What follows are lecture notes from a graduate course given at the University of Texas at Austin in Spring, 2021. The notes are rough in many places, so use at your own risk!

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Mod 2 winding number
Lecture 1: Topological manifolds

I posted notes on “multivariable analysis”. I will not repeat material in those notes. For example, in the first weeks of the course I use basic notions of affine geometry and of differential calculus from those notes.

The first part of this lecture was motivation for the course, which I won’t repeat here. So these notes only cover the last part of the lecture.
(1.1) **Some point set topology.** Let $X$ be a topological space. Recall that $X$ is *Hausdorff* if for all distinct $x_1, x_2 \in X$ there exist disjoint open sets $U_1, U_2 \subset X$ such that $x_i \in U_i$, $i = 1, 2$. Every metric space is Hausdorff since the distance between distinct points is positive, hence distinct points can be separated by open balls (of radius half the distance). The topological space $X$ is *second countable* if it admits a countable basis. That is, there exists a countable collection $\{U_i\}_{i \in I}$ of open subsets $U_i \subset X$ such that any nonempty open set $U \subset X$ is the union $U = \bigcup_{i \in I'} U_i$ for some subset $I' \subset I$. (The empty set is the union with $I' = \emptyset$, so we can omit ‘nonempty’ in the previous sentence.) A metric space is not necessarily second countable; it is if it is separable, which means it has a countable dense set.

**Definition 1.2.** Let $X$ be a topological space.

(i) $X$ is **locally Euclidean** if for all $x \in X$ there exists an open set $U \subset X$ containing $x$ and a homeomorphism of $U$ onto an open subset of a finite dimensional affine space.

(ii) $X$ is a **topological manifold** if it is locally Euclidean, Hausdorff, and second countable.

**Remark 1.3.**

(i) The *data* of a topological manifold is that of a topological space. The definition specifies three *conditions* on the topology.

(ii) Perhaps ‘locally Euclidean’ should be ‘locally affine’, but some terms and notations are ingrained—it would be counterproductive to protest. The affine space in (i) can be chosen to be the standard affine space $\mathbb{A}^n$ of some dimension $n \in \mathbb{Z}_{\geq 0}$. The *invariance of domain* theorem shows that the dimension $n$ is well-defined, i.e., it is the same for all choices of local homeomorphism. In other words, there is no local homeomorphism between open subsets of affine spaces of different dimension.

(iii) A topological manifold $X$ is a *regular* topological space. This means that given a point $x \in X$ and a disjoint closed subset $C \subset X$, there exist disjoint open sets $U, V \subset X$ so that $x \in U$ and $C \subset V$; we can separate points and closed sets. This is stronger than the Hausdorff property assuming points are closed subsets.

(iv) Urysohn’s metrization theorem states that a regular, second countable topological space $X$ in which points are closed sets is metrizable: there exists a metric on $X$ whose underlying topology—set of open sets—agrees with the given topology. In particular, topological manifolds are metrizable.

(v) A subset of a topological manifold is a component if and only if it is a path component.

**Definition 1.4.** The **dimension** of a topological manifold $X$ is the locally constant function

\[
\dim X : X \longrightarrow \mathbb{Z}_{\geq 0}
\]

whose value at $x \in X$ is the dimension of an affine space locally homeomorphic to $X$ at $x$. If $n \in \mathbb{Z}_{\geq 0}$, then a **topological manifold of dimension** $n$, or **topological** $n$-**manifold**, is one for which (1.5) is the constant function with value $n$. 

(1.6) Charts. The local homeomorphisms to affine space on a topological manifold are called coordinate systems or charts.

Definition 1.7. Let $X$ be a topological manifold and $A$ a finite dimensional affine space. An $A$-valued chart on $X$ is a pair $(U, \phi)$ consisting of an open set $U \subset X$ and a continuous map $\phi: U \to A$ which is a homeomorphism onto its image.

A topological manifold admits a covering by charts, i.e., a collection

$$\mathcal{A} = \{(U_\alpha, \phi_\alpha) \text{ charts}\}_{\alpha \in \mathcal{A}}$$

indexed by some set $\mathcal{A}$ such that $X = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$.

