is a circle, and the map f wraps X around the handle transversely to Z. Orient Y as the boundary of the 3-dimensional region which lies below it in the picture, and orient Z and X as indicated by the little arrow. We use the standard

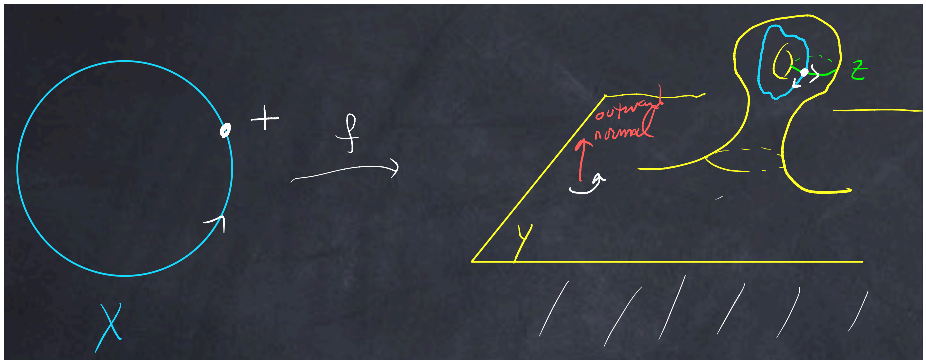


FIGURE 98. Example 27.16

"right hand rule" orientation of affine 3-space. The sign at the intersection point is then $+$, since the ordered basis of 3-space {outward normal, oriented tangent to $f(X)$, oriented tangent to Z} is positively oriented.

(27.17) *Oriented degree redux.* Suppose that in (27.6) we have $\dim Y = \dim X$ and $Z = \{q\} \subset Y$ is a single point. Assume that Y is connected. Then comparing the definitions we see that the oriented intersection number reduces to the degree $\#^Y(f, Z) = \deg f$.

Example 27.18 (transverse intersection of complex manifolds)**.** Let V be a 3-dimensional complex vector space and $W \subset V$ a 2-dimensional subspace. Then $Y = \mathbb{P}V$ is a compact 4-manifold (a complex projective plane) and $X = Z = \mathbb{P}W$ is a compact 2-dimensional submanifold (a complex projective line). Let $i_X \colon X \hookrightarrow Y$ denote the inclusion map. We compute $\#^Y(i_X, Z)$. To do so we need to homotop i_X to a map which is transverse to Z; the inclusion of a distinct complex projective line will do. The problem, then, is to compute the intersection number of two lines in a plane, here the lines and planes are complex projective. We have not yet said how to orient X, Y, and Z. Each is a complex manifold, an object we have not formally defined, and so each tangent space is a complex vector space. In fact, if $L \subset V$ is a line, so $L \in \mathbb{P}V$ a point, then we know $T_L\mathbb{P}V \cong \operatorname{Hom}(L, V/L)$ which is a complex vector space.

In general, if U is a finite dimensional complex vector space and $U_{\mathbb{R}}$ the underlying real vector space, let $I \colon U_{\mathbb{R}} \to U_{\mathbb{R}}$ denote the real linear map which is multiplication by $\sqrt{-1}$ on U. Then if $e_1, \dots, e_m$ is a (complex) basis of U, the $2m$ vectors $e_1, Ie_1, e_2, Ie_2, \dots, e_m, Ie_m$ is a (real) basis of $U_{\mathbb{R}}$. We convene that this is a positively oriented basis of $U_{\mathbb{R}}$. It is easy to check that this orientation of $U_{\mathbb{R}}$ is independent of the basis.

At a transverse intersection point the sign (27.15) is determined by comparing orientations of $df_p(T_pX) \oplus T_{f(p)}Z$ and $T_{f(p)}Y$. Each vector space has a complex structure, and the direct sum is compatible with the complex structures. It follows that the natural orientations agree, so the local oriented intersection number is $+1$.

Remark 27.19. The argument in Example 27.18 proves that transverse intersection points of complex submanifolds of a complex manifold contribute positively to the oriented intersection number. It is nonetheless possible to have negative self-intersection numbers in complex geometry.

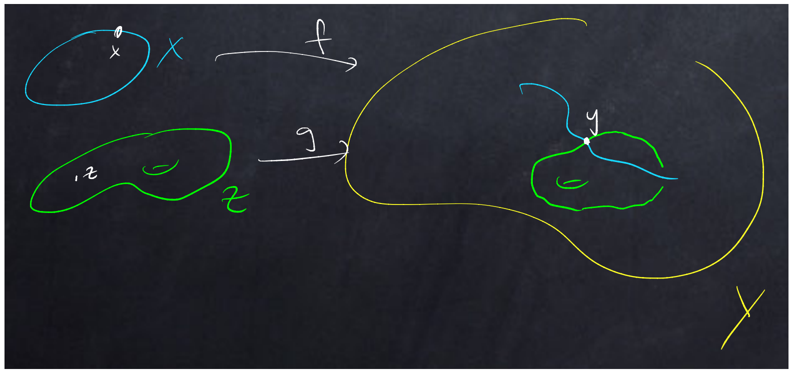


FIGURE 99. Intersection number of two maps

(27.20) *A symmetric version.* We can treat X and Z symmetrically if we assume Z is a compact oriented manifold rather than a closed submanifold. Hence modify (27.6) to:

(27.21)

X, Z	compact oriented manifolds
Y	oriented manifold
$f\colon X \longrightarrow Y$	smooth map
$g\colon Z \longrightarrow Y$	smooth map
$\dim X + \dim Z = \dim Y$	

If $f \pitchfork g$, and $x \in X$, $z \in Z$ satisfy $f(x) = g(z)$ for some $y \in Y$, then define the local intersection number ± 1 according as the isomorphism

$$\operatorname{Det} df_x(T_X X) \otimes \operatorname{Det} dg_z(T_z Z) \cong \operatorname{Det} T_y Y \tag{27.22}$$

is orientation-preserving or orientation-reversing. Define $\#^Y(f,g) \in \mathbb{Z}$ by summing the local intersection numbers. In case g is the inclusion of a submanifold, this agrees with Definition 27.8; see (27.15).

Observe that

$$\big\{(x,z) \in X, Z : f(x) = g(z)\big\} = (f \times g)^{-1}(\Delta), \tag{27.23}$$

where

$$f \times g\colon X \times Z \longrightarrow Y \times Y \tag{27.24}$$

is the Cartesian product of f and g, and $\Delta \subset Y \times Y$ is the diagonal, a closed submanifold. This gives the opportunity to reduce the intersection number $\#^Y(f,g)$ to $\#^{Y\times Y}(f\times g, \Delta)$ as in Definition 27.8, at least up to sign. The following enables this comparison.

Lemma 27.25. *If* $(x,z) \in X \times Z$ *satisfies* $f(x) = g(z) = y \in Y$*, then* $f \times g \pitchfork_{(x,z)} \Delta$ *if and only if* $df_x(T_xX) \oplus dg_z(T_zZ) = T_yY$*. If so, the local intersection numbers are related by*

$$\#^{Y}_{(x,z)}(f,g) = (-1)^{\dim Z}\, \#^{Y\times Y}_{(x,z)}(f\times g, \Delta). \tag{27.26}$$

Proof. Set $A = df_x(T_xX)$, $B = dg_z(T_zZ)$, and $C = T_yY$; then A and B are subspaces of C. Fix oriented bases $a_1, \dots, a_k$ of A and $b_1, \dots, b_\ell$ of B. Set $a = a_1 \wedge \cdots \wedge a_k \in \operatorname{Det} A$ and $b = b_1 \wedge \cdots \wedge b_\ell \in \operatorname{Det} B$. Recall the natural isomorphism $\bigwedge^\bullet(C \oplus C) \cong \bigwedge^\bullet C \otimes \bigwedge^\bullet C$. Then inclusion $A \oplus 0 \hookrightarrow C \oplus C$ induces a linear map $\operatorname{Det} A \hookrightarrow \bigwedge^\bullet C \otimes \bigwedge^\bullet C$, and we denote the image of a as $a^{(1)}$. Including in the other summand we obtain $a^{(2)}$, and similar elements $b^{(1)}$ and $b^{(2)}$. Compute

$$\begin{aligned} a^{(1)}\wedge b^{(2)} \wedge (a_1\otimes 1 + 1\otimes a_1) \wedge \cdots \wedge (a_k \otimes 1 + 1 \otimes a_k) \\ \wedge (b_1\otimes 1 + 1\otimes b_1) \wedge \cdots \wedge (b_\ell \otimes 1 + 1 \otimes b_\ell) \\ = a^{(1)} \wedge b^{(2)} \wedge a^{(2)} \wedge b^{(1)} \\ = (-1)^{\dim A \dim B + \dim B^2 + \dim A \dim B}\, a^{(1)} \wedge b^{(1)} \wedge a^{(2)} \wedge b^{(2)} \\ = (-1)^{\dim B}\, a^{(1)} \wedge b^{(1)} \wedge a^{(2)} \wedge b^{(2)} \end{aligned} \tag{27.27}$$

The first expression is nonzero iff $f \times g \pitchfork_{(x,z)} \Delta$ and the last is nonzero iff $A \oplus B = C$; the first assertion of the lemma follows. Assuming this, $a \wedge b \in \operatorname{Det} C$ is nonzero and is positively or negatively oriented according to the sign $\#^{Y}_{(x,z)}(f,g)$. In either case $a^{(1)} \wedge b^{(1)} \wedge a^{(2)} \wedge b^{(2)} \in \operatorname{Det}(C \oplus C)$ is positively oriented. Let $D \subset C \oplus C$ be the diagonal. Then

$$(a_1\otimes 1 + 1\otimes a_1) \wedge \cdots \wedge (a_k \otimes 1 + 1 \otimes a_k) \wedge (b_1\otimes 1 + 1\otimes b_1) \wedge \cdots \wedge (b_\ell \otimes 1 + 1 \otimes b_\ell) \tag{27.28}$$

in $\operatorname{Det} D$ is positively or negatively oriented according to $\#^{Y}_{(x,z)}(f,g)$. Putting everything together, (27.27) implies (27.26). ∎

Corollary 27.29. *In the setup* (27.21)*,*

$$\#^{Y}(f,g) = (-1)^{\dim Z}\, \#^{Y\times Y}(f\times g, \Delta). \tag{27.30}$$

Finally, from (27.30) we deduce the symmetry property

$$\#^{Y}(g,f) = (-1)^{\dim X \dim Z}\, \#^{Y}(f,g). \tag{27.31}$$

(27.32) *Intersection of submanifolds.* A special case of **(27.20)** is when $f = i_X$ and $g = i_Z$ are inclusions of submanifolds. Then we denote the intersection number as $\#^Y(X,Z)$. Writing it as $\#^Y(f,g)$ enables us to use our theory to wiggle the submanifolds to achieve transversality. The symmetry (27.31) now reads

$$\#^{Y}(Z,X) = (-1)^{\dim X \dim Z}\, \#^{Y}(X,Z). \tag{27.33}$$

This immediately implies the following.

Proposition 27.34. *If* $\dim Y = 4k+2$ *for some* $k \in \mathbb{Z}^{\geqslant 0}$*, and* $X \subset Y$ *is a compact orientable submanifold of dimension* $2k+1$ *such that* $\#^Y_2(X,X) \neq 0$*, then* Y *is not orientable.*

For example, $\mathbb{RP}^{4k+2}$ is not orientable. (We already proved a stronger statement in Corollary 25.50.)

Linking number

Beware that some of the arguments in this section are sketchy as presented here; they can be made rigorous.

(27.35) *The Gauss map.* Let A be an affine space over a 3-dimensional oriented real inner product space V, and let S^2 be the unit sphere in V. Suppose $K_1, K_2 \subset A$ are disjoint oriented compact 1-dimensional submanifolds. Define the *Gauss map* (Figure 100)

$$f\colon K_1 \times K_2 \longrightarrow S^2 \qquad p_1\ ,\ p_2 \longmapsto \frac{p_2 - p_1}{\|p_2 - p_1\|} \tag{27.36}$$

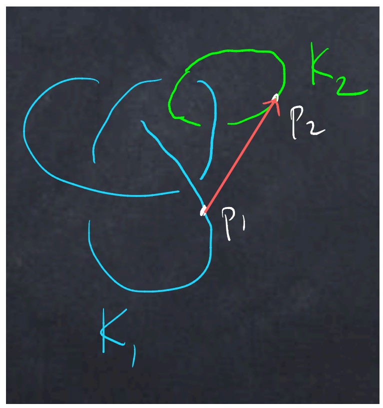


FIGURE 100. Gauss map that defines the linking number

Definition 27.37. The *linking number* of K_1, K_2 is $L(K_1, K_2) = \deg f$.

We use the Cartesian product orientation on $K_1 \times K_2$ and orient $S^2 = S(V)$ as the boundary of the closed unit ball in V. It follows from **(25.42)** that $L(K_2, K_1) = L(K_1, K_2)$. Also, if we reverse orientation, then $L(-K_1, K_2) = L(K_1, -K_2) = -L(K_1, K_2)$.

Remark 27.38. There is no self-linking number $L(K, K)$ without additional data, for example a nonzero normal vector field to $K \subset A$.

Example 27.39 (Hopf link)**.** Choose $A = \mathbb{A}^3_{x,y,z}$ with standard orientation and inner product on $\mathbb{R}^3$, let K_1 be the unit circle in the x-y plane with center $(0,0,0)$, and let K_2 be the unit circle in the x-z plane with center $(-1,0,0)$. Orient K_1, K_2 as in Figure 101. Then $(0,0,1) \in S^2 \subset \mathbb{R}^3$ is a regular value of the Gauss map f whose inverse image is a single point:

$$(p_1, p_2) = \big((-1,0,0), (-1,0,1)\big) \in K_1 \times K_2. \tag{27.40}$$

The differential of f maps $-\partial/\partial y \in T_{p_1}K_1$ to $\partial/\partial y$ and $-\partial/\partial x \in T_{p_2}K_2$ to $-\partial/\partial x$. The outward normal to S^2 at $(0,0,1)$ is $\partial/\partial z$, and since $\partial/\partial z, \partial/\partial y, -\partial/\partial x$ is a positively oriented basis of $\mathbb{R}^3$, the differential of f is orientation-preserving. Hence $L(K_1, K_2) = 1$.

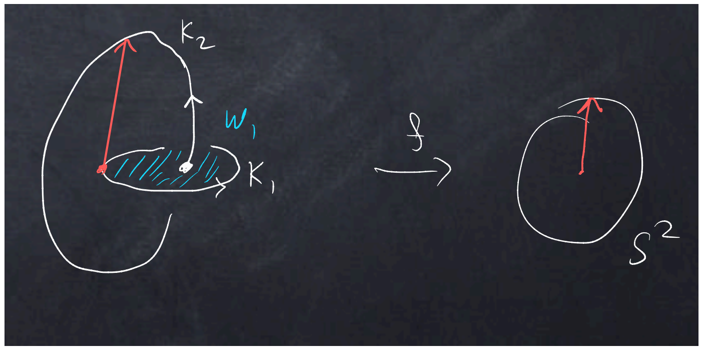


FIGURE 101. Gauss map for the Hopf link

Remark 27.41.

(1) In Figure 101 appears the disk W_1 with boundary K_1. Orient W_1 so that the indicated orientation of K_1 is the boundary orientation. Explicitly, W_1 is the unit disk in the x-y plane, and the ordered basis $\partial/\partial x, \partial/\partial y$ is positively oriented. (Compute the boundary orientation at the point $(1,0,0)$: there $\partial/\partial x$ is the outward normal.) Now observe that the local intersection number of W_1 and K_2 at the intersection point p is $+1$: amalgamate the oriented basis $\partial/\partial x, \partial/\partial y$ of T_pW_1 with the oriented basis $\partial/\partial z$ of T_pK_2 to obtain an oriented basis of $A = \mathbb{A}^3_{x,y,z}$.

(2) With a different choice of orientations we can obtain the linking number of the Hopf link to be -1; see the text following Definition 27.37. Observe that in that case the local intersection number of W_1 and K_2 also changes sign.

(27.42) *Computation by a bounding surface.* We now prove that the linking number can also be computed as an intersection number, but of a curve with a surface *with boundary*, and so we must take care to keep the curve away from the boundary lest an intersection point disappear off the edge of the surface.

Proposition 27.43. *Let A be an affine space over a 3-dimensional oriented real inner product space, let $W_1 \subset A$ be a compact oriented 2-manifold with boundary $\partial W_1 = K_1$, and suppose $K_2 \subset A$ is a compact oriented 1-manifold. Assume $\partial W_1 \cap K_2 = \emptyset$ and $W_1 \pitchfork K_2$. Then $L(K_1, K_2) = \#(W_1, K_2)$ where the latter is the signed count $\#_s\big(W_1 \cap K_2\big)$.*

We need not assume the transversality $W_1 \pitchfork K_2$: we can perturb W_1 and/or K_2 to achieve it. We sketch (literally) a proof, cognizant that we have not filled in all details.

Proof. Write $W_1 \cap K_2 = \{p_1, \dots, p_N\}$. For each $i \in \{1, \dots, N\}$ fix an open ball[38] $B_i \subset W_1$ which contains p_i and such that $B_i \cap B_j = \emptyset$ for all i, j. Set $D_i = \overline{B_i}$, $S_i = \partial D_i$, and $\widetilde{K}_1 = \bigcup_i S_i$. Then $W_1 \backslash \bigcup_i B_i$ is an oriented bordism from $\widetilde{K}_1$ to K_1, and the bordism invariance of degree implies

$$L(K_1, K_2) = L(\widetilde{K}_1, K_2) = \sum_i L(S_i, K_2). \tag{27.44}$$

[38]This means an open subset contained in the domain of a chart whose image is a ball in affine space.

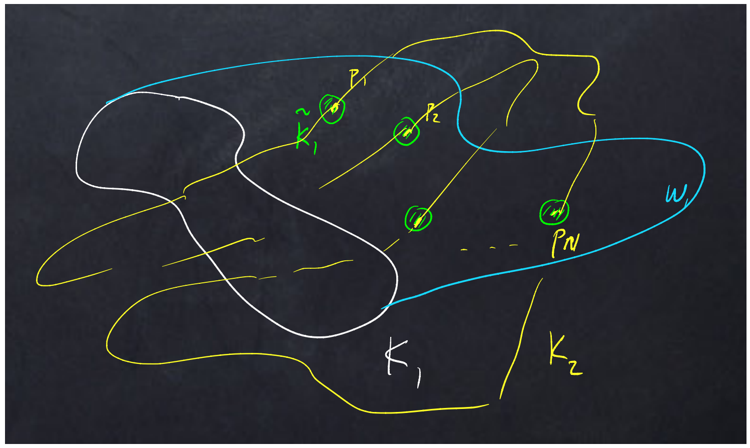


FIGURE 102. Reduction of K_1 to a union of "small" loops

FIGURE 103. Surgery $K_2 \rightsquigarrow M_i \amalg N_i$

To compute $L(S_i, K_2)$ we execute the surgery indicated in Figure 103. Namely, cut out $S^0 \times D^1 \subset K_2$, which is the union of two closed intervals, and glue in $D^1 \times S^0$ along $\partial(K_2 \setminus S^0 \times D^1) \approx S^0 \times S^0$. The result is the disjoint union of a circle M_i and a 1-manifold N_i. There is an oriented bordism $K_2 \to M_i \amalg N_i$ which is a product away from the surgery and is $D^1 \times D^1$ at the surgery point. (We must smooth corners.) The bordism invariance of degree implies

$$L(S_i, K_2) = L(S_i, M_i) + L(S_i, N_i). \tag{27.45}$$

Since $D_i \cap N_i = \emptyset$, we have $L(S_i, N_i) = 0$, again by the bordism invariance of degree. Finally, the circles S_i, M_i form a Hopf link (Example 27.39), and so $L(S_i, M_i) = \pm 1$ where the sign depends on the orientations. Tracing through, S_i is oriented as the boundary of D_i, and D_i inherits its orientation from that of W. K_2 is assumed oriented, and M_i inherits an orientation. It follows from Remark 27.41 that the sign is equal to the local intersection number $\#^A(D_i, M_i)$. □

(27.46) *Links in S^3.* If now S^3 is an oriented 3-sphere, and $K_1, K_2 \subset S^3$ are disjoint oriented compact 1-dimensional submanifolds, then we can[39] bound K_1 by an oriented surface $W_1 \subset S^3$, use transversality to arrange that K_2 intersects W_1 transversely and only in the interior, and so

[39]This requires proof, which we do not give here.

define the linking number as $\#_s(W_1 \cap K_2)$. (Alternatively, we may delete a point of S^3, compute the linking number in $S^3\backslash \mathrm{pt} \approx \mathbb{A}^3$, and prove that the result is independent of the point.)