Remark 1.9. The codomain of a coordinate function $\phi$ is often taken to be standard affine space $\mathbb{A}^n$ for some $n$. We allow more general charts for convenience. However, when it comes to the notion of a maximal atlas on a smooth manifold, we will need to take $\mathbb{A}^n$-valued charts.

(1.10) Examples of topological spaces which fail to be topological manifolds. We first give three examples of topological spaces which are not topological manifolds; each illustrates the failure of precisely one of the three conditions in Definition 1.2.

![Figure 1. Three topological spaces which are not topological manifolds](image)

Example 1.11 (non locally Euclidean). The subspace $X \subset \mathbb{A}^3_{x,y,z}$ defined by

$$X = \{(x^2 + y^2 < 1) \cap z = 0\} \cup \{x = y = 0\}$$

fails to be locally Euclidean at the point $(0,0,0)$.

Example 1.13 (non Hausdorff). The topological space

$$\mathbb{R} \cup_{\mathbb{R}\setminus\{0\}} \mathbb{R}$$

obtained by gluing two copies of $\mathbb{R}$ at every point except 0 is locally Euclidean but fails to be Hausdorff: the two copies of 0 cannot be separated.

Example 1.15 (non second countable). The space $X = \mathbb{R}$ with the discrete topology fails to be second countable.
Examples of topological manifolds. Since countable disjoint unions of topological manifolds are topological manifolds, it suffices to give connected examples.

Example 1.17 (affine space). Any finite dimensional affine space is a topological manifold with its usual topology. (A finite dimensional vector space has a unique topology with respect to which the vector space operations are continuous; see Lecture 4 of the multivariable notes for a closely related theorem.)

Example 1.18 (dimension 1). Any connected topological 1-dimensional manifold is homeomorphic to either $S^1$ or $A^1 = \mathbb{R}$. We will prove this for smooth manifolds later in the course.

Example 1.19 (the 2-sphere). Let

$$\tag{1.20} X = \{(x, y, z) \in \mathbb{A}^3 : x^2 + y^2 + z^2 = 1\}.$$ 

Define a chart $(U, \phi)$ with $\phi: U \to \mathbb{A}^2$ by $U = \{(x, y, z) \in X : x > 0\}$ and

$$\phi: U \to \mathbb{A}^2$$

$$\quad\quad (x, y, z) \mapsto (y, z)$$

The 2-sphere $X$ is covered by six such charts; the domains are the open sets where $x > 0$, $x < 0$, $y > 0$, $y < 0$, $z > 0$, and $z < 0$.

Example 1.22 (dimension 2). The classification theorem for surfaces states that there are two infinite families of compact connected topological 2-manifolds, the first family indexed by $\mathbb{Z}_{\geq 0}$ and the second by $\mathbb{Z}_{> 0}$. The first family starts off with the surfaces $S^2$, $S^1 \times S^1$, the 2-sphere and the 2-torus. The next in the list is the connected sum of two copies of $S^1 \times S^1$, written

$$\tag{1.23} (S^1 \times S^1)^\# = (S^1 \times S^1) \# (S^1 \times S^1).$$

It is formed by removing an open 2-disk from each torus and then gluing the two surfaces remaining along their boundary. For any $g \in \mathbb{Z}_{> 0}$ there is a similarly constructed $(S^1 \times S^1)^{g}$. (By convention, if $g = 0$ the empty connected sum is the 2-sphere $S^2$.)

The second family begins with the real projective plane $\mathbb{R}P^2$, which is the projectivization $\mathbb{P}(\mathbb{R}^3)$ of 3-space, the space of lines through the origin of $\mathbb{R}^3$ with a suitable topology. A closely related description is to take $S^2$ as in (1.20) and let the cyclic group of order 2 act by the antipodal action $(x, y, z) \mapsto (-x, -y, -z)$. Then $\mathbb{R}P^2$ is the quotient space with the quotient topology. For each $g > 0$ we have the surface $(\mathbb{R}P^2)^{g}$, and together with the surfaces $(S^1 \times S^1)^{g}$ these exhaust the possible compact connected 2-manifolds. The surface $\mathbb{R}P^2 \# \mathbb{R}P^2$ is homeomorphic to the Klein bottle, and there is a homeomorphism $(\mathbb{R}P^2)^{3} \approx_{\text{homeo}} (S^1 \times S^1).$