(27.47) *The Hopf invariant.* Suppose $g\colon S^3 \to S^2$ is a smooth map. The *Hopf invariant* is

$$H(g) = L\big(g^{-1}(q_1), g^{-1}(q_2)\big) \tag{27.48}$$

for any distinct regular values $q_1, q_2 \in S^2$ of g. We fix orientations on the spheres and use the inverse image orientations of the closed 1-manifolds $g^{-1}(q_1)$ and $g^{-1}(q_2)$. The Hopf invariant does not change if g undergoes a smooth homotopy.

Example 27.49 (The Hopf fibration)**.** Let $\rho\colon \mathbb{C}^2\backslash\{0\} \to \mathbb{CP}^1$ be the map which assigns to a nonzero vector its span. Then ρ is a fiber bundle with fiber $\mathbb{C}^\times$ the nonzero complex numbers. (Better: $\mathbb{C}^\times$ acts on $\mathbb{C}^2\backslash\{0\}$ by scalar multiplication, and ρ is a quotient map for this group action.) Restrict ρ to the unit sphere $S^3 \subset \mathbb{C}^2\backslash\{0\}$ to obtain a fiber bundle $\pi\colon S^3 \to \mathbb{CP}^1 \approx S^2$. We claim that $H(\pi) = \pm 1$. If so, then π is not smoothly homotopic to a constant map, which clearly has vanishing Hopf invariant.

To compute $H(\pi)$ omit the point $(0,1) \in \mathbb{C}^2$ from S^3 and by stereographic projection identify $S^3\backslash\{(0,1)\} \approx \mathbb{A}^3$. Every $q \in \mathbb{CP}^1$ is a regular value since π is a submersion. Then

$$\begin{aligned} K_1 &:= \pi^{-1}\big([1,0]\big) = \{(\lambda,0) \in S^3 : |\lambda| = 1\} \\ K_2 &:= \pi^{-1}\big([0,1]\big) = \{(0,\mu) \in S^3 : |\lambda| = 1\} \end{aligned} \tag{27.50}$$

Under stereographic projection to $\mathbb{A}^3_{x,y,z}$ we can identify K_1 as the unit circle in the x-y plane and $K_2\backslash\{(0,1)\}$ as the z-axis. This is the Hopf link (minus a point at ∞) so has linking number ± 1, which we can compute by writing K_1 as the boundary of the unit disk W_1 in the x-y plane, and then $W_1 \cap K_2$ is a single point.

Remark 27.51 (Homotopy groups of S^2)*.* Leaving off basepoints, the q^{th} homotopy group, $q \in \mathbb{Z}^{\geqslant 0}$, is the set $[S^q, S^2]$ of homotopy classes of maps $S^q \to S^2$. We can take smooth maps and smooth homotopies. For $q = 0, 1$ we know by Sard's theorem that every map is homotopic to a constant map. For $q = 2$ we have an invariant—the degree—which is a map

$$\deg\colon [S^2, S^2] \longrightarrow \mathbb{Z}. \tag{27.52}$$

In fact, it is an isomorphism, as follows from the Hopf degree theorem. In other words, the degree is a *complete* invariant of $[S^2, S^2]$. Now for $q = 3$ we have sketched a map

$$H\colon [S^3, S^2] \longrightarrow \mathbb{Z}, \tag{27.53}$$

the Hopf invariant, and this too turns out to be an isomorphism. The degree and Hopf invariant illustrate how differential topology—the application of calculus to global geometry—can be used to construct interesting and effective invariants.

It turns out that $[S^q, S^2]$ has cardinality greater than one for infinitely many q.

Lecture 28: Euler numbers and Lefschetz numbers

In this lecture we take up the Euler number, a basic invariant of a smooth manifold. We define and study it not just for manifolds, but also for real vector bundles whose rank equals the dimension of the base manifold. A generalization is the Lefschetz number of a self map on a manifold, and we use it to illustrate some computations.

Throughout the base manifold X is compact, and we also assume it is oriented, though the orientation can be eliminated for the Euler number of a manifold and the Lefschetz number of a map; see Remark 28.4.

Euler numbers

(28.1) *Euler number of a compact oriented manifold.*

Definition 28.2. Let X be a compact oriented manifold. The *Euler number* of X is

$$\chi(X) = \#^{X\times X}(\Delta,\Delta), \tag{28.3}$$

where $\Delta \subset X\times X$ is the diagonal.

The manifold $X\times X$ has the Cartesian product orientation, and we transport the orientation of X to Δ via the diffeomorphism $X\to\Delta$ given by $p\mapsto(p,p)$. The symmetry property (27.33) implies $\chi(X)=0$ if $\dim X$ is odd.

Remark 28.4. Imagine the first copy of Δ in (28.3) is perturbed to be transverse to Δ at some point $(p,p)\in X\times X$. Then infinitesimally we have the isomorphism

$$W\oplus D \xrightarrow{\ \cong\ } V\oplus V, \tag{28.5}$$

where $V=T_pX$ and $W\subset V\oplus V$ is transverse to the diagonal $D\subset V\oplus V$. We can take W to be the graph of a linear map $V\to V$, and it is oriented by its projection to $V\oplus 0$. The local intersection number at (p,p) is ±1 according as (28.5) is orientation-preserving or orientation-reversing. The answer to that question is independent of the orientation of V, and this explains why orientations are not necessary for Euler numbers of manifolds, as we now explain.

(28.6) *Euler number of a compact (unoriented) manifold.* We can define the Euler number of an unoriented manifold by applying Definition 28.2 to its double cover. This relies on the following.

Lemma 28.7. *Let X be a compact manifold and let $\hat\pi\colon \widehat X\to X$ be its orientation double cover.*

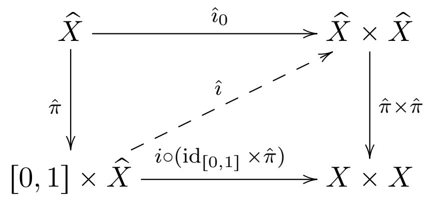
(1) *$\widehat X$ is oriented and $\chi(\widehat X)$ is even.*
(2) *If X is oriented, then $\chi(\widehat X)=2\chi(X)$.*

The Euler numbers which appear in Lemma 28.7 are defined using (28.3).

Proof. At a point $\hat{p} \in \widehat{X}$ the differential of the projection $\hat{\pi}$ is an isomorphism $T_{\hat{p}}\widehat{X} \xrightarrow{\cong} T_pX$, where $p = \hat{\pi}(\hat{p})$, and furthermore at $\hat{p}$ we have an orientation of T_pX, hence of $T_{\hat{p}}\widehat{X}$. It follows that $\widehat{X}$ carries a canonical orientation.

Let $i\colon [0,1] \times X \to X \times X$, $0 \leqslant t \leqslant 1$, be a smooth homotopy from the inclusion of the diagonal $i_0\colon \Delta \hookrightarrow X \times X$ to a map i_1 such that $i_1 \pitchfork \Delta$. Let $\hat{\imath}_0\colon \widehat{\Delta} \hookrightarrow \widehat{X} \times \widehat{X}$ be the inclusion of the diagonal into the Cartesian square of the total space of the orientation double cover. Then there is a unique smooth homotopy $\hat{\imath}\colon [0,1] \times \widehat{X} \to \widehat{X} \times \widehat{X}$ which covers i; it fits into the diagram:

(28.8)



The existence of $\hat{\imath}$ is the *homotopy lifting property* of the covering map $\hat{\pi} \times \hat{\pi}$. For $\hat{x} \in \widehat{X}$ the restriction of $\hat{\imath}$ to $[0,1] \times \{\hat{p}\}$ is the lift of the path $i \circ (\mathrm{id}_{[0,1]} \times \hat{\pi})\big|_{[0,1]\times\{\hat{p}\}}$ in $X \times X$ to the total space $\widehat{X} \times \widehat{X}$ of the covering $\hat{\pi} \times \hat{\pi}$. (Recall that covering spaces satisfy *path lifting.*) In terms of this perturbation we compute the Euler number of $\widehat{X}$ as an oriented intersection number:

$$\chi(\widehat{X}) = \#^{\widehat{X}\times\widehat{X}}(\hat{\imath}_1, \widehat{\Delta}). \tag{28.9}$$

Set

$$\begin{aligned} S &= \{p \in X : i_1(p) = (p,p)\} \\ \widehat{S} &= \{\hat{p} \in \widehat{X} : \hat{\imath}_1(\hat{p}) = (\hat{p},\hat{p})\} \end{aligned} \tag{28.10}$$

By transversality and compactness these are finite sets. Furthermore, since $\hat{\imath}$ is the lift of i, we have $\widehat{S} = \hat{\pi}^{-1}(S)$. Also, it follows from Remark 28.4 that the local intersection number of $\hat{\imath}_1$ and $\widehat{\Delta}$, which is a function $\widehat{S} \to \{\pm 1\}$, is constant on the fibers of $\widehat{S} \to S$: for $p \in S$, reversing the orientation of T_pV does not affect the computation of the local intersection number which tells if (28.5) is orientation-preserving or orientation-reversing. The conclusions of the lemma now follow. ☐

In view of Lemma 28.7, the right hand side of (28.12) below is an integer and it equals the right hand side of (28.3) if X is oriented.

Definition 28.11. Let X be a compact manifold and $\hat{\pi}\colon \widehat{X} \to X$ its orientation double cover. The *Euler number* of X is

$$\chi(X) = \frac{1}{2}\#^{\widehat{X}\times\widehat{X}}(\widehat{\Delta}, \widehat{\Delta}), \tag{28.12}$$

where $\widehat{\Delta} \subset \widehat{X} \times \widehat{X}$ is the diagonal.

(28.13) *Euler number of a real vector bundle.* Let $\pi_E\colon E\to X$ be a real vector bundle. It has a canonical section $s_0\colon X\to E$, the zero section, whose image $Z_E = s_0(X)\subset E$ is a closed submanifold which is also called the 'zero section'. The section s_0 and projection π_E give inverse diffeomorphisms of X and Z_E. For $p\in X$, let $0_p = s_0(p)\subset E_p$ denote the zero vector of the vector space E_p. There are two natural submanifolds which pass through $0_p\in E$: the zero section Z_E and the fiber E_p. Infinitesimally, that leads to a direct sum decomposition of the tangent space to the total space:[40]

$$\begin{aligned} T_{0_p}E &\cong T_{0_p}Z_E\oplus T_{0_p}E_p \\ &\cong\ T_pX\ \oplus\ E_p \end{aligned} \tag{28.15}$$

Here we use the canonical isomorphism of the tangent space of the vector space E_p with E_p. The set of two orientations of each fiber E_p form a double cover $\mathfrak{o}(E)\to X$, the *orientation double cover*; an *orientation* of E is a section of $\mathfrak{o}(E)\to X$. Note from (28.15) that an orientation of X and of π_E induce an orientation of the total space E. Finally, recall that the rank of π_E is a locally constant function $\operatorname{rank} E\colon X\to\mathbb{Z}$.

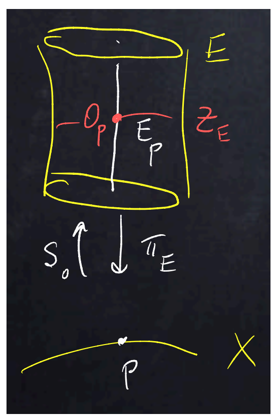


FIGURE 104. A vector bundle π_E and its zero section s_0 with image Z_E

Definition 28.16. Let X be a compact oriented manifold and $\pi_E\colon E\to X$ an oriented real vector bundle. Assume $\operatorname{rank}\pi_E = \dim X = n$ for some positive integer n. Then the *Euler number* of π_E is

$$\chi(\pi_E) = \#^E(Z_E, Z_E). \tag{28.17}$$

The symmetry property (27.33) implies $\chi(\pi_E)=0$ if n is odd.

[40] At any point $e\in E$ there is a short exact sequence

$$0\longrightarrow E_p\longrightarrow T_eE\xrightarrow{d(\pi_E)_e} T_pX\longrightarrow 0, \tag{28.14}$$

where $p=\pi_E(e)$. For $e\in Z_E$ there is a canonical splitting, written in (28.15), and we observe the 'Quotient Before Sub' rule in writing (28.15).

(28.18) *Perturbing sections.* The intersection number in (28.17) is defined to be $\#^E(s_0, Z_E)$, and it is computed by homotoping $s_0\colon X \to E$ to a map transverse to Z_E. In fact, we can perturb to a transverse *section*. Observe that any section $s_1\colon X \to E$ of π_E is smoothly homotopic to the zero section s_0 through sections via the homotopy $s_t(p) = ts(p)$, $t \in [0,1]$, $p \in X$.

Lemma 28.19. *There exists a section $s_1\colon X \to E$ of π_E which is transverse to Z_E.*

Proof. By Corollary 16.11 there exists a map $f\colon X \to E$ such that $f \pitchfork Z_E$ and f is homotopic to s_0. Recall the construction embeds s_0 in a family of maps $f_b\colon X \to E$ parametrized by a ball B, and Sard's theorem implies that transversality is achieved for b in a dense subset of B. The Stability Theorem 12.17(vii) implies that $\pi_E \circ f_b\colon X \to X$ is a diffeomorphism for all $b \in B$ in a neighborhood of $0 \in B$ (if we take $f_0 = s_0$). Choose b in this neighborhood so that $f_b \pitchfork Z_E$, and set $f = f_b$. Then $s := f \circ (\pi_E \circ f)^{-1}$ is a section of π_E that is transverse to X. □

(28.20) *Properties of the Euler number.* We record two elementary facts which follow immediately from the definition of the intersection number and its symmetry property.

Proposition 28.21. *Let $\pi_E\colon E \to X$ be an oriented real vector bundle over a compact oriented manifold, and assume* $\operatorname{rank} \pi_E = \dim X$.

(1) *If* $\dim X$ *is odd, then* $\chi(\pi_E) = 0$.
(2) *If π_E admits a nowhere vanishing section, then* $\chi(\pi_E) = 0$.

The converse of (2) is also true, though we do not prove it.

(28.22) *Computation of local intersection numbers.*

Lemma 28.23. *Let $s\colon X \to E$ be a section of π_E, suppose $s(p) = 0$ for some $p \in X$, and assume $s \pitchfork_p Z_E$. Assume $n = \dim X = \operatorname{rank} E$ is even. Then the local intersection number $\#_p(s, Z_E)$ equals ± 1 according as the isomorphism*

$$I_p\colon T_pX \xrightarrow{ds_p} T_{0_p}E \xrightarrow{\text{proj}} E_p \tag{28.24}$$

is orientation-preserving $(+1)$ or orientation-reversing (-1).

The projection $T_{0_p}E \to E_p$ is defined by the splitting (28.15). It is convenient to refer to the *sign* of the isomorphism (28.24). If n is odd, then the Euler number vanishes so we have no need for the local intersection number.

Proof. Let $\epsilon_p = \#_p(s, Z_E) = \pm 1$; it is computed by requiring that the isomorphisms

$$\begin{aligned} \operatorname{Det} T_{0_p}E &\cong \epsilon_p \operatorname{Det} ds_p(T_pX) \otimes \operatorname{Det} T_pX \\ &\cong \epsilon_p \operatorname{Det} T_pX \otimes \operatorname{Det} T_pX \end{aligned} \tag{28.25}$$

be orientation-preserving. The sign $\delta_p = \pm 1$ of I_p makes the isomorphisms

$$\begin{aligned} \operatorname{Det} T_{0_p}E &\cong \operatorname{Det} T_pX \otimes \operatorname{Det} E_p \\ &\cong \operatorname{Det} T_pX \otimes \delta_p \operatorname{Det} T_pX \end{aligned} \tag{28.26}$$

orientation-preserving. The lemma follows by comparing (28.25) and (28.26), bearing in mind that n is even so we can swap the factors in the tensor product without incurring a sign penalty. (To do the comparison, one must bear in mind what the isomorphisms are.) □

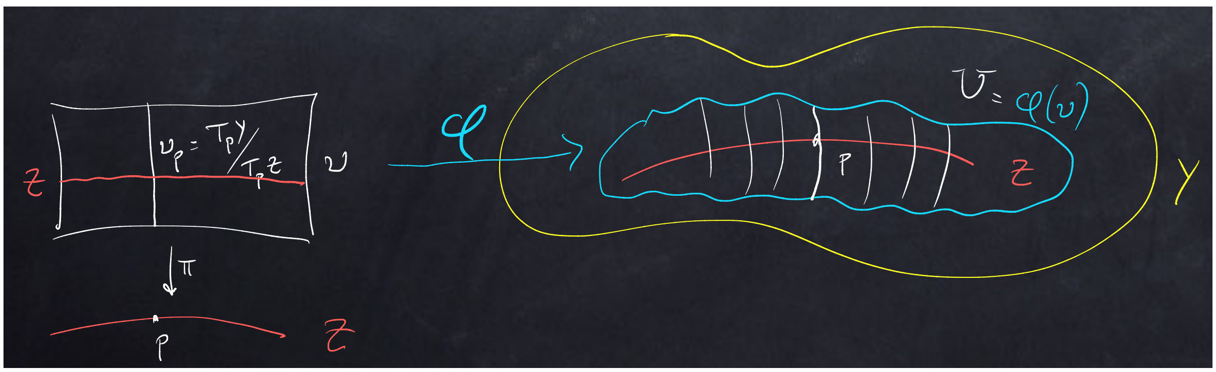


FIGURE 105. A tubular neighborhood U of the submanifold $Z \subset Y$

(28.27) *The tubular neighborhood theorem.* A neighborhood of each point in a submanifold has a normal form; indeed, that is the very definition of a submanifold (Definition 6.19). The following theorem is a global version which gives control of an open neighborhood of the entire submanifold. Theorem 16.8 is the tubular neighborhood theorem for submanifolds of affine space; see Remark 16.26.

Theorem 28.28. *Let Y be a smooth manifold, $Z \subset Y$ a submanifold, and $\nu = \nu(Z \subset Y) \to Z$ the normal bundle. Then there exists an embedding $\varphi \colon \nu \to Y$ such that $\varphi\big|_Z = \mathrm{id}_Z$ and $\varphi(\nu) \subset Y$ is an open subset.*

The theorem is illustrated in Figure 105. We identify Z with the image $Z_\nu \subset \nu$ of the zero section. Theorem 28.28 can be proved from Theorem 16.8, but we do not do so in these notes.