Example 1.24 (empty set). The empty set $\emptyset$ satisfies the conditions of Definition 1.2: it is trivially locally Euclidean, Hausdorff, and second countable (‘trivial’ meaning ‘nothing to check’). It is convenient to regard $\emptyset$ as a manifold of any dimension, and even to allow the dimension to be a negative integer.
Lecture 2: Smooth manifolds

I will sometimes use letters like ‘M, N,...’ for manifolds and other times use ‘X, Y,...’.

(2.1) $C^\infty$ concepts/objects. In this class I use the word ‘smooth’ synonymously with ‘$C^\infty$’. A smooth manifold is an abstract space on which one has “$C^\infty$ concepts/objects” in affine space, i.e., concepts/objects defined on open subsets of affine space which are invariant under $C^\infty$ diffeomorphisms. (A $C^\infty$ diffeomorphism $\varphi: U \to U'$ between open subsets of affine spaces is a bijective map such that both $\varphi$ and $\varphi^{-1}$ are smooth.) These are the concepts/objects of main interest in differential topology. A non-obvious example is the concept of measure zero: a subset $S \subset U$ has measure zero iff its image under a $C^\infty$ diffeomorphism has measure zero.

$C^\infty$-related charts

Recall Definition 1.7 in which charts are defined.

**Definition 2.2.** Let $M$ be a topological manifold. Let $V,W$ be vector spaces and $A,B$ affine spaces over $V,W$, respectively. Suppose $(U,x)$ is an $A$-valued chart and $(U',y)$ is a $B$-valued chart. We say $(U',y)$ is $C^\infty$-related to $(U,x)$ if

$$y \circ x^{-1}: x(U \cap U') \to B$$

is a $C^\infty$ map.

The map (2.3) is called the overlap or transition function. We illustrate in Figure 2. A smooth manifold is built from open subsets of affine space, glued together by smooth transition functions.

![Figure 2. The transition function](image)

**Example 2.4.** Let $M$ be the 2-sphere with charts as defined in Example 1.19. As previously, let $U$ be the chart where $x > 0$, and now let $U'$ be the chart where $y > 0$. Let $u,v$ be the coordinates in $U$ and $\alpha, \beta$ the coordinates in $V$. Then the transition function is given by the formulas

$$\alpha = \sqrt{1 - u^2 - v^2}$$

$$\beta = v$$
The domain of (2.5) is \( \{(u, v) : u > 0 \text{ and } u^2 + v^2 < 1\} \); the image is \( \{ (\alpha, \beta) : \alpha > 0 \text{ and } \alpha^2 + \beta^2 < 1\} \).

**Proposition 2.6.** There does not exist a covering of \( S^2 \) with a single chart.

**Proof.** Suppose \( (S^2, x) \) is a chart, where \( x : S^2 \to \mathbb{A}^2_{(x^1, x^2)} \) is a homeomorphism onto the open subset \( x(S^2) \subset \mathbb{A}^2 \). Then since \( S^2 \) is compact so too is \( x(S^2) \), hence \( x(S^2) \subset \mathbb{A}^2 \) is closed and bounded. But then \( x(S^2) \subset \mathbb{A}^2 \) is open and closed and nonempty, hence since \( \mathbb{A}^2 \) is connected we conclude \( x(S^2) = \mathbb{A}^2 \). This contradicts the boundedness of \( x(S^2) \). \qed

**Review of calculus**

I will rely on you to study the notes on multivariable analysis. The immediately relevant parts are: Lecture 1 (basic definitions), Lecture 2 (bases and coordinates), Lecture 3 (basic definitions), Lecture 5 (shapes and functions to end of lecture), Lecture 6, Lecture 7 (chain rule).