Remark 28.29. A variation of the tubular neighborhood theorem holds for a neighborhood of the boundary ∂X of a manifold X with boundary. Namely, there exists a *collar*: an embedding $[0, 1) \times \partial X \hookrightarrow X$ which is the identity on $\{0\} \times \partial X$.

(28.30) *The normal bundle to the diagonal.* Let X be a smooth manifold. The diagonal $\Delta \subset X \times X$ is a submanifold, so there is a short exact sequence

$$0 \longrightarrow T\Delta \longrightarrow TX \oplus TX \longrightarrow \nu(\Delta \subset X \times X) \longrightarrow 0 \tag{28.31}$$

of vector bundles over Δ. For $p \in X$ the coset of $(\xi_1, \xi_2) \in T_pX \oplus T_pX$ under the diagonal action of T_pX by translation contains a unique vector of the form $(\xi, 0)$. Hence there is a splitting $\nu(\Delta \subset X \times X) \to TX \oplus TX$ of (28.31) whose image is $TX \oplus 0$. This proves the following.

Lemma 28.32. *The normal bundle $\nu(\Delta \subset X \times X) \to \Delta$ is canonically isomorphic to the tangent bundle $TX \to X$.*

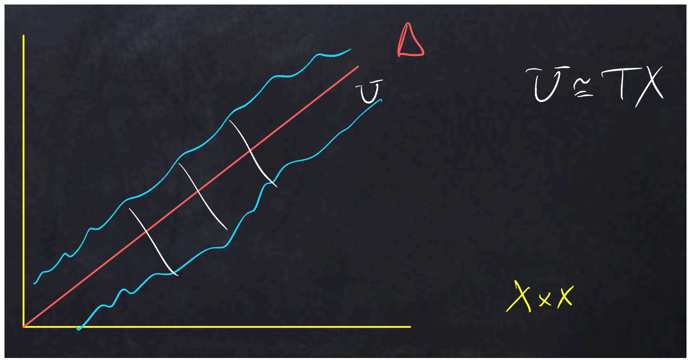


FIGURE 106. A tubular neighborhood of $\Delta \subset X \times X$

(28.33) *The Euler number and vector fields.* We combine **(28.27)** and **(28.30)** to prove the following. Recall that a section of the tangent bundle to a smooth manifold is a vector field.

Theorem 28.34. *Let X be a compact oriented manifold and $\pi_{TX}\colon TX \to X$ its tangent bundle. Then we have the equality of Euler numbers*

$$\chi(X) = \chi(\pi_{TX}). \tag{28.35}$$

Proof. Let $\xi\colon X \to TX$ be a vector field which is transverse to the zero section (Lemma 28.19), and fix a tubular neighborhood $\varphi\colon TX \hookrightarrow X \times X$ of the diagonal Δ.

$$\chi(\pi_{TX}) = \#^{TX}(\xi, Z_{TX}) = \#^{X\times X}(\varphi\circ\xi, \Delta) = \chi(X). \tag{28.36}$$

□

Corollary 28.37. *Let X be a compact oriented manifold.*

(1) *If $\dim X$ is odd, then $\chi(X) = 0$.*
(2) *If X admits a nowhere vanishing vector field, then $\chi(X) = 0$.*

(28.38) *A special case of Poincaré-Hopf.* If ξ is a vector field which vanishes at $p \in X$, then (28.24) is the composition

$$T_pX \xrightarrow{d\xi_p} T_{0_p}TX \xrightarrow{\text{proj}} T_pX \tag{28.39}$$

which is usually identified as the differential of the vector field. By Lemma 28.23 it is invertible iff ξ is transverse to the zero section at p. In that case its sign is called the *index* of ξ at p, denoted $\operatorname{ind}_p \xi = \pm 1$.

Theorem 28.40. *Let X be a compact oriented manifold and ξ a vector field which is transverse to the zero section. Then*

$$\chi(X) = \sum_{p\in \mathrm{Zero}(\xi)} \mathrm{ind}_p\,\xi. \tag{28.41}$$

There is a generalization of the index to an isolated zero of a vector field for which (28.41) still holds. We discuss that, or at least its analog for isolated fixed points of a map $X \to X$, in the next lecture.

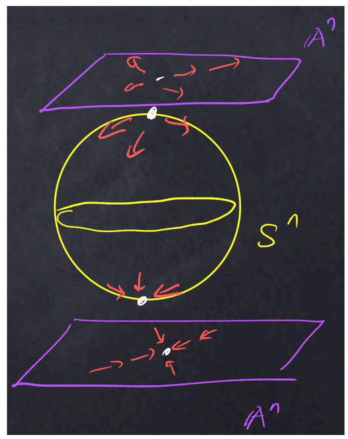


FIGURE 107. Computation of $\chi(S^n)$

Example 28.42 (Euler number of S^n)**.** Construct S^n by the surjection $\mathbb{A}^n \amalg \mathbb{A}^n \to S^n$ in which two copies of affine space are glued on the complement of a point by inversion. Namely, identify $\mathbb{A}^n\backslash\{0\} \approx \mathbb{R}^{>0} \times S^{n-1}$ ("polar coordinates"). Then the overlap map is

$$\begin{aligned} \mathbb{R}^{>0} \times S^{n-1} &\longrightarrow \mathbb{R}^{>0} \times S^{n-1} \\ (r,\Theta) &\longmapsto (r^{-1},\Theta) \end{aligned} \tag{28.43}$$

More simply the map is $s = r^{-1}$, and so $ds = -r^{-2}dr$ from which $r\,\partial/\partial r \mapsto -s\,\partial/\partial s$ under (28.43). The latter radial vector field glues then to a global vector field ξ on S^n which vanishes transversely at the two poles of S^n. The differential of $r\,\partial/\partial r$ at $r = 0$ is the identity map, so for ξ the differential at one pole is id and at the other is $-\,$id. Now Theorem 28.40 gives

$$\chi(S^n) = 1 + (-1)^n = \begin{cases} 0, & n \text{ odd;} \\ 2, & n \text{ even.} \end{cases} \tag{28.44}$$

Lefschetz numbers

The basic definition is a variant of Definition 28.2.

Definition 28.45. Let X be a compact oriented manifold and $f\colon X \to X$ a smooth map. The *Lefschetz number* of f is

$$L(f) = \#^{X\times X}\bigl(\Gamma(f), \Delta\bigr), \tag{28.46}$$

where $\Gamma(f) \subset X \times X$ is the graph of f.

Observe that $\Gamma(f) \cap \Delta = \mathrm{Fix}(f)$ is the fixed point set of f.

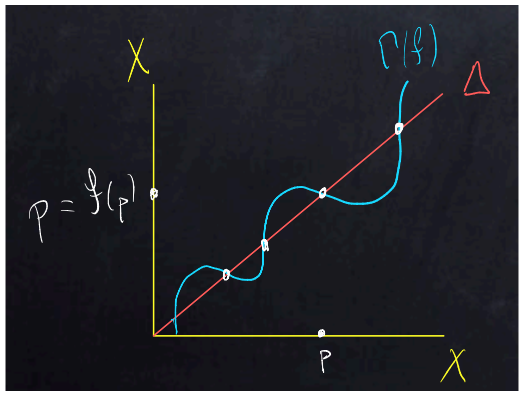


FIGURE 108. The Lefschetz number of a self map

Remark 28.47. A variation of the discussion in **(28.6)** extends the Lefschetz number to self maps of compact (unoriented) manifolds.

The following properties are immediate from Proposition 27.10(1) and Definition 28.2.

Proposition 28.48.

(1) *If $f_0 \simeq f_1$ are smoothly homotopic maps, then $L(f_0) = L(f_1)$.*
(2) *If $f \simeq \mathrm{id}_X$, then $L(f) = \chi(X)$.*
(3) *If $L(f) \neq 0$, then $\mathrm{Fix}(f) \neq \emptyset$.*

Assertion (2) leads to effective computations of the Euler number of a manifold, as we illustrate below. Assertion (3) is a fixed point theorem, effective if we have a method for computing the Lefschetz number. We discuss this more in the next lecture.

Example 28.49. Let G be a positive dimensional Lie group and $g \in G$ a non-identity element which is connected to the identity by a smooth path. Let $L_g\colon G \to G$ be left multiplication by g, i.e., the diffeomorphism $x \mapsto gx$. Then $L_g \simeq \mathrm{id}_G$ and $\mathrm{Fix}(L_g) = \emptyset$. Therefore, $\chi(G) = 0$.

(28.50) *Lefschetz fixed points.* The following is in keeping with our theme that transverse intersections are special.

Definition 28.51. Let X be a compact oriented manifold and $f\colon X \to X$ a smooth map. Then $p \in \mathrm{Fix}(f)$ is a *Lefschetz fixed point* of f if $\Gamma(f) \pitchfork_{(p,p)} \Delta$.

We compute the local intersection number at a Lefschetz fixed point.

Proposition 28.52. *Let X be a compact oriented manifold and $f\colon X \to X$ a smooth map.*

(1) *$p \in \mathrm{Fix}(f)$ is Lefschetz iff $1 - df_p\colon T_pX \to T_pX$ is invertible.*

(2) *If p is a Lefschetz fixed point, then $\#^{X\times X}_{(p,p)}\bigl(\Gamma(f),\Delta\bigr) = \deg(1-df_p) = \pm 1$.*

Here and hereafter we use '1' in place of 'id_{T_pX}' for ease of reading. If p is a Lefschetz fixed point, then we define the *local Lefschetz number*

$$L_p(f) = \deg(1-df_p) = \pm 1. \tag{28.53}$$

Note that this equals $\operatorname{sign}\det(1-df_p)$. In the next lecture we generalize the local Lefschetz number from Lefschetz fixed points to general isolated fixed points.

Remark 28.54. The map $1 - df_p$ is invertible iff the map df_p has no nonzero fixed vector. In other words, p is a Lefschetz fixed point of f iff the linearization of f at p has a single fixed point: the zero vector. In still other words, the condition is that 1 is *not* an eigenvalue of the differential df_p.

Proof. Set $V = T_pX$ and $T = df_p\colon V \to V$. The nonlinear maps $X \to X\times X$ with image $\Gamma(f)$ and Δ, respectively, have differentials the linear maps $V \longrightarrow V\oplus V$ given by

$$\begin{aligned} \xi &\longmapsto (\xi, T\xi) \\ \xi &\longmapsto (\xi,\xi) \end{aligned} \tag{28.55}$$

Compose with the orientation-preserving automorphism

$$\begin{aligned} V\oplus V &\longrightarrow V\oplus V \\ (\xi_1,\xi_2) &\longmapsto (\xi_1-\xi_2,\, \xi_2) \end{aligned} \tag{28.56}$$

to obtain the maps

$$\begin{aligned} \xi &\longmapsto \bigl((1-T)\xi,\, T\xi\bigr) \\ \xi &\longmapsto (0,\xi) \end{aligned} \tag{28.57}$$

The images are transverse iff $1-T$ is invertible, and if so the map

$$\begin{aligned} V\oplus V &\longrightarrow V\oplus V \\ (\xi_1,\xi_2) &\longmapsto \bigl((1-T)\xi_1,\, T\xi_1+\xi_2\bigr) \end{aligned} \tag{28.58}$$

preserves or reverses orientation according as $1-T\colon V\to V$ preserves or reverses orientation. □

(28.59) *Lefschetz maps.* More variations on our theme that transversality is generic follow.

Definition 28.60. Let X be a compact oriented manifold and $f\colon X \to X$ a smooth map. Then f is *Lefschetz* if $\Gamma(f) \pitchfork \Delta$.

If f is Lefschetz, then its global Lefschetz number is the sum of local Lefschetz numbers:

$$L(f) = \sum_{p \in \mathrm{Fix}(f)} L_p(f). \tag{28.61}$$

Theorem 28.62. *Let X be a compact manifold and $f\colon X \to X$ a smooth map. Then there exists a smooth homotopy $f_t\colon X \to X$, $t \in [0,1]$ such that $f_0 = f$ and f_1 is Lefschetz.*

Proof. By Theorem 11.11 we can and do embed X in an affine space A over a normed linear space V. Let $S = B_1(0) \subset V$ be the unit ball. In Corollary 16.9 we constructed a submersion $F\colon S \times X \to X$ such that $F(0,p) = f(p)$ for all $p \in X$ and the partial differential $dF^1_{(s,p)}\colon V \to T_{F(s,p)}X$ is surjective for all $s \in S$ and $p \in X$. Define

$$\begin{aligned} G\colon S \times X &\longrightarrow X \times X \\ (s,p) &\longmapsto \big(p, F(s,p)\big) \end{aligned} \tag{28.63}$$

Then the differential of G has the form $dG = \left(\begin{smallmatrix} 0 & \mathrm{id} \\ dF^1 & dF^2 \end{smallmatrix}\right)$, so G is a submersion and hence $G \pitchfork \Delta$. It follows from Theorem 15.18 that for a dense set of $s \in S$ the map $p \mapsto \big(p, F(s,p)\big)$ is transverse to Δ. For any such s set $f_t(p) = F(ts, p)$. ∎

(28.64) *Computations of Euler number via Lefschetz maps.* We illustrate on real projective space and the first real Grassmannian which is not a projective space. You can generalize to other Grassmannians and to the complex case as well.

Example 28.65 (Euler number of real projective space)**.** Let

$$T = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n+1 \end{pmatrix}. \tag{28.66}$$

This linear transformation of $\mathbb{R}^{n+1}$ induces the eigenspace decomposition

$$\mathbb{R}^{n+1} \cong L_1 \oplus \cdots \oplus L_{n+1} \tag{28.67}$$

where L_i is the i^{th} coordinate line and T acts as multiplication by i on L_i. The linear map $T\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ induces a projective linear map $f = f_T\colon \mathbb{RP}^n \to \mathbb{RP}^n$ which is Lefschetz (as we will see) with $\mathrm{Fix}(f) = \{L_1, \dots, L_{n+1}\}$. Recall that for any vector space V and line $L \subset V$ we have $T_L\mathbb{P}V \cong \mathrm{Hom}(L, V/L)$. In this case we identify

$$T_{L_i}\mathbb{RP}^n \cong \mathrm{Hom}(L_i, L_1) \oplus \cdots \oplus \mathrm{Hom}(L_i, L_i)\!\!\!\!\!\!\!\!\!\!\!\!/\;\;\;\;\;\; \oplus \cdots \oplus \mathrm{Hom}(L_i, L_{n+1}). \tag{28.68}$$

The differential of f is the map induced on the Hom spaces by the linear map T, which acts by conjugation on an element of $\mathrm{Hom}(L_i, L_j)$, so acts as multiplication by j/i. In other words, (28.68) is the eigenspace decomposition of df_{L_i}, so too of $1 - df_{L_i}$:

$$1 - df_{L_i} = \Big(1 - \frac{1}{i}\Big) \oplus \Big(1 - \frac{2}{i}\Big) \oplus \cdots \oplus \Big(1 \not- \frac{i}{i}\Big) \oplus \cdots \oplus \Big(1 - \frac{n+1}{i}\Big). \tag{28.69}$$

Thus $1 - df_{L_i}$ is invertible, and the local Lefschetz number is the parity of the number of negative eigenvalues, which is $(-1)^{n+1-i}$. Since T is homotopic to the identity matrix, it follows that $f \simeq \mathrm{id}_{\mathbb{RP}^n}$ and so

$$\chi(\mathbb{RP}^n) = L(f) = \sum_{i=1}^{n+1} (-1)^{n+1-i} = \begin{cases} 0, & n \text{ odd;} \\ 1, & n \text{ even.} \end{cases} \tag{28.70}$$

Remark 28.71. Note that $\mathbb{RP}^n$ is not orientable if n is even, but nonetheless the computation is valid; see Remark 28.47. Also, the Euler number is multiplicative for a finite covering space, and that is borne out in this example by comparing (28.70) with (28.44).

Example 28.72 (The Grassmannian $\mathrm{Gr}_2(\mathbb{R}^4)$)**.** Set $X = \mathrm{Gr}_2(\mathbb{R}^4)$, and let $f\colon X \to X$ be the map on 2-planes induced from the linear transformation

$$T = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix} \tag{28.73}$$

of $\mathbb{R}^4$. Then the fixed point set consists of the 6 coordinate 2-planes

$$\mathrm{Fix}(f) = \big\{L_1 \oplus L_2,\ L_1 \oplus L_3,\ L_1 \oplus L_4,\ L_2 \oplus L_3,\ L_2 \oplus L_4,\ L_3 \oplus L_4\big\} \tag{28.74}$$

The tangent space is hom from the sub to the quotient, so for example

$$\begin{aligned} T_{L_1 \oplus L_2}X &\cong \mathrm{Hom}(L_1 \oplus L_2, L_3 \oplus L_4) \\ &\cong \mathrm{Hom}(L_1, L_3) \oplus \mathrm{Hom}(L_1, L_4) \oplus \mathrm{Hom}(L_2, L_3) \oplus \mathrm{Hom}(L_2, L_4). \end{aligned} \tag{28.75}$$

The differential $df_{L_1 \oplus L_2}$ is computed from (28.73) as in the previous example, so (28.75) is the eigenspace decomposition and $df_{L_1 \oplus L_2}$ acts as $\frac{3}{1} \oplus \frac{4}{1} \oplus \frac{3}{2} \oplus \frac{4}{2}$. We see that f is Lefschetz at this fixed point (and at the other 5); the Lefschetz number counts the parity of the number of eigenvalues greater than 1. Hence $L_{L_1\oplus L_2}(f) = +1$. The result at all fixed points:

$$\begin{aligned} L_{L_1\oplus L_2}(f) &= +1\\ L_{L_1\oplus L_3}(f) &= -1\\ L_{L_1\oplus L_4}(f) &= +1\\ L_{L_2\oplus L_3}(f) &= +1\\ L_{L_2\oplus L_4}(f) &= -1\\ L_{L_3\oplus L_4}(f) &= +1 \end{aligned} \tag{28.76}$$

Therefore,

$$\chi\bigl(\mathrm{Gr}_2(\mathbb{R}^4)\bigr) = L(f) = 2. \tag{28.77}$$

Lecture 29: More on Lefschetz numbers

In the first part of this lecture we develop a formula for computing the global Lefschetz number of a self map with isolated fixed points which need not be Lefschetz. As preparation for the next step we state the basic theorems about flows of vector fields on a manifold. Then we deduce the Poincaré-Hopf formula for the Euler number in terms of a vector field with isolated zeros.[41] Finally, we introduce de Rham cohomology and state a theorem which identifies the global Lefschetz number of a self map in terms its induced action on de Rham cohomology. We conclude with a brief discussion of fixed point theorems.

Isolated fixed points

(29.1) *An example.* We begin with an echo of **(26.46)** and **(26.56)**. Fix $m \in \mathbb{Z}^{>0}$ and consider the function

$$\begin{aligned} f\colon \mathbb{C} &\longrightarrow \mathbb{C}\\ z &\longmapsto z + z^m \end{aligned} \tag{29.2}$$

Then $\mathrm{Fix}(f) = \{0\}$. Then

$$1 - df_0 = \begin{cases} -1, & m = 1;\\ 0, & m > 1, \end{cases} \tag{29.3}$$

[41]Guillemin-Pollack give an argument which does not rely on the theorems in ODE which underlie flows. However, these theorems are fundamental, which is why I introduce you to them here.

so 0 is a Lefschetz fixed point iff $m = 1$. For $m > 1$ we make the perturbation

$$f_\epsilon(z) = z + z^m - \epsilon^m, \qquad \epsilon \in \mathbb{R}. \tag{29.4}$$

If $\epsilon \neq 0$, then $\mathrm{Fix}(f_\epsilon) = \{\epsilon, \epsilon\omega, \dots, \epsilon\omega^{m-1}\}$, where $\omega = e^{2\pi i/m}$ is a primitive n^{th} root of unity. Each of these fixed points is Lefschetz: at ϵz^k we compute $1 - df_\epsilon$ is multiplication by $-m\epsilon^{m-1}\omega^{k(m-1)}$, which is invertible. This complex linear transformation on $\mathbb{C}$ preserves the orientation of the underlying real vector space $\mathbb{R}^2$, so each local Lefschetz number is $+1$ and the total Lefschetz number is m.

Remark 29.5. The intuition is that under a generic perturbation, an isolated fixed point of a self-map breaks up (explodes) into a constellation of Lefschetz fixed points. The sum of the local Lefschetz numbers of those fixed points is an invariant we attach to the unperturbed fixed point, which may not be Lefschetz. We develop this idea in general and return to (29.2) in Example 29.26 after developing the theory.

(29.6) *Two linear algebra lemmas.* The proof which follows is based on (and proves) the "polar decomposition" of a linear transformation.

Lemma 29.7. *Let V be a finite dimensional real inner product space and $T\colon V \to V$ an invertible linear transformation. Then T is homotopic to an orthogonal transformation.*

Proof. The endomorphism $P = T^*T$ is positive and self-adjoint, hence it is diagonalizable. Decompose $V = V_1 \oplus \dots \oplus V_r$ into the orthogonal eigenspaces of P, so that P acts as multiplication by a scalar $\lambda_i > 0$ on V_i. Define $Q\colon V \to V$ to act as $\sqrt{\lambda_i}$ on V_i. Then $O := TQ^{-1}$ is orthogonal:

$$(TQ^{-1})^*(TQ^{-1}) = Q^{-1}T^*TQ^{-1} = Q^{-1}Q^2Q^{-1} = \mathrm{id}_V\,. \tag{29.8}$$

Define a homotopy $Q_t\colon V \to V$, $t \in [0,1]$, in which Q_t acts as multiplication by $(1-t) + t\sqrt{\lambda_i}$ on V_i; the desired homotopy of T to O is $T_t = OQ_t$. □

The following lemma compares the sign of the determinant of an invertible endomorphism, which is used to compute intersection numbers, with the degree of a map on spheres.[42]

Lemma 29.9. *Let V be a finite dimensional inner product space and $T\colon V \to V$ an invertible linear transformation. There is an induced map $\frac{T}{\|T\|}\colon S(V) \to S(V)$ on the unit sphere. Then*

$$\operatorname{sign}\det T = \deg\left(\frac{T}{\|T\|}\right). \tag{29.10}$$

Proof. Both sides of (29.10) are invariant under homotopy of T, so by Lemma 29.7 we may replace T with an orthogonal transformation O. By composition with a rotation, which does not change the

[42] I use an inner product, but point out the following more canonical construction. If V is a finite dimensional real vector space, then the space of rays in V, which is the quotient of $V \backslash \{0\}$ by the scaling action of $\mathbb{R}^{>0}$, is diffeomorphic to the unit sphere with respect to any inner product. A linear transformation $T\colon V \to V$ induces a self map of this space of rays.

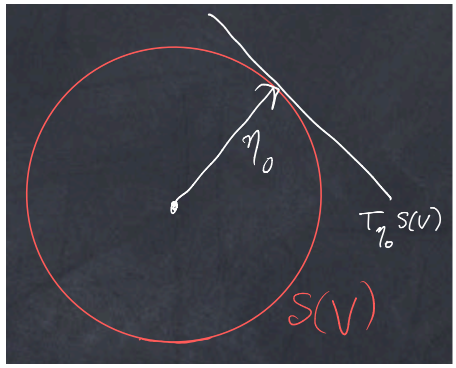


FIGURE 109. The proof of Lemma 29.9

degree (Proposition 25.39(4)), we may assume that O has a fixed vector η_0; see Figure 109. The degree of $\frac{T}{\|T\|}$ is ± 1 according to whether the differential of $O\big|_{S(V)}$ at η_0 preserves or reverses orientation on $T_{\eta_0}S(V)$. There is an orthogonal decomposition $V = \mathbb{R}\cdot\eta_0 \oplus T_{\eta_0}S(V)$ which is invariant under O, and since $O\big|_{\mathbb{R}\cdot\eta_0} = \mathrm{id}$, it follows that $\det O = \det O\big|_{T_{\eta_0}S(V)}$. Finally, observe that the differential of $O\big|_{S(V)}$ at η_0 equals $O\big|_{T_{\eta_0}S(V)}$ and that $\det O = \pm 1$ since O is orthogonal. $\square$

(29.11) *Isolated fixed point in affine space.* We work locally: on an open subset of affine space.

Theorem 29.12. *Let V be a finite dimensional real inner product space, A an affine space over V, $U \subset A$ an open subset, $f\colon U \to A$ a smooth map, $p \in \mathrm{Fix}(f) \cap U$ a fixed point of f, and ϵ a positive real number such that $\mathrm{Fix}(f) \cap \overline{B_\epsilon(p)} = \{p\}$. Let $S_\epsilon(p)$ be the sphere of radius ϵ about p and define*

$$\varphi_{p,\epsilon}(f)\colon S_\epsilon(p) \longrightarrow S(V) \qquad q \longmapsto \frac{q - f(q)}{\|q - f(q)\|}, \tag{29.13}$$

where $S(V) \subset V$ is the unit sphere. Then

(1) *There exists a homotopy $f_t\colon U \to A$, $t \in [0,1]$, such that $f_0 = f$ on U, $f_t = f$ on $U \backslash \overline{B_\epsilon(p)}$ for all $t \in [0,1]$, and $g = f_1\big|_{B_\epsilon(p)}$ is Lefschetz.*
(2) *The sum of Lefschetz numbers of the fixed points of f_1 in $B_\epsilon(p)$ is*

$$\deg \varphi_{p,\epsilon}(f) = \sum_{p' \in \mathrm{Fix}(g)} L_{p'}(g). \tag{29.14}$$

The left hand side of (29.14) measures the local contribution of the isolated fixed point p of f to the global Lefschetz number, once we transfer to a compact manifold.

Proof. Let $\rho\colon U \to [0,1]$ be a smooth function such that $\rho \equiv 1$ on $B_{\epsilon/2}(p)$ and $\mathrm{supp}(\rho) \subset B_\epsilon(p)$. For $\xi \in V$ set

$$f_t^\xi(q) = f(q) + t\rho(q)\xi, \qquad t \in [0,1]. \tag{29.15}$$

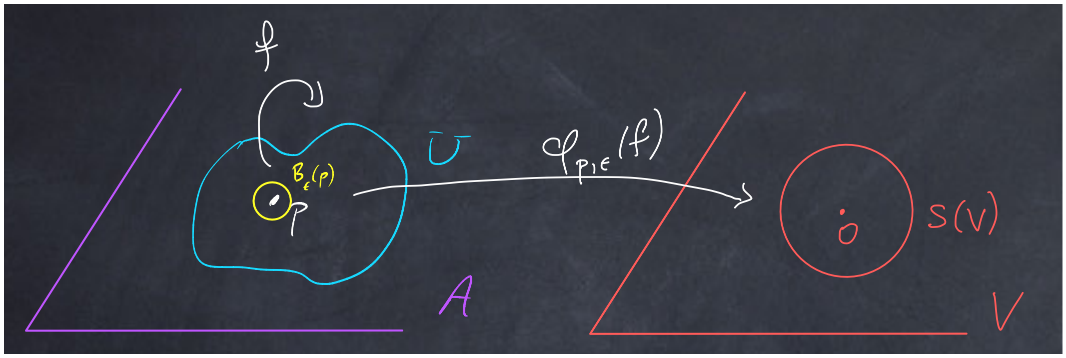


FIGURE 110. Local contribution to the Lefschetz number

Then $f_0^\xi = f$ on U and $f_t^\xi = f$ on $U\backslash B_\epsilon(p)$ for all $t \in [0,1]$. If $q \in B_\epsilon(p)\backslash B_{\epsilon/2}(p)$, then

$$\|q - f_t^\xi(q)\| \geqslant \|q - f(q)\| - |t|\,\|\xi\|. \tag{29.16}$$

Since f has no fixed points on $\overline{B_\epsilon(p)}$ other than p, we can and do choose $\delta > 0$ such that $\|q - f(q)\| > \delta$ on $\overline{B_\epsilon(p)} \setminus B_{\epsilon/2}(p)$. Then from (29.16), if $\|\xi\| < \delta/2$ and $q \in B_\epsilon(p)\backslash B_{\epsilon/2}(p)$ we have $q \neq f_t^\xi(q)$. Therefore, $\mathrm{Fix}\bigl(f_t^\xi(q)\bigr) \cap B_\epsilon(p) \subset B_{\epsilon/2}(p)$. Observe[43] that $f_t^\xi(q) = f(q) + t\xi$ on $B_{\epsilon/2}(p)$. Choose $\xi \in B_{\delta/2}(0) \subset V$ to be a regular value of

$$\begin{aligned} B_{\epsilon/2}(p) &\longrightarrow V \\ q &\longmapsto q - f(q) \end{aligned} \tag{29.17}$$

and then set $f_t = f_t^\xi$ and $g = f_1\big|_{B_\epsilon(p)}$. If $p' \in \mathrm{Fix}(g) \cap B_{\epsilon/2}(p)$, then since ξ is a regular value of (29.17) it follows that the map $1 - dg_{p'} \colon V \to V$ is an isomorphism, i.e., p' is a Lefschetz fixed point of g. This proves (1). Note that $\mathrm{Fix}(g) \cap \overline{B_\epsilon(p)} = \{p_1, \dots, p_N\}$ is a finite set since Lefschetz fixed points are isolated and $\overline{B_\epsilon(p)}$ is compact.

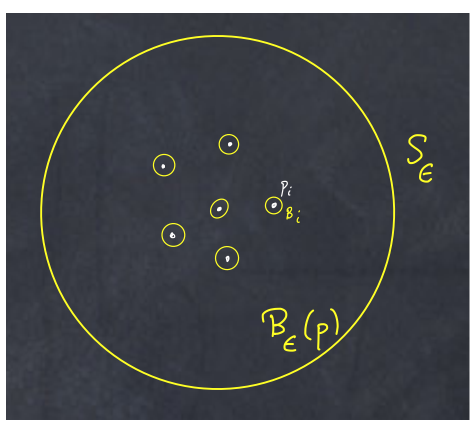


FIGURE 111. Reduction of the computation to Lefschetz fixed points

[43]Deformation by a translation is also what we used in (29.4).

By the homotopy invariance of degree, $\deg \varphi_{p,\epsilon}(f_t)$ is constant in t and equals the left hand side of (29.14) at $t = 0$. (Note that there are no fixed points of f_t on $S_\epsilon(p)$, so $\varphi_{p,\epsilon}(f_t)$ is well-defined.) Choose an open ball $B_{\epsilon_i}(p_i) \subset B_\epsilon(p)$ about p_i and arrange that the $B_{\epsilon_i}(p_i)$ are pairwise disjoint. The bordism invariance of degree (Proposition 25.39(3)) applied to $\overline{B_\epsilon(p)} \setminus \bigsqcup_i B_{\epsilon_i}(p_i)$ (see Figure 111) implies

$$\deg \varphi_{p,\epsilon}(g) = \sum_i \deg_{\partial B_{\epsilon_i}(p_i)} \left(q \longmapsto \frac{q - g(q)}{\|q - g(q)\|} \right). \tag{29.18}$$

Fix i and define the homotopy of functions $\partial B_{\epsilon_i}(p_i) \to V$:

$$g_t(p_i + \epsilon_i \eta) = \begin{cases} \dfrac{g(p_i + t\epsilon_i \eta) - p_i}{t}, & t \in (0,1]; \\ dg_{p_i}(\epsilon_i \eta), & t = 0, \end{cases} \tag{29.19}$$

where $\eta \in S(V)$. Under this homotopy the degree in the i^{th} term on the right hand side of (29.18) does not change, so it is equal to the degree of $\frac{T}{\|T\|}\big|_{S(V)}$ for $T = 1 - dg_{p_i} \colon V \to V$. By Lemma 29.9 this equals $\operatorname{sign}\det T$, which is the local Lefschetz number. □

Corollary 29.20. *In the situation of Theorem 29.12, if p is a Lefschetz fixed point of f, then $L_p(f) = \deg \varphi_{p,\epsilon}(f)$.*

We have succeeded in finding a formula at an isolated fixed point—the degree of (29.13)—which generalizes the local Lefschetz number. We turn now to its globalization on a manifold.

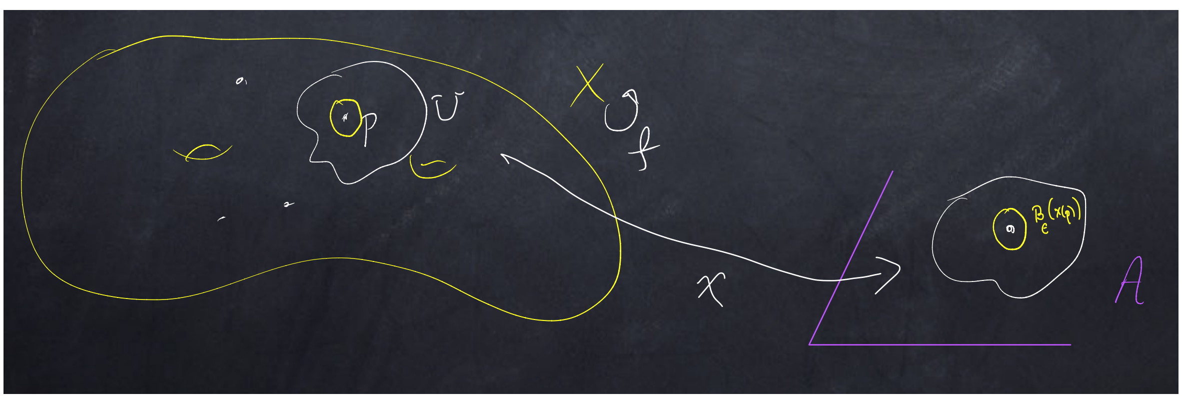


FIGURE 112. Exploding an isolated fixed point on a manifold

(29.21) *Isolated fixed points on a smooth manifold.* Let X be a compact manifold, assumed oriented for convenience, and suppose $f \colon X \to X$ is a smooth map with isolated fixed points. (A fixed point $p \in X$ is *isolated* if there exists an open neighborhood $N \subset X$ of p such that $\operatorname{Fix}(f) \cap N = \{p\}$.) For each $p \in \operatorname{Fix}(f)$ we want to define a local Lefschetz number $L_p(f) \in \mathbb{Z}$ so that the global Lefschetz number $L(f)$ is the sum of the local Lefschetz numbers. We sketch the construction now.

Choose a coordinate chart (U,x) such that $p \in U$, say $x\colon U \to A$ for an affine space A over an inner product space, set $\tilde p = x(p)$, and fix $\epsilon > 0$ such that $x^{-1}\bigl(B_\epsilon(\tilde p)\bigr)$ contains no fixed points other than p and $f\bigl(x^{-1}\bigl(B_\epsilon(\tilde p)\bigr)\bigr) \subset U$. Then f transports via x to a map $\tilde f\colon B_\epsilon(\tilde p) \to A$ with $\tilde p$ as its unique fixed point.

Definition 29.22. Define

$$L_p(f) = \deg \varphi_{p,\epsilon}(\tilde f), \tag{29.23}$$

where $\varphi_{p,\epsilon}(\tilde f)$ is the map (29.13).

We claim that $L_p(f)$ is independent of the choice of chart (U,x) and of $\epsilon > 0$. To see this, observe that Theorem 29.12 identifies $L_p(f)$ with the sum of the local Lefschetz numbers at *Lefschetz* fixed points of a perturbation of f supported in $x^{-1}\bigl(B_\epsilon(\tilde p)\bigr)$. Since these local Lefschetz numbers are defined intrinsically on X, without reference to a coordinate chart, it follows that (29.23) is independent of the choice of coordinate chart and of the choice of ϵ.

The following assertion about the global Lefschetz number follows from (29.14).

Theorem 29.24. *Let X be a compact manifold and suppose $f\colon X \to X$ is a smooth map with isolated fixed points. Then*

$$L(f) = \sum_{p \in \mathrm{Fix}(f)} L_p(f). \tag{29.25}$$

Example 29.26. As usual, write $\mathbb{CP}^1 = \mathbb{C} \amalg \{\infty\}$, and define the smooth map $f\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ to agree with (29.2) on $\mathbb{C}$ and send $\infty \mapsto \infty$. Then $\mathrm{Fix}(f) = \{0,\infty\}$. The perturbation (29.4) and subsequent computation prove that the local Lefschetz number of f at 0 is

$$L_0(f) = m. \tag{29.27}$$

We can also derive (29.27) directly as the degree of (29.13), which is the map $z \mapsto -z^m$ on the unit circle $\{|z| = 1\}$. This is the composition of a half-turn and $z \mapsto z^m$, so has degree m. To compute $L_\infty(f)$ we write $w = 1/z$ and then f has the local form

$$w \longmapsto \frac{1}{\frac{1}{w} + \frac{1}{w^m}} = \frac{w^m}{1 + w^{m-1}} \tag{29.28}$$

near $w = 0$. If $m > 1$, then the differential of (29.28) at $w = 0$ vanishes, from which we see $w = 0$ is a Lefschetz fixed point and the local Lefschetz number is 1. If $m = 1$ the differential is multiplication by $1/2$, so in this case too we conclude

$$L_\infty(f) = 1. \tag{29.29}$$

Now Theorem 29.24 implies that the global Lefschetz number is $L(f) = 1 + m$. Note that $\deg f = m$, and so

$$L(f) = 1 + \deg f. \tag{29.30}$$

This formula holds for *any* self-map of $\mathbb{CP}^1$, as follows from Theorem 29.82 below, though we will not prove (29.30) here.

Interlude of vector fields, integral curves, and flows

We first reprise the discussion of integral curves from (**14.27**) and (**14.31**). Then we move on to state the basic theorems about flows, both local and global.

(29.31) *Integral curves.* The basic ordinary differential equation on a manifold is for a motion with prescribed velocity.

Definition 29.32. Let X be a smooth manifold, let ξ be a vector field on X, and suppose $(a,b)\subset\mathbb{R}$ is an open interval. Then a motion $\gamma\colon (a,b)\to X$ is an *integral curve* of ξ if

$$\dot\gamma(t)=\xi_{\gamma(t)},\qquad t\in(a,b). \tag{29.33}$$

A globalized form of the fundamental theorem of ODEs is the existence and uniqueness of a maximal integral curve with given initial value.

Theorem 29.34. *Let X be a smooth manifold, let ξ be a vector field on X, and fix $x\in X$. Then there exists a unique maximal integral curve*

$$\gamma_x\colon \big(a(x),b(x)\big)\longrightarrow X \tag{29.35}$$

such that $\gamma_x(0)=x$. Furthermore, $-\infty\leqslant a(x)<0<b(x)\leqslant+\infty$.

The maximality and uniqueness mean that if $\delta\colon(c,d)\to X$ is any integral curve of ξ satisfying $\delta(0)=x$, then $a(x)\leqslant c<0<d\leqslant b(x)$ and $\delta=\gamma_x\big|_{(c,d)}$.

Definition 29.36. The vector field ξ is *complete* if $a(x)=-\infty$ and $b(x)=+\infty$ for all $x\in X$.