**Atlases and differential structures**

**Remark 2.7.** I defined a chart to have values in an abstract affine space. That is convenient in practice—we will experience this for Grassmannian manifolds in the next lecture—and also conceptually: we can separate affine coordinates from the manifestation of the locally affine property of a manifold. But for a **maximal** atlas, defined below, we use charts with values in standard affine space.\(^1\) I will call these **standard charts**.

**Definition 2.8.** Let \( M \) be a topological manifold.

1. An **atlas** on \( M \) is a collection \( \mathcal{A} = \{(U_\alpha, x_\alpha)\}_{\alpha \in A} \) of charts such that
   
   (i) \( \bigcup_{\alpha \in A} U_\alpha = X \),
   
   (ii) for all \( \alpha_1, \alpha_2 \in A \) the charts \( (U_{\alpha_1}, x_{\alpha_1}) \) and \( (U_{\alpha_2}, x_{\alpha_2}) \) are \( C^\infty \)-related.\(^2\)

2. An atlas \( \mathcal{A} \) is a **differential structure** if the charts take values in standard affine space and
   
   (iii) \( \mathcal{A} \) is maximal in the sense that if \( (U, x) \) is a standard chart which is \( C^\infty \)-related to all \( (U_\alpha, x_\alpha) \in \mathcal{A} \), then \( (U, x) \in \mathcal{A} \).

3. A **smooth manifold** is a pair \( (M, \mathcal{A}) \) consisting of a topological manifold \( M \) and a differential structure \( \mathcal{A} \).

In practice we only need an atlas to define a smooth manifold; we do not need a maximal atlas. This is due to the following completion theorem.

**Theorem 2.9.** Let \( M \) be a topological manifold and \( \mathcal{A} = \{(U_\alpha, x_\alpha)\}_{\alpha \in A} \) an atlas on \( M \). Define

\[
\overline{\mathcal{A}} = \{(U, x) \text{ standard charts on } M : (U, x) \text{ is } C^\infty \text{-related to all charts in } \mathcal{A} \}.
\]

Then \( \overline{\mathcal{A}} \) is a maximal atlas on \( M \), i.e., a differential structure.

---

\(^1\)For a topological manifold, the collection of pairs \( (U, x) \) consisting of an open subset \( U \subset M \) and a continuous map \( x : U \to \mathbb{A}^n \) is a set \( S(M) \), and we can define maximal subsets of \( S(M) \). (We can and should let \( n \) vary as well.) It is more difficult to control if we replace \( \mathbb{A}^n \) by an arbitrary affine space.

\(^2\)Standard charts are charts which take values in standard affine space \( \mathbb{A}^n \).
Proof. I gave you this on Homework #2, so I won’t spoil the fun.

(2.11) Useful picture of an atlas. Let $M$ be a topological manifold equipped with an atlas $\mathcal{A} = \{(U_\alpha, x_\alpha)\}_{\alpha \in \mathcal{A}}$. In view of Theorem 2.9 we consider such a pair a smooth manifold. It comes equipped with a canonical surjective map

$$\bigcup_{\alpha \in \mathcal{A}} x_\alpha(U_\alpha) \to M$$

The domain is a smooth manifold if the indexing set $\mathcal{A}$ is finite or countable. Any surjective map expresses the codomain as a quotient of the domain; the map encodes an equivalence relation on the domain. So for (2.12) we “see” (Figure 3) a manifold as sewn together from opens in affines. The sewing maps are precisely the transition functions depicted in Figure 2.

![Figure 3. A quilt of open subsets of affine spaces](https://example.com/figure3.png)

Lecture 3: Examples; tangent space; smooth functions

Examples of smooth manifolds

To specify a smooth manifold $M$ we need to give a topological manifold together with an atlas $\mathcal{A}$. Theorem 2.9 shows that this data determines a smooth manifold.