In other words, all integral curves of X exist for all time.

Remark 29.37.

1. If X is compact, then every vector field on X is complete.
2. The vector field $x^2\,\partial/\partial x$ on $\mathbb{R}$ is not complete. Nor is the vector field $\partial/\partial x$ on $\mathbb{R}^{\neq 0}$.

(29.38) *Global and local flows.* Fix a vector field $\xi\in\mathfrak{X}(X)$. We consider simultaneously all maximal integral curves of ξ and show (well, state) that they assemble into a *flow*.

Definition 29.39. A *global flow* $\varphi\colon\mathbb{R}\times X\to X$ on a smooth manifold X is a smooth function which is also a homomorphism $\mathbb{R}\to\mathrm{Diff}(X)$.

Here $\mathrm{Diff}(X)$ is the group of diffeomorphisms $X\to X$. The diffeomorphism at $t\in\mathbb{R}$ is denoted φ_t, i.e., for $t\in\mathbb{R}$ and $x\in X$ we write $\varphi_t(x)=\varphi(t,x)$. A complete vector field determines a global flow, but a general vector field determines a flow defined on an open subset of $\mathbb{R}\times X$. Set

$$\mathcal{D}_t=\big\{x\in X: t\in\big(a(x),b(x)\big)\big\},\qquad t\in\mathbb{R}, \tag{29.40}$$

and define

$$\varphi(t,x) = \varphi_t(x) = \gamma_x(t), \qquad \text{if } x \in \mathcal{D}_t. \tag{29.41}$$

The following theorem, which is essentially Theorem 1.48 in Warner's *Foundations of Differentiable Manifolds and Lie Groups*, gives the main properties of $\mathcal{D}_t$ and φ. I defer to that reference for the proof, assuming the fundamental theorem of ODE.

Theorem 29.42.

(1) *$\mathcal{D}_t \subset X$ is open and $\bigcup_{t>0} \mathcal{D}_t = X$.*
(2) *$\varphi_t \colon \mathcal{D}_t \to \mathcal{D}_{-t}$ is a diffeomorphism with inverse φ_{-t}.*
(3) *For $t_1, t_2 \in \mathbb{R}$, we have $\varphi_{t_1}^{-1}(\mathcal{D}_{t_2}) \cap \mathcal{D}_{t_1} \subset \mathcal{D}_{t_1+t_2}$ and $\varphi_{t_2} \circ \varphi_{t_1} = \varphi_{t_1+t_2}$ on their common domain. (The two domains are equal if t_1, t_2 have the same sign.)*
(4) *If $U \subset X$ is an open subset and $x \in U$, then there exist $\epsilon \in \mathbb{R}^{>0}$ and an open subset $U' \subset U$ with $x \in U'$ such that $\varphi\big((-\epsilon,\epsilon) \times U'\big) \subset U$.*
(5) *There exists an open subset $\mathcal{U} \subset \mathbb{R} \times X$ and a smooth function $\widehat{\varphi} \colon \mathcal{U} \to X$ such that $\{0\} \times X \subset \mathcal{U}$ and if $(t,x) \in \mathcal{U}$ then $x \in \mathcal{D}_t$ and $\widehat{\varphi}(t,x) = \varphi_t(x)$.*
(6) *If ξ is complete, then $\mathcal{D}_t = X$ for all $t \in \mathbb{R}$ and a global flow exists.*

We use 'φ' in place of '$\widehat{\varphi}$'; it is the local flow. The local flow is unique in the sense that any two agree on their common domain. We say that ξ is the *generator* of the flow φ, or that ξ *generates* the flow.

(29.43) *The flow in local coordinates.* We continue with a vector field ξ and the flow φ it generates. Let $(U; x^1, \dots, x^n)$ be a standard chart, and write $\xi = \xi^i \frac{\partial}{\partial x^i}$ for the vector field in the chart. As in Theorem 29.42(4) we choose $\epsilon > 0$ and $U' \subset U$ such that if $t \in (-\epsilon, \epsilon)$ and $x \in U'$, then $\varphi(t,x) \in U$. In the formulæ below the arguments t, x are implicitly restricted to $(-\epsilon,\epsilon) \times U'$. Write $(t;x) = (t; x^1, \dots, x^n)$ and

$$\varphi(t,x) = \big(\varphi^1(t;x), \dots, \varphi^n(t;x)\big). \tag{29.44}$$

The elementary properties of the flow imply

$$\begin{aligned} \varphi^i(0,x) &= x^i \\ \dot\varphi^i(t;x) &= \xi^i_{\varphi(t;x)} \\ \frac{\partial \varphi^i}{\partial x^j}\bigg|_{t=0} &= \delta^i_j \\ \frac{\partial^2 \varphi^i}{\partial x^k \partial x^j}\bigg|_{t=0} &= 0. \end{aligned} \tag{29.45}$$

We abbreviate the second equation: $\dot\varphi^i = \xi^i$.

The Poincaré-Hopf theorem

We now prove the generalization of Theorem 28.40 which relaxes the transversality condition to the condition that the vector field ξ have isolated zeros. To begin, we derive a local index at the zeros of a vector field from the formula (29.23) for the local Lefschetz number.

(29.46) *The local index.* Let V be a finite dimensional real inner product space, suppose $U \subset V$ is an open neighborhood of $0 \in V$, and let $\xi\colon U \to V$ be a smooth map whose only fixed point is 0. We interpret ξ as a vector field on U with an isolated zero.

Definition 29.47. The *index* of ξ at 0, denoted $\operatorname{ind}_0(\xi)$, is the degree of the map

$$\frac{\xi}{\|\xi\|}\colon S_\delta(V) \longrightarrow S(V) \tag{29.48}$$

for sufficiently small $\delta > 0$.

Here $S_\delta(V)$ is the sphere of radius δ about 0, and we take $\delta \in (0, \epsilon)$ where $B_\epsilon(0) \subset U$. Orient the spheres consistently, so that the homothety which maps $S_\delta(V)$ to $S(V)$ is orientation-preserving. The degree is the same for both choices of compatible orientations. The degree is also independent of δ by homotopy invariance.

The differential of ξ at 0 is a linear map $d\xi_0\colon V \to V$.

Lemma 29.49. *If $d\xi_0$ is an isomorphism, then*

$$\operatorname{ind}_0(\xi) = \operatorname{sign}\det d\xi_0. \tag{29.50}$$

Proof. Write (29.48) as the map

$$\eta \longmapsto \frac{\xi(\delta\eta)}{\|\xi(\delta\eta)\|} = \frac{\xi(\delta\eta)/\delta}{\|\xi(\delta\eta)/\delta\|} \tag{29.51}$$

on $S(V)$ and take the limit as $\delta \to 0$ to obtain the map $\dfrac{d\xi_0}{\|d\xi_0\|}$. Now apply Lemma 29.9. $\square$

Remark 29.52. I highly recommend looking in Guillemin-Pollack and other books for pictures and computations of the local index in low dimensions. Note that Lemma 29.49 is the transverse case when the index is ± 1, as in Lemma 28.23 and Theorem 28.40.

(29.53) *The local index and the local Lefschetz number.* Now apply Theorem 29.42 to construct a local flow $\varphi_t\colon (-\epsilon, \epsilon) \times U' \to U$ for some $\epsilon > 0$ and open subset $U' \subset U$ which contains 0. Then for each $t \in (-\epsilon, \epsilon)$, the map $\varphi_t\colon U' \to U$ has 0 as its unique fixed point. Recall the local Lefschetz number (29.13).

Proposition 29.54. *The local index of ξ is the local Lefschetz number of φ_t up to a sign:*

$$L_0(\varphi_t) = \operatorname{ind}_0(-\xi) \tag{29.55}$$

for all t.

The sign is unfortunate. Observe that the index of $-\xi$ is the index of the composition of (29.48) with the antipodal map on the sphere. Thus if $\dim V$ is even we have $\operatorname{ind}_0(-\xi) = \operatorname{ind}_0(\xi)$. In our application to the Euler number we can restrict to even-dimensional manifolds—the Euler number of an odd-dimensional manifold vanishes—and so we can replace $-\xi$ with ξ in (29.55).

Proof. The local Lefschetz number $L(\varphi_t)$ is the degree of the self map

$$\eta \longmapsto \frac{\delta\eta - \varphi_t(\delta\eta)}{\|\delta\eta - \varphi_t(\delta\eta)\|}, \qquad \eta \in S(V), \tag{29.56}$$

for any $\delta \in (-\epsilon, \epsilon)$. Now

$$\frac{\delta\eta - \varphi_t(\delta\eta)}{\|\delta\eta - \varphi_t(\delta\eta)\|} = \frac{\dfrac{\delta\eta - \varphi_t(\delta\eta)}{t}}{\left\|\dfrac{\delta\eta - \varphi_t(\delta\eta)}{t}\right\|} \xrightarrow{t\to 0} -\frac{\xi(\delta\eta)}{\|\xi(\delta\eta)\|}, \tag{29.57}$$

which, up to a sign, is the expression (29.51) whose degree is the local index of ξ. ∎

(29.58) *The local index on a smooth manifold.* Now suppose X is a smooth manifold and ξ is a vector field with an isolated zero at $p \in X$. Introduce a coordinate chart about p with values in a vector space V such that p maps to $0 \in V$. Transport ξ to the chart and use Definition 29.47 to compute its index as a degree. The degree is independent of the chart, since under a change of chart the map (29.48) is conjugated by a diffeomorphism of spheres, and the index of a diffeomorphism is ± 1.

(29.59) *The Poincaré-Hopf theorem.* The following is now a corollary of Theorem 29.24.

Theorem 29.60. *Let X be a compact manifold and suppose ξ is a vector field on X with isolated zeros. Then*

$$\chi(X) = \sum_{p \in \operatorname{Zero}(\xi)} \operatorname{ind}_p(\xi). \tag{29.61}$$

Proof. Since X is compact, by Theorem 29.42(6) there is a global flow φ_t generated. Furthermore, for sufficiently small $|t|$ we have $\operatorname{Fix}(\varphi_t) = \operatorname{Zero}(\xi)$. For such t we use Proposition 29.54 to equate the local Lefschetz number of φ_t at a fixed point p with the local index of ξ at p, with a minus sign if $\dim_p X$ is odd. Now the homotopy invariance of the Lefschetz number implies $\chi(X) = L(\varphi_t)$ for all t. We apply it for sufficiently small $|t|$ to deduce (29.61), but with a sign $(-1)^{\dim_p X}$ in the summand. Since the Euler number of each odd dimensional component of X vanishes, the formula is true without the sign. ∎

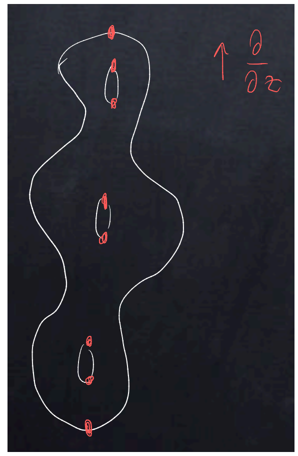


FIGURE 113. A surface of genus g

Example 29.62. Let $X = \Sigma_g$ be a surface of genus g, embedded generically into standard Euclidean 3-space $\mathbb{E}^3$ with coordinates x, y, z; see Figure 113. Let ξ be the vector field on X whose value at $p \in X$ is the orthogonal projection of $\partial/\partial z$ onto the tangent plane $T_pX \subset \mathbb{R}^3$. We can arrange that ξ vanishes at isolated points. These are the critical points of the restriction of $z\colon \mathbb{E}^3 \to \mathbb{R}$ to X, and occur when $T_pX \subset \mathbb{R}^3$ is the (x, y)-plane. Near such a point $p \in x$ we can locally write X as the graph of a function $f\colon T_pX \to \mathbb{R}$. Identify T_pX with $\mathbb{R}^2$ and use standard coordinates x, y. Then the function $f = f(x, y)$ satisfies $f(0, 0) = 0$, $f_x(0, 0) = 0$, $f_y(0, 0) = 0$, where the subscripts denote partial derivatives. The tangent space $T_{(x,y)}X \subset \mathbb{R}^3$ for small x, y is the span of $\partial/\partial x + f_x\partial/\partial z$ and $\partial/\partial y + f_y\partial/\partial z$. It is isomorphic by orthogonal projection to the span of $\partial/\partial x$ and $\partial/\partial y$. Under that identification we compute

$$\xi_{(x,y)} = f_x \frac{\partial}{\partial x} + f_y \frac{\partial}{\partial y}. \tag{29.63}$$

Therefore, the differential is represented by the Hessian matrix

$$d\xi_{(0,0)} = \begin{pmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{pmatrix}$$

If this is nondegenerate, then the sign of the determinant is $+1$ at maxima and minima of f and the sign of the determinant is -1 at saddle points. As in the picture, this nondegeneracy can be arranged, and furthermore we can arrange for a unique maximum, a unique minimum, and $2g$ saddle points. Therefore, (29.61) computes

$$\chi(\Sigma_g) = 2 - 2g. \tag{29.64}$$

Introduction to de Rham cohomology

(29.65) *Definition of de Rham cohomology.* Let X be a smooth n-dimensional manifold. Recall the de Rham complex (22.53)

$$0 \longrightarrow \Omega^0(X) \xrightarrow{\;d\;} \Omega^1(X) \xrightarrow{\;d\;} \Omega^2(X) \xrightarrow{\;d\;} \cdots \xrightarrow{\;d\;} \Omega^n(X) \longrightarrow 0, \tag{29.66}$$

in which $d^2 = 0$.

Definition 29.67.

(1) A differential form $\alpha \in \Omega^\bullet(X)$ is *closed* if $d\alpha = 0$.
(2) A differential form $\alpha \in \Omega^\bullet(X)$ is *exact* if there exists $\beta \in \Omega^\bullet(X)$ such that $\alpha = d\beta$.

Since $d^2 = 0$ we have for each $k \in \{0, 1, \dots, n\}$ the inclusions

$$d\Omega^{k-1}(X) \subset \Omega^k_{\text{closed}}(X) \subset \Omega^k(X), \tag{29.68}$$

where $d\Omega^{k-1}(X)$ is the space of exact k-forms.

Definition 29.69. The *de Rham cohomology* in degree k is the vector space

$$H^k_{\text{dR}}(X) = \frac{\Omega^k_{\text{closed}}(X)}{d\Omega^{k-1}(X)}. \tag{29.70}$$

Example 29.71. A function $f \in \Omega^0(X)$ is closed iff f is locally constant, and it is exact iff it vanishes. Hence $H^0_{\text{dR}}(X)$ is the vector space of locally constant functions on X.

Example 29.72. We indicate an isomorphism

$$H^1_{\text{dR}}(\mathbb{R}/\mathbb{Z}) \xrightarrow{\;\cong\;} \mathbb{R} \tag{29.73}$$

Namely, a function on $\mathbb{R}/\mathbb{Z}$ lifts under the covering $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ to a periodic function $f\colon \mathbb{R} \to \mathbb{R}$, i.e., one for which $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Similarly, a 1-form $\alpha \in \Omega^1(\mathbb{R}/\mathbb{Z})$ lifts to a 1-form $\alpha = g(x)dx \in \Omega^1(\mathbb{R})$ in which g is a periodic function. Every 1-form on a 1-manifold is closed, and we claim α is exact iff

$$\int_{\mathbb{R}/\mathbb{Z}} \alpha = \int_0^1 g(x)dx = 0. \tag{29.74}$$

The isomorphism (29.73) is integration over $\mathbb{R}/\mathbb{Z}$. We leave the reader to fill in the details.

(29.75) *Pullbacks.* Let $f\colon X' \to X$ be a smooth map of smooth manifolds. As noted in Theorem 22.56 there is an induced pullback map of differential forms

$$f^*\colon \Omega^k(X) \longrightarrow \Omega^k(X'), \tag{29.76}$$

and $df^* = f^*d$. This latter implies that f^* maps closed forms to closed forms and exact forms to exact forms, hence induces a map

$$f^*\colon H^k_{\text{dR}}(X) \longrightarrow H^k_{\text{dR}}(X') \tag{29.77}$$

on de Rham cohomology.

(29.78) *The de Rham theorem.* The utility of de Rham cohomology arises from a comparison with other cohomology theories on a smooth manifold, say the singular theory. The book by Frank Warner has a very nice treatment of the following.

Theorem 29.79. *Let X be a smooth manifold. Then there exists a natural isomorphism*

$$H^k_{\mathrm{dR}}(X) \longrightarrow H^k(X;\mathbb{R}). \tag{29.80}$$

The codomain of (29.80) is the singular cohomology with real coefficients, which is isomorphic to $\mathrm{Hom}\big(H_k(X),\mathbb{R}\big)$, where $H_k(X)$ is the singular homology group. The map (29.80) is constructed by integration over "smooth singular chains".

Fixed point theorems

(29.81) *The Lefschetz fixed point theorem.* We have defined the Lefschetz number of a self map $f\colon X \to X$ in (28.46) as an intersection number, and in **(29.21)** have a formula in terms of local data at fixed points in case every fixed point of f is isolated. This has more power if we can compute the global intersection number effectively. The following theorem does this in terms of de Rham cohomology and the induced map (29.77). Implicit is the assertion that the de Rham cohomology vector spaces of a *compact* manifold are finite dimensional.

Theorem 29.82. *Let $f\colon X \to X$ be a self map of a compact oriented manifold. Then*

$$L(f) = \sum_{k=0}^{\dim X} (-1)^k \operatorname{Tr}\left(f^*\Big|_{H^k_{\mathrm{dR}}(X)} \right). \tag{29.83}$$

Sadly, it is beyond the scope of this course to prove Theorem 29.82.

Remark 29.84. Fixed point theorems are important in many parts of geometry and beyond. For example, recall that we use the contraction fixed point theorem to prove the inverse function theorem and to construct solutions to ordinary differential equations, so more geometrically to construct integral curves of vector fields. There are also infinite dimensional analogs of degree (Leray-Schauder) and of fixed point theorems (Schauder) with many applications to integral and partial differential equations; see *Topics in Nonlinear Functional Analysis* by Louis Nirenberg.

Remark 29.85. The relationship between fixed points and cohomology expressed in the Lefschetz fixed point theorem has a powerful arithmetic cousin introduced by André Weil. It uses the Frobenius map of a variety defined over a finite field.

Remark 29.86. Atiyah-Bott prove a generalization of Theorem 29.82 for linear elliptic differential operators, a theorem with diverse applications in topology, geometry, and representation theory.

Problem Set # 1

M382D: Differential Topology

Due: January 27

There will be weekly homework assignments due each Thursday at the beginning of class on Gradescope. Please work the problems neatly. There is no need to copy over the problem or hand in the problem sheet. Do not show scratch work. Try the problems on your own first. Then feel free to discuss them and work together with classmates, friends, parents, etc. However, I expect you to write up your own solutions to the problems. Please come and discuss the problems (and the class generally) with me during office hours.

Do the best you can on these problems. Please write your scratch work on scratch paper and only hand in coherent, readable arguments and calculations. You will have to produce good mathematical writing on your tests, prelims, and in your future mathematical writing, so why not practice now on these homeworks? Good mathematical writing is concise, so don't write volumes of material. Some of the problems are computational, others conceptual. Some may involve ideas you are not familiar with. (Here is one place where your classmates may be able to help you.) Often I leave problems open-ended. Feel free to explore. Some are meant to be challenging, so do not get discouraged if you find them difficult. I certainly don't expect anyone to do all of the problems or even to come close. They are a guide for your learning of the material in the lectures and the book.