Example 3.1 (affine space). Let $A$ be a finite dimensional real affine space. (Recall that a vector space is an example of an affine space.) The $A$ is locally Euclidean, Hausdorff, and second countable in its usual topology, so it is a topological manifold. It admits an atlas with a single chart $(A, \text{id}_A)$, which defines a smooth manifold structure. This smooth structure is understood when we treat $A$ as a smooth manifold.
Remark 3.2. In fact, if \( \text{dim } A \neq 4 \), then up to diffeomorphism this is the only smooth structure on \( A \). By contrast, \( \mathbb{R}^4 \) (and therefore any 4-dimensional real affine space) admits infinitely many inequivalent smooth structures. In fact, some come in continuous families, so there are uncountably many. This rather shocking state of affairs was discovered in the early 1980’s by Mike Freedman, who combined his own work on the 4-dimensional Poincaré conjecture for topological manifolds with Simon Donaldson’s thesis on smooth 4-manifolds. Freedman proved the existence of a single exotic smooth structure. Soon after, Bob Gompf constructed infinitely many.

Example 3.3 (sphere). We treated the 2-dimensional sphere in Example 1.19 and Example 2.4. Define the \( n \)-sphere as the unit sphere in affine space:

\[
S^n = \{ (x^0, x^1, \ldots, x^n) \in \mathbb{R}^{n+1} : (x^0)^2 + \cdots + (x^n)^2 = 1 \}.
\]

Cover \( S^n \) with \( 2(n+1) \) charts: for each \( - \leq i \leq n \) there is a chart with domain \( \{ x^i > 0 \} \cap S^n \) and a chart with domain \( \{ x^i < 0 \} \cap S^n \). Then as in Example 2.4 you can check that all overlap functions are \( C^\infty \).

Remark 3.5. There is an atlas of \( S^n \) with 2 charts; the domain of each is the complement of a single point in \( S^n \). The coordinate functions are constructed via stereographic projection.

Example 3.6 (disjoint unions). Let \( A \) be a finite or countable set and \( \{ M_\alpha \}_{\alpha \in A} \) a collection of smooth manifolds. Then the disjoint union

\[
M = \bigsqcup_{\alpha \in A} M_\alpha
\]

has a natural smooth structure: an atlas on \( M \) is constructed as the union of atlases on each \( M_\alpha \).

Example 3.8 (Cartesian product). Let \( A \) be a finite set and \( \{ M_\alpha \}_{\alpha \in A} \) a collection of smooth manifolds. Then the Cartesian product

\[
M = \times_{\alpha \in A} M_\alpha
\]

has a natural smooth structure. (A point of \( \times_{\alpha \in A} M_\alpha \) is a function \( p : A \rightarrow \bigsqcup_{\alpha \in A} M_\alpha \) which satisfies \( p(\alpha) \in M_\alpha \) for all \( \alpha \in A \).) Given atlases \( \mathcal{A}_\alpha, \alpha \in A \), one obtains an atlas on \( M \) by taking Cartesian products of charts. (The Cartesian product of maps \( x_\alpha : U_\alpha \rightarrow A_\alpha \) is a map \( \times_\alpha x_\alpha : \times_\alpha U_\alpha \rightarrow \times_\alpha A_\alpha \).)

So a finite product of spheres, such as the torus \( S^1 \times \cdots \times S^1 \) \( (n \text{ factors for any } n \in \mathbb{Z}^{\geq 0}) \) is a smooth manifold.

Example 3.10 (open subset). Let \( M \) be a smooth manifold and \( N \subset M \) an open subset. Then \( N \) is a topological manifold and it inherits an atlas from an atlas \( \mathcal{A}_M \) of \( M \). Namely, for each chart \( (U, x) \in \mathcal{A}_M \) we introduce a chart \( (U|_{U \cap N}, x|_{U \cap N}) \) on \( N \). These have \( C^\infty \) overlaps and comprise an atlas \( \mathcal{A}_N \) of \( N \).

\[\text{2The notation \{\(x^i > 0\}\} is shorthand for the open subset \( \{(x^0, \ldots, x^n) \in \mathbb{R}^{n+1} : x^i > 0 \} \) of affine \( (n+1) \)-space.}\]
Example 3.11 (general linear group). As a special case of the preceding, let $M_n \mathbb{R}$ be the vector space of real $n \times n$ matrices. By Example 3.1 it has a natural smooth manifold structure. Let $GL_n \mathbb{R} \subset M_n \mathbb{R}$ be the subset of invertible matrices. It is an open subset, since it is the inverse image of the open subset $\mathbb{R}^{\neq 0} \subset \mathbb{R}$ under the continuous determinant map $M_n \mathbb{R} \to \mathbb{R}$.