You should be reading my lecture notes as well as Warner and Guillemin/Pollack along with the class, even if I do not give specific reading assignments. (But follow the definitions given in lecture!) Please skim through the book immediately to see if you are comfortable with it and think it is at the right level for you. (We will only cover Chapters 1, 2, and 4 of Warner.) And don't forget Milnor's *Topology from the Differentiable Viewpoint*—it's a wonderful book and a wonderful piece of mathematical writing. Feel free to ask questions about the book during class or office hours. You should now be reading the first chapter in each of the text books. There are plenty of problems in each, and I suggest you try some of them.

All vector spaces and manifolds studied in this class are assumed to be finite dimensional. The coefficients are real. In this class, and in some parts of the mathematical world, the word 'manifold' is short for 'smooth manifold', which we use synonymously with 'C^∞ manifold'.

Problems

1. Suppose X, Y are manifolds and $f\colon X \to Y$ a smooth map.
   (a) Prove that $X \times Y$ is also a manifold.
   (b) Show that the *graph* $\Gamma(f) \subset X \times Y$ of f, defined by
   $$\Gamma(f) := \big\{(x, y) \in X \times Y : y = f(x)\big\},$$
   is a manifold.
   (c) Now suppose $U \subset X$ is an open subset. Define a manifold structure on U.

2. There are three different ways in which functions correspond to shapes. Let X, Y be sets and $f\colon X \to Y$. (You might prefer to think that X, Y are topological spaces and f is continuous, but it doesn't matter for this part.) The *graph* of f, which is a subset of $X \times Y$, is defined in the previous problem. The *image* of f is $f(X) \subset Y$. The fiber of f at $c \in Y$ is $f^{-1}(x) \subset X$. Graphs were discussed in the previous problem; now we consider fibers and images, all in the context of smooth maps of smooth manifolds.

   (a) Suppose $f\colon \mathbb{A}^3 \to \mathbb{R}$ is a smooth function. Define $X_c = f^{-1}(c)$ for all $c \in \mathbb{R}$. Is X_c necessarily a manifold? Think carefully about what that statement means. For a fixed f what can you say about the set of c for which X_c is a manifold? Try many examples. You might also want to try this problem with $\mathbb{A}^2$ replacing $\mathbb{A}^3$.

   (b) Repeat with $f\colon \mathbb{R} \to \mathbb{A}^3$ and $X = f(\mathbb{R}) \subset \mathbb{A}^3$. Here there is no parameter ('c' in the previous), so you'll have to vary the map f.

3. (a) Suppose A is an n dimensional affine space over a vector space V. Let $\gamma\colon (a,b) \to A$ be a smooth curve, where $(a,b) \in \mathbb{R}$ is an open set. For $t \in (a,b)$ define the tangent vector $\dot\gamma(t) \in V$.

   (b) Take $n = 2$ and $A = \mathbb{A}^2$ so we write $\gamma(t) = \big(x(t), y(t)\big)$. What is the formula for the tangent vector in terms of the real-valued functions x, y? (What is the vector space V?)

   (c) Suppose the image of γ lies in the open set $\mathbb{A}^2 \setminus \{(x,0) : x \geqslant 0\}$. Introduce polar coordinates (r,θ) on this open set and write the curve as $\big(r(t), \theta(t)\big)$. How precisely are the functions r, θ defined? What is the tangent vector to the curve in terms of the functions r, θ?

4. Review the general form of the chain rule; here you will use it in a specific example. Define $f, g\colon \mathbb{A}^3 \to \mathbb{A}^3$ by $f(x,y,z) = (x^2, y^2, z^2)$ and $g(x,y,z) = (yz, xz, xy)$.

   (a) Write a formula for $g \circ f$ and compute $d(g \circ f)$ at a general point (x,y,z) from the formula.

   (b) Now compute dg and df separately and use the chain rule to compute $d(g \circ f)$. Compare your answer to that in part (a).

5. Prove the following assertions, which should look familiar.

   (a) Suppose $U \subset \mathbb{A}^n$ is a connected open set and $f\colon U \to \mathbb{A}^m$ is a smooth function whose differential df_x vanishes for all $x \in U$. Prove that f is constant.

   (b) Let $U, V \subset \mathbb{A}^n$ be open subsets and $f\colon U \to \mathbb{A}^n$ and $g\colon V \to \mathbb{A}^n$ be smooth maps such that the compositions $f \circ g$ and $g \circ f$ are defined and equal to the identity map. Prove that for each $x \in U$ the differential df_x is an invertible map.

   (c) Let U be a connected open subset of an affine space A and $f\colon U \to B$ a smooth map to an affine space B. Prove that f extends to an affine map $A \to B$ if and only if the differential $df\colon U \to \operatorname{Hom}(V,W)$ is constant. Here V, W are the vector spaces associated to the affine spaces A, B and $\operatorname{Hom}(V,W)$ is the vector space of linear maps from V to W.

6. Recall that $GL_n\mathbb{R}$ is the open subset of $n \times n$ real matrices which are invertible, so it inherits a manifold structure from that of the vector space of $n \times n$ matrices. Show that multiplication and inversion are smooth maps

$$GL_n\mathbb{R} \times GL_n\mathbb{R} \longrightarrow GL_n\mathbb{R}$$
$$GL_n\mathbb{R} \longrightarrow GL_n\mathbb{R}$$

Repeat for the group $GL_n\mathbb{C}$ of invertible complex matrices. This proves that $GL_n\mathbb{R}$ and $GL_n\mathbb{C}$ are *Lie groups*. At some point during the semester I strongly recommend reading Chapter 3 of Warner for some basics on Lie groups.

7. Fix positive numbers r and R with $r < R$. Let the torus T be the surface of revolution in $\mathbb{A}^3$ (with coordinates x, y, z) obtained by revolving the circle

$$y = 0, \quad (x - R)^2 + z^2 = r^2$$

about the z-axis.

(a) Show that T is a 2-manifold.

(b) Define the *Gauss map* $g\colon T \to S^2$ to the unit sphere in $\mathbb{A}^3$ by mapping a point $p \in T$ to the unit normal vector to T at p, viewed as a point of S^2. (Here I am relying on your geometric intuition, not on definitions we have discussed in this class.) Show that g is smooth. Compute its differential in some coordinate system.

8. Let V be a finite dimensional real vector space and $k \in \mathbb{Z}^{\geqslant 0}$. The *Grassmannian* is

$$\mathrm{Gr}_k(V) = \{W \subset V \text{ subspace of dimension } k\}.$$

Topologize $\mathrm{Gr}_k(V)$. Construct an atlas on $\mathrm{Gr}_k(V)$ following the start given in lecture. Can you do both at once?

9. Let X denote the set of affine lines in $\mathbb{A}^2$. Topologize X and show that it is a topological manifold. What is $\dim X$? Is X connected? Is X compact? Is X simply connected? Can you recognize X as a familiar topological manifold: do you know a familiar topological manifold which is homeomorphic to X?

Problem Set # 2

M382D: Differential Topology

Due: February 4

For the problems in Guillemin/Pollack, I would like you to use the definitions (of a manifold, tangent space, etc.) that we use in lecture and in the lecture notes. That said, please do read Guillemin/Pollack.

Problems in Guillemin/Pollack

Chapter 1, §2 (p. 11): 4, 10, 11
Chapter 1, §3 (p. 18): 5

Other Problems

1. Let M be a topological manifold and $\mathscr{A} = \{(U_\alpha, x_\alpha)\}_{\alpha \in A}$ an atlas. Define

$$\overline{\mathscr{A}} = \mathscr{A} \,\cup\, \{(U,x) \text{ standard charts on } M : (U,x) \text{ is } C^\infty\text{-related to all charts in } \mathscr{A}\,\}.$$

Prove that $\overline{\mathscr{A}}$ is a maximal atlas on M, i.e., a differential structure.

2. (a) Let $P(z) := a_n z^n + \cdots + a_1 z + a_0$ be a polynomial in a single complex variable; the coefficients a_i are complex numbers. Consider the family of equations $P(z) = s$ for a variable complex number s. Suppose that for some z_0, s_0 we have $P(z_0) = s_0$ and z_0 is a simple root of $P(z) - s_0$. Let $t \mapsto s_t$ be a smooth curve through s_0. Prove that there is a smooth curve $t \mapsto z_t$ around z_0 so that $P(z_t) = s_t$. For what values of t is this curve guaranteed to exist? What happens if z_0 is a double root?

   (b) A theorem in classical Euclidean geometry, named after the great Napolean, goes as follows. Suppose A, B, C are points in $\mathbb{E}^2$, where $\mathbb{E}^2$ is the *Euclidean plane,* the affine plane with the standard notion of distance and angle. Let C' be the point external to the triangle ABC such that the triangle ABC' is equilateral. Similarly, define equilateral triangles $A'BC$ and $AB'C$. Let A'', B'', C'' be the centers of the triangles BCA', CAB', ABC'. The theorem states that $A''B''C''$ is equilateral. You will have fun proving that if you haven't seen it before—there is a very elegant argument for it. But the statement I'm asking you to prove here is that the length of the side of $A''B''C''$ is a smooth function of the points A, B, C. You can do so without computing a formula.

3. Consider the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{A}^3 : x^2 + y^2 + z^2 = 1\}.$$

(a) There is an obvious inclusion $i\colon S^2 \to \mathbb{A}^3$. Show that the differential di_p at any point $p \in S^2$ is an injection $di_p\colon T_pS^2 \to \mathbb{R}^3$ and identify the image.

(b) More generally, suppose M is a smooth manifold and $i\colon M \to A$ an immersion into an affine space A over a vector space V. For each $p \in M$, show how to identify T_pM as a linear subspace of V.

(c) On the upper hemisphere $\{z > 0\}$ of S^2 consider the functions x, y to be coordinate function. As a second coordinate system we take spherical coordinates θ, ϕ defined by solving the equations

$$\begin{aligned} x &= \sin\phi\cos\theta \\ y &= \sin\phi\sin\theta. \end{aligned}$$

Identify a (maximal) subset of the upper hemisphere on which θ, ϕ is a coordinate system. (You may want to translate: replace θ, ϕ by $\theta - \theta_0$, $\phi - \phi_0$ for some θ_0, ϕ_0.) On that subset express the vector field $\partial/\partial x$ in terms of $\partial/\partial\theta$ and $\partial/\partial\phi$.

4. Let $x^1, \dots x^n$ and $y^1, \dots, y^n$ be two local coordinate systems (charts) on a smooth manifold, and suppose the domains agree. Let f be a smooth real-valued function defined on this common domain.

(a) Show

$$\frac{\partial f}{\partial y^j} = \frac{\partial x^i}{\partial y^j}\frac{\partial f}{\partial x^i}.$$

(b) Verify from (a) and other equations that

$$df = \frac{\partial f}{\partial x^i}dx^i = \frac{\partial f}{\partial y^j}dy^j.$$

(c) Compute

$$\frac{\partial^2 f}{\partial y^j \partial y^k}$$

in terms of partial derivatives of f in the x-coordinate system. Does

$$\frac{\partial^2 f}{\partial y^j \partial y^k}dy^j dy^k$$

behave nicely under coordinate change? You'll have to invent multiplication rules for the differentials to answer this. Spell out functionally (or theoretically if you want) what rules you are following.

5. This problem is a standard and important corollary of the inverse function theorem, called the implicit function theorem. It states a condition under which we can solve an equation of two variables implicitly for one variable as a function of the other.

   Suppose X, Y, Z are manifolds and $F\colon X \times Y \to Z$ a smooth map with $F(x_0, y_0) = z_0$ for some $x_0 \in X$, $y_0 \in Y$, and $z_0 \in Z$. Assume that the restriction of the differential $dF_{(x_0,y_0)}$ to $T_{y_0}Y \subset T_{(x_0,y_0)}(X \times Y)$ is an isomorphism onto $T_{z_0}Z$. Prove that there exists a neighborhood U of x_0 and V of y_0 and a smooth function $f\colon U \to V$ such that

$$F\big(x, f(x)\big) = z_0$$

   for all $x \in U$. (More generally, we can find a function $f_z\colon U \to V$ which solves the equation $F\big(x, f_z(x)\big) = z$ for z in a neighborhood of z_0.)

6. Let V be a finite dimensional real vector space of dimension at least 3. Specify an open subset $M \subset \mathbb{P}V \times \mathbb{P}V$ on which the map $f\colon M \to \mathrm{Gr}_2(V)$ which maps a pair of lines (ℓ_1, ℓ_2) to the 2-plane they span is defined. Compute the differential of f and prove that it is surjective at all points of M.

Problem Set # 3

M382D: Differential Topology

Due: February 11

Over the next few weeks we will cover Sard's theorem, the Whitney embedding theorem, and partitions of unity. As we proceed you should read over all of Chapter 1 in Guillemin/Pollack as well as the discussion on p. 22–35 of Warner. But be warned that his use of 'submanifold' is not standard and in particular is not ours; his is a '1:1 immersion'.

Problems in Guillemin/Pollack

Chapter 1, §4 (p. 25): 2, 5

Other Problems

1. This exercise is preparation for our discussion of partitions of unity.

   (a) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} e^{-1/x^2}, & x > 0; \\ 0, & x \leqslant 0. \end{cases}$$

   Prove that f is C^∞. Sketch the graph of f. Compare f to its Taylor series at $x = 0$.

   (b) Given real numbers $a < b$ show that

   $$g(x) := f(x-a)f(b-x)$$

   is smooth and vanishes outside the interval (a, b).

   (c) Given real numbers $a < b$, construct a C^∞ function $h\colon \mathbb{R} \to \mathbb{R}$ such that: (i) $h(x) = 0$ for $x \leqslant a$, (ii) $h(x) = 1$ for $x \geqslant b$, and (iii) h is monotonic nondecreasing.

   (d) Given real numbers $a < b < c < d$, construct a C^∞ function $k\colon \mathbb{R} \to \mathbb{R}$ so that (i) $k(x) = 0$ for $x \leqslant a$, (ii) $k(x) = 1$ for $b \leqslant x \leqslant c$, and (iii) $k(x) = 0$ for $x \geqslant d$.

   (e) Given real numbers $a^i < b^i < c^i < d^i$, $i = 1, \dots, n$, construct a C^∞ function $k\colon \mathbb{A}^n \to \mathbb{R}$ so that (i) $k(x^1, \dots, x^n) = 0$ if any $x^i \leqslant a^i$; (ii) $k(x^1, \dots, x^n) = 1$ if $b^i \leqslant x^i \leqslant c^i$ for all $i = 1, \dots, n$; and (iii) $k(x^1, \dots, x^n) = 0$ if any $x^i \geqslant d^i$.

   (f) Prove that on every manifold X there is a nonconstant C^∞ function $f\colon X \to \mathbb{R}$.

2. Consider the function

$$f(x, y, z) = x^4 + y^4 + z^4$$

defined on $\mathbb{A}^3$.

(a) Determine the *critical points* of f, that is, the points where the differential of f vanishes.

(b) Compute the differential of f in *cylindrical coordinates* r, z, θ given by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z. \end{aligned}$$

Do this two ways. First: write f in cylindrical coordinates and then differentiate. Second: differentiate f in rectangular coordinates and then change to cylindrical coordinates. Your answers should agree.

(c) Let $g\colon S^2 \to \mathbb{R}$ be the restriction of f to the unit sphere. What is the maximum value of g? Where is it attained? Can you do a complete analysis of the critical points, i.e., determine the maxima, minima, and saddle points? How many critical points are there? How many *critical values* (values of g at the critical points)?

3. (a) Define complex projective space $\mathbb{CP}^n$ as the manifold of equivalence classes

$$\mathbb{CP}^n = \left\{[z^0, z^1, \ldots, z^n] : z^i \in \mathbb{C},\ (z^0, z^1, \ldots, z^n) \neq (0, 0, \ldots, 0)\right\} /\!\sim,$$

where

$$[z^0, \ldots, z^n] \sim [z'^0, \ldots, z'^n] \qquad \text{if and only if} \qquad z'^i = \lambda z^i$$

for some nonzero complex number λ. Show that $\mathbb{CP}^n$ is a manifold. (Consider $U_i = \{[z^0, \ldots, z^n] : z^i \neq 0\}$.)

(b) Construct a diffeomorphism between $\mathbb{CP}^1$ and the standard 2-sphere.

(c) Identify the 3-sphere with the unit sphere in $\mathbb{C}^2$:

$$S^3 = \{(z^1, z^2) \in \mathbb{C}^2 : |z^1|^2 + |z^2|^2 = 1\}.$$

Prove that the map

$$\begin{aligned} f\colon S^3 &\longrightarrow \mathbb{CP}^1 \\ (z^1, z^2) &\longmapsto [z^1, z^2] \end{aligned}$$

is a submersion. What is the inverse image of a point? What can you say about the inverse images of two distinct points? How are they situated in S^3? The map f is the *Hopf fibration*.

4. Let A_t be a curve of symmetric $n \times n$ real matrices, defined for $t \in (-\epsilon, \epsilon)$, and suppose A_0 has a simple eigenvalue λ with corresponding eigenspace $L_0 \subset \mathbb{R}^n$. Consider L_0 as a point of $\mathbb{RP}^{n-1}$. Show that there is a smooth parametrized curve $t \mapsto L_t \subset \mathbb{RP}^{n-1}$ so that L_t is an eigenspace of A_t. The domain of this motion is an open interval $(-\delta, \delta)$ for some $\delta < \epsilon$.

5. What manifold parametrizes great circles in the standard unit sphere $S^2 \subset \mathbb{A}^3$?

6. Let X, Y be finite dimensional real vector spaces and $W \subset Y$ a subspace. A linear map $L\colon X \to Y$ is *transverse* to W if $L(X) + W = Y$, that is, if any vector in Y is a sum (possibly nonuniquely) of a vector in $L(X)$ and a vector in W.

   (a) Let $\pi\colon Y \to Y/W$ be the quotient map. Prove that L is transverse to W if and only if $\pi \circ L$ is surjective.

   (b) If L is transverse to W then compute the dimension of $L^{-1}(W) \subset X$.

   (c) Prove that the set of linear maps transverse to W is an open subset of $\mathrm{Hom}(X, Y)$.

Problem Set # 4

M382D: Differential Topology

Due: February 17

Problems in Guillemin/Pollack

Chapter 1, §4 (p. 25): 10, 12
Chapter 1, §7 (p. 45): 4, 6 (assume f is smooth)

Other Problems

1. Let M, N be smooth manifolds and $f\colon M \to N$ an injective proper immersion. (A map is *proper* if for all $C \subset N$ compact, the inverse image $f^{-1}(C) \subset M$ is compact.) Prove that f is an embedding.

2. Produce a smooth map $f\colon (-\delta,\delta) \times S^1 \to S^1$ for some $\delta > 0$ with the following property: If $f_t = f\big|_{\{t\}\times S^1}$, and $q \in S^1$ is a chosen point, then $\# f_t^{-1}(q)$ is nonconstant as a function of t. What causes the jump in this function? Give other examples for other compact domains and arbitrary codomains. Can you see a topological invariant in this situation?

3. For each of the following construct an example.
   (a) A compact manifold X and a smooth manifold Y with $\dim X = \dim Y = 2$, and a smooth map $f\colon X \to Y$ such that if $R \subset Y$ is the subset of regular values and $\#\colon R \to \mathbb{Z}^{\geqslant 0}$ the function which assigns to $q \in R$ the cardinality of $f^{-1}(q)$, then $\#$ takes on three distinct values. (Recall from lecture that $\#$ is locally constant.)
   (b) An embedding $f\colon X \to Y$ which is not proper
   (c) A non-simply connected compact 4-manifold
   (d) A surjective local diffeomorphism of 3-manifolds which is not a diffeomorphism

4. (a) A 3×3 rotation matrix always has a fixed line—that is, an eigenspace—which is actually pointwise fixed—the eigenvalue is 1. Show that this is so. (A 3×3 rotation matrix is an orthogonal matrix with determinant 1. The Lie group of all such matrices is denoted SO_3.) Except for the identity matrix I, this line is unique. Show that the map $f\colon \mathrm{SO}_3 \setminus \{I\} \to \mathbb{RP}^2$ so defined is a submersion. What is the inverse image of a point?
   (b) Show that $\mathbb{RP}^3$ may be constructed from the unit ball $B^3 \subset \mathbb{A}^3$ by identifying antipodal points of the boundary S^2.
   (c) Construct a diffeomorphism $f\colon \mathbb{RP}^3 \to \mathrm{SO}_3$. Hint: Take the ball in part (b) to have radius π.
   (d) The manifold underlying the Lie group SO_2 of orthogonal 2×2 matrices of determinant 1 is also familiar. What is it? What manifold underlies O_2?

5. Real projective space $\mathbb{RP}^n$ may be defined as the set of nonzero real $(n+1)$-tuples $x = [x^0, x^1, \dots, x^n]$ up to an equivalence which identifies two $(n+1)$-tuples if one is obtained from the other using scalar multiplication by a nonzero constant.

   (a) Let $F = F(x^0, \dots, x^n)$ be a homogeneous real-valued function: $F(\lambda x) = \lambda^r F(x)$ for some real number r and all nonzero $\lambda \in \mathbb{R}$. How does the equation $F = 0$ define a subset of $\mathbb{RP}^n$?

   (b) What condition on F guarantees that this subset is a submanifold?

   (c) Homogeneous polynomials are particular examples of homogeneous functions. Show that any linear polynomial F satisfies the condition you found in part (b). What is the corresponding submanifold of $\mathbb{RP}^n$?

   (d) Now investigate (homogeneous) quadratic and cubic polynomials. You might try the case $n = 2$ first to see what sort of submanifolds you get.

6. Let V be a finite dimensional real inner product space. Define the *Stiefel manifold*

$$\mathrm{St}_2(V) = \big\{ b \colon \mathbb{R}^2 \to V : b \text{ is an isometry} \big\}.$$

   Construct a smooth manifold structure on $\mathrm{St}_2(V)$.

Problem Set # 5

M382D: Differential Topology

Due: February 24

Problems in Guillemin/Pollack

Chapter 1, §8 (p. 55): 3, 4, 7, 8

Other Problems

1. Let $f\colon X \to Y$ be an embedding of a smooth manifold X into a smooth manifold Y. Prove that f is a proper map iff $f(X) \in Y$ is closed.

2. Let $X = \{(x, y, z) \in \mathbb{A}^3 : x^2 + y^2 + z^2 = 1,\ z \neq \pm 1\}$ and $Y = \{(x', y', z') \in \mathbb{A}^3 : (x')^2 + (y')^2 = 1\}$ be the twice punctured 2-sphere and cylinder in affine 3-space. Let $f\colon X \to Y$ be radial projection from the z-axis. Compute df. Specify the domain and codomain, the charts you use if you use them, etc.

3. (a) Consider the diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X \end{array}$$

   in which π is a fiber bundle and f a smooth map. Construct $E', \pi', \tilde{f}$ in the diagram

$$\begin{array}{ccc} E' & \overset{\tilde{f}}{\dashrightarrow} & E \\ {\scriptstyle \pi'}\,\vdots & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X \end{array}$$

   so that the diagram commutes and $\tilde{f}$ restricts to a diffeomorphism $\pi'(p') \to \pi^{-1}(f(p'))$ for all $p' \in X'$. Show that π' is a fiber bundle. It is the *pullback* of π along f.

   (b) Let X be a smooth manifold and $\pi_i\colon E_i \to X$, $i = 1, 2$, be fiber bundles. Construct a fiber bundle $\pi\colon E_1 \times_X E_2 \to X$, the *fiber product* of π_1 and π_2, whose fibers are the Cartesian products of the fibers of π_1 and π_2.

   (c) Prove that if $f\colon X \to Y$ is a smooth map, then the differential $df = f_*\colon TX \to TY$ is also a smooth map. (Check in charts.)

4. (a) Let V be a finite dimensional real vector space. Recall that an *inner product* on V is a function $\langle -, - \rangle \colon V \times V \to \mathbb{R}$ which is linear in each variable separately, symmetric, and positive definite: for $\xi, \xi_1, \xi_2, \eta \in V$ and $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} \langle \xi_1 + \lambda\xi_2, \eta \rangle &= \langle \xi_1, \eta \rangle + \lambda\langle \xi_2, \eta \rangle \\ \langle \xi, \eta \rangle &= \langle \eta, \xi \rangle \\ \langle \xi, \xi \rangle &> 0 \qquad \text{if } \xi \neq 0 \end{aligned}$$

(b) Show that the space of maps $V \times V \to \mathbb{R}$ which satisfy the first two equations above is a vector space. What is its dimension (in terms of $\dim V$)? Show that the subset of maps which in addition satisfy the positive definiteness condition is convex.

(c) A *Riemannian metric* on a smooth manifold X is a smoothly varying assignment of inner products on the tangent spaces T_pX. How do we formalize 'smoothly varying' in the previous sentence?

(d) Construct a Riemannian metric on $U \subset X$ if U is the domain of a coordinate chart $(U; x^1, \dots, x^n)$.

(e) Use a partition of unity to construct a Riemannian metric on X.

5. (a) Construct a smooth map $\pi \colon E \to M$ which is a surjective submersion, which has compact fibers, and yet which is not a fiber bundle.

(b) Give an example of a fiber bundle which is not proper.

(c) Prove that a covering space is a fiber bundle.

(d) Is every local diffeomorphism a fiber bundle?

(e) Can an immersion be a fiber bundle?

6. Consider the real quadratic equation

$$(*) \qquad ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

where $a, b, c, d, e, f \in \mathbb{R}$ and x, y are standard coordinates in $\mathbb{A}^2$.

(a) Under what conditions is 0 a regular value of the quadratic function on the left hand side? If so, when is the resulting submanifold of $\mathbb{A}^2$ compact?

(b) Recall that the real projective plane $\mathbb{RP}^2$ is a compactification of the real affine plane $\mathbb{A}^2$ that has a real projective line $\mathbb{RP}^1$ at infinity. If x, y, z are homogeneous coordinates on $\mathbb{RP}^2$ we identify $\mathbb{A}^2$ as the subset of points of the form $[x, y, 1]$. (Recall the equivalence relation $[x, y, z] \sim [\lambda x, \lambda y, \lambda z] \in \mathbb{RP}^2$ for $\lambda \in \mathbb{R}^{\neq 0}$.) Show that the set X of solutions to the *homogeneous* quadratic equation

$$(**) \qquad ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$

intersects $\mathbb{A}^2$ in the subset defined by $(*)$. What is the intersection with the $\mathbb{RP}^1$ at infinity?

(c) What are the possible topologies for X in part (b)?

(d) We can also study the *complex* solutions to $(**)$, that is, the space of solutions $X_{\mathbb{C}} \subset \mathbb{CP}^2$. Can you identify the manifold $X_{\mathbb{C}}$ when 0 is a regular value of the quadratic function in $(*)$?

(e) Show that $[1, \pm\sqrt{-1}, 0] \in X_{\mathbb{C}}$ if and only if $(*)$ defines a (Euclidean) circle in $\mathbb{A}^2$.

Problem Set # 6

M382D: Differential Topology

Due: March 3

Next week, in lieu of a problem set, you will have a take home midterm exam. The only difference with the problem sets is that you are to do it on your own. Also, the more difficult problems will be extra credit.

We will not have any lecture on March 10. Instead, I will post a recorded lecture (on the classification of 1-manifolds) and ask that you watch it at some point.

Problems in Guillemin/Pollack

Chapter 1, §5 (p. 32): 2, 9, 10

Other Problems

1. Construct a nontrivial rank one real vector bundle over the circle S^1 by gluing the ends of $[0,1] \times \mathbb{R}$ using the linear map $\xi \mapsto -\xi$. In other words, identify $(0,\xi) \sim (1,-\xi)$. From this construct a vector bundle $\pi \colon E \to S^1$. Can you identify the 2-manifold which is the total space (the 'E' in $\pi \colon E \to S^1$) of this bundle?

2. Let M be a smooth manifold and $\pi \colon T^*M \to M$ its cotangent bundle. Introduce the notation

$$\Omega^0(M) = \{f \colon M \to \mathbb{R} \text{ smooth}\}$$
$$\Omega^1(M) = \{\text{sections of } \pi \colon T^*M \to M\}$$

   (a) Construct vector space structures on each of these sets. Show that the differential is a linear map

$$(*) \qquad \Omega^0(M) \xrightarrow{d} \Omega^1(M).$$

   (We will soon construct "higher" versions of these vector spaces and of the differential.)

   (b) Let $M = \mathbb{R}$. Is the map $(*)$ injective? Is it surjective? If not, identify the kernel and cokernel.

   (c) Repeat (b) for $M = S^1$.

3. (a) Let $f = f(x, y, z)$ and $g = g(x, y, z)$ be smooth functions defined on an open set $U \subset \mathbb{A}^3$, and suppose each has 0 as a regular value. Then $X = f^{-1}(0)$ and $Y = g^{-1}(0)$ are submanifolds of $\mathbb{A}^3$ of dimension 2. Then X and Y intersect transversely if and only if a certain condition on f and g holds. What is it?

   (b) Check your answer for the specific functions

$$\begin{aligned} f &= x^2 + y^2 + z^2 - 1 \\ g &= (x-a)^2 + y^2 + z^2 - 1 \end{aligned}$$

   where a is a real parameter. For what values of a is the intersection transverse? Think about the geometric picture as well as the equations.

4. (a) Recall that O_n, the set of orthogonal $n \times n$ matrices, is a Lie group. It acts on the sphere S^{n-1} of unit vectors in $\mathbb{R}^n$. Show that the map

$$\begin{aligned} O_n &\longrightarrow S^{n-1} \\ g &\longmapsto g \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} \end{aligned}$$

   is a fiber bundle. Try $n = 1, 2, 3$.

   (b) Let V be a finite dimensional real inner product space. Recall (from Homework #4) the Stiefel manifold $\mathrm{St}_k(V)$ for $k \in \{1, 2, \dots, \dim V\}$. What is $\mathrm{St}_1(V)$? What is $\mathrm{St}_n(V)$ if $n = \dim V$? (Is there a sensible definition of $\mathrm{St}_0(V)$?) For any k, construct a map

$$\mathrm{St}_k(V) \longrightarrow \mathrm{Gr}_k(V)$$

   and prove that it is a fiber bundle. Construct a fiber bundle

$$\mathrm{St}_k(V) \longrightarrow \mathrm{St}_{k-1}(V)$$

   and so a sequence of fiber bundles

$$\mathrm{St}_k(V) \longrightarrow \mathrm{St}_{k-1}(V) \longrightarrow \cdots \longrightarrow \mathrm{St}_0(V).$$

   (c) What are the fibers of each map in this problem?

5. Define

$$X = \{[x,y,z] \in \mathbb{CP}^2 : x^2 + y^2 - z^2 = 0\} \subset \mathbb{CP}^2.$$

(a) Prove that X is a 2-dimensional submanifold of $\mathbb{CP}^2$.

(b) Consider the *pencil* (= 1-dimensional family) of *projective lines*

$$Y_t = \{[x,y,z] \in \mathbb{CP}^2 : x + y - tz = 0\} \subset \mathbb{CP}^2.$$

Here $t \in \mathbb{C}$. Define a projective line Y_∞ by taking the limit as $t \to \infty$. Write an equation for Y_∞.

(c) For which t is $X \pitchfork Y_t$? For those t identify the manifold $X \cap Y_t$.

(d) Redo the problem with $\mathbb{RP}^2$ replacing $\mathbb{CP}^2$.

Problem Set # 7

M382D: Differential Topology

Due: March 24

This is homework set is due *after* spring break. Please be sure to take a break!

Problems in Guillemin/Pollack

Chapter 2, §2 (p. 66): 1, 2, 3, 4

Other Problems

1. Let X be a manifold with boundary. Construct a smooth function $f\colon X \to \mathbb{R}$ such that 0 is a regular value, $f^{-1}(0) = \partial X$, and $f < 0$ on $X \backslash \partial X$.

2. For each of the following construct an example (with justification) or show that it does not exist.
   (a) For each $n \geqslant 1$ two maps $S^n \to S^n$ which are not homotopic.
   (b) For each $n \geqslant 1$ two maps $\mathbb{RP}^n \to \mathbb{RP}^n$ which are not homotopic.
   (c) A map $f\colon S^1 \times S^1 \to S^2$ with $\deg_2 f \neq 0$.
   (d) A map $f\colon S^2 \to S^1 \times S^1$ with $\deg_2 f \neq 0$.
   (e) A map $f\colon S^2 \to \mathbb{RP}^2$ with $\deg_2 f \neq 0$.
   (f) A map $f\colon \mathbb{RP}^2 \to S^2$ with $\deg_2 f \neq 0$.

3. A *knot* is the image of an embedding $f\colon S^1 \to \mathbb{A}^3$. Suppose we have two disjoint knots, which are the images of maps $f, g\colon S^1 \to \mathbb{A}^3$. Define the mod 2 *linking number* as the mod 2 degree of the map
$$\begin{aligned} f \times g\colon S^1 \times S^1 &\longrightarrow S^2 \\ s \times t &\longmapsto \frac{f(s) - g(t)}{|f(s) - g(t)|} \end{aligned}$$
   (a) Compute the mod 2 linking number of the unit circle in the x-y plane centered at the origin with the unit circle in the y-z plane centered at the point $y = 1/2$, $z = 0$, where x, y, z are standard coordinates.
   (b) Suppose that f extends to a map $F\colon D^2 \to \mathbb{A}^3$, where D^2 is the unit disk with boundary S^1. By one of our basic theorems we may assume, possibly after perturbation, that F is transverse to $g(S^1)$. Prove that the mod 2 linking number is the number of points in $F^{-1}\big(g(S^1)\big)$ mod 2.

Problem Set # 8

M382D: Differential Topology

Due: March 31

Problems in Guillemin/Pollack

Chapter 2, §4 (p. 82): 3, 5, 8, 11, 13
Chapter 2, §6 (p. 93): 1, 2

Other Problems

1. Let V be a 4-dimensional real vector space. Set $Y = \mathrm{Gr}_2(V)$, the Grassmannian of 2-dimensional subspaces of V. What dimension is Y?
   (a) Let $U \subset V$ be a 3-dimensional subspace, and set $X = \mathrm{Gr}_2(U)$. Explain how to regard $X \subset Y$ as a submanifold (of what dimension?) and compute $\#_2(X, X)$.
   (b) Let $L \subset V$ be a 1-dimensional subspace, and let $Z \subset Y$ be the set of $W \in \mathrm{Gr}_2(V)$ such that $L \subset W$. Prove that $Z \subset Y$ is a submanifold (of what dimension?) and compute $\#_2(Z, Z)$.
   (c) Compute $\#_2(X, Z)$.

2. Let $\mathcal{X}, Y$ be smooth manifolds, S a connected smooth manifold, $Z \subset Y$ a closed submanifold, and suppose that in the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{G} & Y \\ \big\downarrow{\scriptstyle F} & & \\ S & & \end{array}$$

   the map G is an embedding and F is proper. Assume that $(\dim \mathcal{X} - \dim S) + \dim Z = \dim Y$. Suppose that $s_0, s_1 \in S$ are regular values of F. Prove that $\#_2\big(G\big|_{F^{-1}(s_0)}, Z\big) = \#_2\big(G\big|_{F^{-1}(s_1)}, Z\big)$.

3. Proof or counterproof. (Or if no proof, at least intuitive reasoning indicating why you think the given statement is true or false.)
   (a) There exists a 2-dimensional submanifold $X \subset \mathbb{RP}^2 \times \mathbb{RP}^2$ with $\#_2(X, X) \neq 0$.
   (b) For $n \geqslant 1$ there exist compact submanifolds $X, Z \subset S^n$ with $\#_2(X, Z) \neq 0$.
   (c) Let Y be a closed 2-manifold of genus 2, i.e., a "sphere with two holes". Then there exists a compact 1-dimensional submanifold $X \subset Y$ such that $\#_2(X, X) \neq 0$.
   (d) Let W be a 4-dimensional complex vector space and set $Y = \mathrm{Gr}_2(W)$, the Grassmannian of 2-dimensional complex subspaces of W. Then there exists a compact submanifold $X \subset Y$ such that $\#_2(X, X) \neq 0$.

Problem Set # 9

M382D: Differential Topology

Due: April 7

There are many problems, but many are short, straightforward, and computational: give them a try!. You need practice with exterior algebra and differential forms if you have never worked with them. You only need to follow the rules

$$df \wedge dg = -dg \wedge df$$
$$d^2 f = 0$$

to do the computation; you do not need to wait for all of the theory we are developing.

Problems in Warner

Chapter 2 (p. 77): 2, 9, 10, 12, 15

Other Problems

1. Let S be a set. Recall from lecture (notes) the definition of a free vector space generated by S. In this problem you prove existence.
   (a) First, if V_1, V_2 are vector spaces construct the *direct sum* $V_1 \oplus V_2$ vector space. Its underlying set is the Cartesian product $V_1 \times V_2$, for example.
   (b) Now let $\{V_s\}_{s\in S}$ be a set of vector spaces indexed by S. Construct the direct sum vector space $\bigoplus_{s\in S} V_s$ whose underlying set is the set of *finitely supported* functions $\xi\colon S \to \coprod_{s\in S} V_s$ with the property $\xi(s) \in V_s$. What is vector addition? Scalar multiplication? (Here $\coprod$ denotes disjoint union, a tricky operation. . .)
   (c) Define the direct product $\prod_{s\in S} V_s$ by dropping the support condition. What is the direct product in the special case that all V_s equal a fixed vector space V?
   (d) Use (some of) these constructions to prove existence of a free vector space generated by S. Verify the universal property.
   (e) For the categorically minded, formulate universal properties for the direct sum and direct product.

2. In this problem we work in $\mathbb{A}^n$ with standard coordinates $x^1, x^2, \dots, x^n$. Or, you can imagine that the x^i are local coordinates on an n-dimensional manifold.
   (a) Take $n = 3$, call the coordinates x, y, z, and set

   $$\alpha = x dx + y dy$$
   $$\beta = z dz$$
   $$\gamma = dx \wedge dy + x dz$$

   Compute $\alpha \wedge \beta$, $\alpha \wedge \gamma$, and $\gamma \wedge \gamma$.

(b) Compute $d\alpha$, $d\beta$, and $d\gamma$.

(c) Now write an arbitrary 1-form ω in $\mathbb{A}^n$ and compute $d\omega$.

(d) For a function $f\colon \mathbb{A}^n \to \mathbb{R}$ verify explicitly that $d(df) = 0$. (To ease notation in the last two problems, you may want to try n small first.)

3. Let $P, Q\colon U \to \mathbb{R}$ be smooth functions on an open set $U \subset \mathbb{A}^2$, and consider the differential form

$$\alpha = P\,dx + Q\,dy,$$

where we restrict the global coordinates x, y on the affine plane $\mathbb{A}^2$ to U.

(a) Compute $d\alpha$.

(b) Consider a parametrized curve $\gamma\colon [0,T] \to \mathbb{A}^2$, which we can write in coordinates as a pair of functions $\big(x(t), y(t)\big)$. Compute $\gamma^*\alpha$, which is a 1-form on $[0,T]$.

(c) What can you say if $\alpha = df$, where f is a smooth function on $\mathbb{A}^2$?

(d) Are you reminded of some integration theorems from advanced calculus?

4. Consider the differential form

$$\beta = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$$

on an open set $U \subset \mathbb{A}^3$ in affine space with standard coordinates x, y, z, where $P, Q, R\colon U \to \mathbb{R}$.

(a) Compute $d\beta$.

(b) Consider a parametrized surface $\sigma\colon V \to \mathbb{A}^3$, where V is an open set in $\mathbb{A}^2$ with coordinates u, v. This is given by writing x, y, z as functions of u, v. Compute $\sigma^*\beta$.

(c) What can you say if $\beta = d\alpha$ for α a 1-form on U?

(d) Are you reminded of some integration theorems from advanced calculus?

5. (a) Consider a 1-form $\alpha = g(x)dx$ on the affine line $\mathbb{A}^1$. Prove that there exists a function $f(x)$ so that $\alpha = df$.

(b) Now try the same problem with $\mathbb{A}^1$ replaced by the circle S^1. Equivalently, replace α and f with a periodic 1-form and a periodic function.

6. Consider the 1-form

$$\omega = xdy + ydx$$

on the affine plane $\mathbb{A}^2$ with standard coordinates x, y.

(a) Compute $d\omega$.

(b) Is there a function f so that $\omega = df$?

(c) Now repeat for the form

$$\omega' = \frac{xdy - ydx}{x^2 + y^2}$$

on the punctured affine plane $\mathbb{A}^2 \backslash \{0\}$ with standard coordinates x, y.

7. In this problem we work in $\mathbb{A}^n$ with standard coordinates $x^1, x^2, \ldots, x^n$. Compute d of the following differential forms.

(a) $\gamma = \sum_{i=1}^{n} (-1)^{i-1}\, x^i\, dx^1 \wedge dx^2 \wedge \cdots \wedge d\not{x}^i \wedge \cdots dx^n$.

(b) $r^{-n}\gamma$ where $r^2 = (x^1)^2 + \cdots + (x^n)^2$.

(c) $\sin(r^2) \sum_{i=1}^{n} x^i\, dx^i$.

Problem Set # 10

M382D: Differential Topology

Due: April 14

Problems in Warner

Chapter 2 (p. 77): 13, 16
Chapter 4 (p. 157): 12

Other Problems

1. Suppose V is a vector space with inner product $\langle -, - \rangle$. Define an induced inner product on $\bigwedge^2 V$. You may want to consider V finite dimensional with orthonormal basis $e_1, \dots, e_n$. Then what property does the basis $e_1 \wedge e_2,\ e_1 \wedge e_3,\ \dots,\ e_1 \wedge e_n,\ e_2 \wedge e_3,\ \dots$ of $\bigwedge^2 V$ have? Suppose $\xi_1 \wedge \xi_2$ represents a parallelogram. What is the geometric interpretation of the norm $\|\xi_1 \wedge \xi_2\|$? What about the inner product between two parallelograms?

2. Let V, W be vector spaces and $T\colon V \to W$ a linear map. Recall that for each $k \in \mathbb{Z}^{\geqslant 0}$ there is an induced map
$$\textstyle\bigwedge^k T\colon \bigwedge^k V \longrightarrow \bigwedge^k W$$
characterized by $\bigwedge^k T(\xi_1 \wedge \cdots \wedge \xi_k) = T\xi_1 \wedge \cdots \wedge T\xi_k$. Suppose $V = W$ is finite dimensional and T is diagonalizable. Compute the trace of $\bigwedge^k T$. Compute
$$\sum_{k=0}^{n} (-1)^k t^k \operatorname{Tr} \textstyle\bigwedge^k T$$
where t is a "dummy variable". Can you formulate and prove a formula which holds even if T is not diagonalizable?

3. Let V be a real vector space.
   (a) Suppose $k \in \mathbb{Z}^{\geqslant 0}$ and $\Xi \in \bigwedge^k V$. Prove there exists a finite dimensional subspace $V' \subset V$ such that Ξ is in the image of the inclusion $\bigwedge^k V' \to \bigwedge^k V$ (induced as in #2 from the inclusion $V' \to V$).
   (b) Let $\Xi \in \bigwedge^2 V$. Prove there exists a finite linearly independent set $\{\xi_1, \xi_2, \dots, \xi_{2m}\}$ of even cardinality such that
$$\Xi = \xi_1 \wedge \xi_2 \ +\ \xi_3 \wedge \xi_4 \ +\ \cdots\ +\ \xi_{2m-1} \wedge \xi_{2m}.$$

4. In this problem you will study differential forms on Euclidean 3-space $\mathbb{E}^3$ and relate the exterior derivative d to div, grad, and curl. (Euclidean space $\mathbb{E}^3$ is the standard affine space $\mathbb{A}^3$ in which the underlying vector space $\mathbb{R}^3$ is endowed with the standard inner product.) Suppose

$$\xi = P(x,y,z)\frac{\partial}{\partial x} + Q(x,y,z)\frac{\partial}{\partial y} + R(x,y,z)\frac{\partial}{\partial z}$$

is a vector field on $\mathbb{E}^3$. We associate a 1-form α_ξ and a 2-form β_ξ by the formulas

$$\begin{aligned} \alpha_\xi &= Pdx + Qdy + Rdz \\ \beta_\xi &= Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy \end{aligned}$$

These formulas give isomorphisms

$$\mathfrak{X}(\mathbb{E}^3) \cong \Omega^1(\mathbb{E}^3) \cong \Omega^2(\mathbb{E}^3),$$

where $\mathfrak{X}(\mathbb{E}^3)$ is the vector space of vector fields on $\mathbb{E}^3$, i.e., functions $\mathbb{E}^3 \to \mathbb{R}^3$. Also, we can associate a 3-form ω_f to a function $f\colon \mathbb{E}^3 \to \mathbb{R}$ by the formula

$$\omega = f(x,y,z)\, dx \wedge dy \wedge dz.$$

(a) These isomorphisms are made pointwise, so belong to linear algebra. That is, they are derived from similar isomorphisms for a 3-dimensional real inner product space V. Choose an orthonormal basis for V and define isomorphisms $V \cong V^* \cong \bigwedge^2 V^*$ by imitating the formulas above. Check that these isomorphisms are independent of the choice of basis. Relate to the star operator you studied in Problem #13 in Warner? Can you generalize to higher dimensions? What is the linear algebra manifestation of the identification of functions and 3-forms stated above?

(b) Identify the composition

$$\Omega^0(\mathbb{E}^3) \xrightarrow{d} \Omega^1(\mathbb{E}^3) \longrightarrow \mathfrak{X}(\mathbb{E}^3)$$

with the gradient of a function. (The second map is the isomorphism above.) Generalize to $\mathbb{E}^n$ for any n.

(c) Identify the composition

$$\mathfrak{X}(\mathbb{E}^3) \longrightarrow \Omega^1(\mathbb{E}^3) \xrightarrow{d} \Omega^2(\mathbb{E}^3) \longrightarrow \mathfrak{X}(\mathbb{E}^3)$$

with the curl. (The first and last maps are the isomorphisms above.)

(d) Identify the composition

$$\mathfrak{X}(\mathbb{E}^3) \longrightarrow \Omega^2(\mathbb{E}^3) \xrightarrow{d} \Omega^3(\mathbb{E}^3) \longrightarrow \Omega^0(\mathbb{E}^3)$$

with the divergence.

5. Consider the 1-form

$$\omega = xdy + ydx$$

on the affine plane $\mathbb{A}^2$ with standard coordinates x, y.

(a) Compute $d\omega$.

(b) Is there a function f so that $\omega = df$? Exhibit or proof.

(c) Repeat for the 1-form

$$\omega' = \frac{xdy - ydx}{x^2 + y^2}$$

on the punctured affine plane $\mathbb{A}^2 \backslash \{0\}$ with standard coordinates x, y.

6. What is the orientation double cover of $\mathbb{RP}^n$? Of $\mathbb{CP}^n$?

Problem Set # 11

M382D: Differential Topology

Due: April 21

Problems in Guillemin/Pollack

Chapter 4, §4 (p. 171): 2, 3, 8, 12
Chapter 4, §7 (p. 185): 2, 3, 4, 7, 8, 9, 13

Other Problems

1. Let

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

be a short exact sequence of finite dimensional real vector spaces.

(a) Let $e'_1, \dots, e'_k$ be a basis of V' and $e''_1, \dots, e''_\ell$ be a basis of V''. Choose vectors $\tilde{e}''_1, \dots, \tilde{e}''_\ell$ in V which map to the corresponding vectors in V''. Show that $\tilde{e}''_1, \dots, \tilde{e}''_\ell, e'_1, \dots, e'_k$ is a basis of V. (Identify vectors in V' with their image in V''.)

(b) Let $T\colon V \to V$ be a linear map such that $T(V') \subset V'$. Then T induces an endomorphism T' of V' and T'' of V''. What is the relationship of $\det T$ to $\det T'$ and $\det T''$? What kind of matrix represents T in the basis of (a)?

(c) Use the bases in (a) to define an isomorphism

$$\operatorname{Det} V'' \otimes \operatorname{Det} V' \longrightarrow \operatorname{Det} V$$

Prove that the isomorphism is independent of the choices.

(d) Use the isomorphism in (c) to give a rule which oriented the third of V, V', V'' if the other two are oriented. You might call your rule "quotient before sub".

2. Let V be a finite dimensional real inner product space.

(a) Define the volume of a k-dimensional parallelepiped in V for all nonnegative integers k.

(b) An affine space E over V is called a *Euclidean space*. Suppose $\gamma\colon [a, b] \to E$ is an embedding onto a smooth 1-manifold with boundary $C \subset E$. Define the length of C.

(c) Generalize to higher dimensional submanifolds, or at least to some special cases.

3. Let V be a 4-dimensional vector space. Does there exist $\omega \in \bigwedge^2 V^*$ such that the restriction of ω to every 2-dimensional $W \subset V$ is nonzero?

4. Let $\mathbb{C}$ be the complex (affine) line[1] with coordinate z. Write $z = x + iy$ where $x, y \in \mathbb{R}$ and $i^2 = -1$. Recall the complex conjugate $\bar{z} = x - iy$. The coordinates x, y identify $\mathbb{C}$ with the real affine plane $\mathbb{A}^2$.

   (a) Write x, y in terms of $z, \bar{z}$. We use $z, \bar{z}$ as (complex) coordinates on $\mathbb{A}^2$.

   (b) We use complex differential forms, which are linear combinations of dx, dy with complex coefficients. Express $dz, d\bar{z}$ in terms of dx, dy. Define the basis $\partial/\partial z, \partial/\partial \bar{z}$ dual to $dz, d\bar{z}$ and express it in terms of $\partial/\partial x, \partial/\partial y$.

   (c) Let $U \subset \mathbb{C}$ be an open set. Show that a C^1 function $f : U \to \mathbb{C}$ is analytic (holomorphic) if and only if $\partial f/\partial \bar{z} = 0$.

   (d) Continuing, define the complex 1-form $\alpha \in \Omega^1(U; \mathbb{C})$ by

   $$\alpha = f(z, \bar{z})dz.$$

   Show that $d\alpha = 0$ if and only if f is holomorphic.

   (e) Apply Stokes' theorem to the 1-form α on a bounded open subset of $\mathbb{C}$ whose closure has smooth boundary. Is the result familiar from complex analysis?

5. (Some of this problem appeared prematurely on the previous problem set.) What is the orientation double cover of $\mathbb{RP}^n$? Of $\mathbb{CP}^n$? Of a Klein bottle K? Of $\mathbb{RP}^2 \times K$? Of a Möbius band? Of $\mathrm{Gr}_k(\mathbb{R}^n)$? (Here n is a positive integer and $k \in 1, \dots, n-1$.)

[1] It is a complex *line*: we navigate with a single complex number, just as we navigate on the real line with a single real number. A complex plane requires two complex numbers to locate a point.

Problem Set # 12

M382D: Differential Topology

Due: April 28

For the problems in Guillemin/Pollack, you may use the definitions from lectures/notes.

Problems in Guillemin/Pollack

Chapter 3, §2 (p. 103): 12, 14, 17, 26
Chapter 3, §3 (p. 116): 6, 8, 9, 11

Other Problems

1. For each of the following construct an example or prove that none exists.
   (a) A map $f\colon S^1 \times S^1 \to S^2$ of degree 3.
   (b) A map $f\colon S^2 \to S^1 \times S^1$ of degree 3.
   (c) A map $f\colon S^5 \to S^5$ of degree 3.
   (d) A map $f\colon \mathbb{RP}^5 \to \mathbb{RP}^5$ of degree 3.
   (e) A map $f\colon X \to S^n$ of any given degree $d \in \mathbb{Z}$, where X is a compact oriented n-manifold.

2. Let $\alpha\colon S^n \to S^n$ be the antipodal map, and suppose $f\colon S^n \to S^n$ satisfies $f(p) = f\big(\alpha(p)\big)$ for all $p \in S^n$. Prove that $\deg f$ is even.

3. Consider the differential 2-form $\omega \in \Omega^2(S^2)$ defined in Example 24.42 of the class notes. For each integer d write a map $f_d\colon S^2 \to S^2$ of degree d. Compute $\deg f_d$ by computing the integral of the pullback of the differential form ω.

4. Consider the manifold $\mathbb{CP}^n$, which is a (real) manifold of dimension $2n$.
   (a) Construct submanifolds $\mathbb{CP}^k \subset \mathbb{CP}^n$ for $k = 1, 2, \dots, n-1$.
   (b) These manifolds are complex in a natural sense. All we need at the moment is that the tangent spaces have a natural complex structure and a fact about orientations. Namely, suppose V is a complex vector space and $V_{\mathbb{R}}$ the underlying real vector space. Show that $V_{\mathbb{R}}$ has a natural orientation: if $e_1, e_2, \dots, e_m$ is any complex basis of V, then take $e_1, \sqrt{-1}e_1, e_2, \sqrt{-1}e_2, \dots, e_m, \sqrt{-1}e_m$ to be a positively oriented basis of $V_{\mathbb{R}}$. Show that this orientation is independent of the complex basis. Can you express the argument in terms of determinant lines?
   (c) A complex linear isomorphism $T\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ determines a map $f_T\colon \mathbb{CP}^n \to \mathbb{CP}^n$. Compute $\deg f_T$ using the orientation constructed in (b).
   (d) Does the degree in (c) depend on which orientation of $\mathbb{CP}^n$ you choose?

5. Recall the *Hopf fibration* $h\colon S^3 \to S^1$ defined as

$$\begin{aligned} h\colon S^3 \subset \mathbb{C}^2 &\longrightarrow \mathbb{CP}^1 \\ (z,w) &\longmapsto [z,w] \end{aligned}$$

(a) Omit the point $\infty = (0,1) \in \mathbb{C}^2$ from S^3 and identify the complement with $\mathbb{A}^3$. Draw a picture of some fibers of h. Observe the linking of distinct fibers.

(b) Construct an analogous Hopf fibration which replaces $\mathbb{C}$ with $\mathbb{R}$. Do you recognize that map? It is not homotopic to a constant map: prove it. What about using the division algebra $\mathbb{H}$ of quaternions, in which case you need to explain carefully what the quaternionic projective line is. Is the resulting Hopf fibration homotopic to a constant map? Can you predict what map you get if you use the octonians?

Problem Set # 13

M382D: Differential Topology

Due: May 5

This is the final homework set. Next Thursday you will receive the final exam (online); it is due Tuesday, May 10. You should begin reviewing the course from the beginning, using the lecture notes and the books as well as your own notes and the homework sets.

Problems in Guillemin/Pollack

Chapter 3, §3 (p. 116): 14, 16, 17, 19, 20
Chapter 5, §3 (p. 138): 5, 7

Other Problems

1. Compute the intersection number $I(\mathbb{CP}^k, \mathbb{CP}^{n-k})$ for $0 \leqslant k \leqslant n$. Note the case $n = 2k$, a self-intersection.

2. Let W be an n-dimensional complex vector space for some $n \in \mathbb{Z}^{>0}$, and let $W_\mathbb{R}$ be the underlying $2n$-dimensional real vector space. Multiplication by $i \in \mathbb{C}$ on W is a real linear map $I\colon W_\mathbb{R} \to W_\mathbb{R}$ which satisfies $I^2 = -\operatorname{id}_{W_\mathbb{R}}$. For any basis $e_1, \dots, e_n$ of W as a complex vector space, there is an associated real basis $e_1, Ie_1, e_2, Ie_2, \dots$ of $W_\mathbb{R}$, and so a nonzero vector $e_1 \wedge Ie_1 \wedge \dots$ in the real determinant line $\operatorname{Det} W_\mathbb{R}$.
   (a) Suppose another basis $f_1, \dots, f_n$ of W is related by $e_j = A^i_j f_i$ for some complex $n \times n$ matrix (A^i_j). What is the change of basis of the associated bases of $W_\mathbb{R}$? How are the induced nonzero elements of $\operatorname{Det} W_\mathbb{R}$ related?
   (b) Conclude that $W_\mathbb{R}$ has a canonical orientation. Can you express it in terms of the complex determinant line $\operatorname{Det} W$?
   (c) If W' is another complex vector space, then a complex linear map $T\colon W' \to W$ induced a real linear map $T_\mathbb{R}\colon W'_\mathbb{R} \to W$. Prove that if T is an isomorphism, then $T_\mathbb{R}$ is orientation-preserving.
   (d) What is the relevance of this problem to the proof of the fundamental theorem of algebra given in lecture?

3. Let X_0, X_1 be compact manifolds. A *bordism* $W\colon X_0 \to X_1$ is a *compact* manifold with boundary together with a diffeomorphism $\partial X \xrightarrow{\cong} X_0 \amalg X_1$. If X_0, X_1 are oriented, then an oriented bordism $W\colon X_0 \to X_1$ is a compact oriented manifold with boundary together with an orientation-preserving diffeomorphism $\partial X \xrightarrow{\cong} -X_0 \amalg X_1$. Two manifolds are bordant if there exists a bordism between them.
   (a) Show that one circle is bordant to two circles, even as oriented manifolds.

(b) Let Y be an oriented manifold, $Z \subset Y$ an oriented submanifold, $W\colon X_0 \to X_1$ an oriented bordism between oriented manifolds, and suppose each manifold has a dimension and that $\dim X_0 + \dim Z = \dim Y$. Let $f\colon W \to Y$ be a smooth map, and denote its restrictions to X_0, X_1 as f_0, f_1, respectively. Prove that $I_Y(f_0, Z) = I_Y(f_1, Z)$. This generalizes the smooth homotopy invariance of the oriented intersection number (and so the oriented degree) to oriented bordism invariance.

(c) Use this—or any technique you like—to compute the Euler characteristic of S^2. (By definition this is $I_{Y\times Y}(\Delta, \Delta)$ for $Y = S^2$. As sketched in lecture, there are submanifolds $S^2 \times \mathrm{pt}$ and $\mathrm{pt} \times S^2$ in $S^2 \times S^2$, and Δ is bordant to a manifold obtained from the union of $S^2 \times \mathrm{pt}$ and $\mathrm{pt} \times S^2$ by a small "surgery" which eliminates the non-manifold point $\mathrm{pt} \times \mathrm{pt}$ in the union.)

4. The matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathbb{Z}$ determines a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ which preserves the integral lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. Therefore, it induces a self map f_A of the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$. Compute the Lefschetz number of the map f_A.