The Grassmannian manifold

We introduce a more abstract manifold, one which does not come to us embedded in affine space. The natural charts take values in affine spaces which are not standard affine space, and indeed they vary from chart to chart.

Let $V$ be a real vector space of dimension $n$. (The same construction works for complex or quaternion vector spaces.) Fix $k \in \{1, \ldots, n-1\}$. Define the Grassmannian as the set

$$(3.12) \quad \text{Gr}_k(V) = \{ W \subset V \text{ subspaces of dimension } k \}.$$  

For $k = 1$ the Grassmannian is called the projectivization of $V$, and this projective space is denoted $$(3.13) \quad \mathbb{P} V = \text{Gr}_1(V).$$

So far $\text{Gr}_k(V)$ is a set. We simultaneously construct a topology and an atlas.

For each $X \in \text{Gr}_{n-k}(V)$—that is, for each subspace $X \subset V$ of dimension $(n-k)$—define

$$(3.14) \quad V_X = \text{Hom}(V/X, X), \quad A_X = \{ W \in \text{Gr}_k(V) : W \cap X = 0 \}.$$  

We define on $A_X$ the structure of an affine space over the vector space $V_X$. Namely, any $W \in A_X$ is a linear complement to $X$. Equivalently, $V = W \oplus X$. Or, in another formulation, the restriction of the quotient map $V \to V/X$ to $W$ is an isomorphism $\theta_W : W \to V/X$. Then given $T \in V_X$, define $W + T \in A_X$ to be the graph of the linear map $T \circ \theta_W : W \to X$. This graph is a subspace of $W \oplus X = V$ of dimension $k$, and it intersects $X$ in the zero vector. The reader can easily check that this defines a simply transitive action of $V_X$ on $A_X$. Choose a finite set $\{X_i\}_{i=1}^N \subset \text{Gr}_{n-k}(V)$ so that $A_{X_1} \cup \cdots \cup A_{X_N} = \text{Gr}_k(V)$. (For example, choose a basis $e_1, \ldots, e_n$ of $V$ and take the spans of all cardinality $(n-k)$ subsets of the basis.) The surjective map

$$(3.15) \quad \bigcup_{i=1}^N A_{X_i} \longrightarrow \text{Gr}_k(V)$$

induces the quotient topology on $\text{Gr}_k(V)$. Then for all $X \in \text{Gr}_{n-k}(V)$ the pair $(A_X, \text{id}_{A_X})$ is a chart on $\text{Gr}_k(V)$. We claim that the overlap functions are smooth. Fix $X, Y \in \text{Gr}_{n-k}(V)$, and choose $W_0 \in A_X \cap A_Y$. Using $W_0$ as a basepoint we identify the affine space $A_X$ with the vector space $\text{Hom}(W_0, X)$ and similarly identify $A_Y$ with $\text{Hom}(W_0, Y)$. Let $\Phi : W_0 \oplus X \to W_0 \oplus Y$ be the linear
map transported from \( \text{id}_V \) under the identifications \( V \cong W_0 \oplus X \cong W_0 \oplus Y \). Then the transition function, defined on a subset \( U \subset \text{Hom}(W_0, X) \) is

\[
\begin{align*}
U &\rightarrow \text{Hom}(W_0, W_0 \oplus X) &\rightarrow &\text{Hom}(W_0, W_0 \oplus Y) &\rightarrow &\text{Hom}(W_0, Y) \\
T &\rightarrow &\text{id}_{W_0 \oplus T} &\rightarrow &\Phi \circ (\text{id}_{W_0} \oplus T) &\rightarrow &\pi \circ \Phi \circ (\text{id}_{W_0} \oplus T)
\end{align*}
\]

where \( \pi: W_0 \oplus Y \rightarrow Y \) is projection. The map (3.16) is a composition of linear maps, hence is smooth. Therefore,

\[
\mathcal{A} = \{(A_X, \text{id}_{A_X})\}_{X \in \text{Gr}_{n-k}(V)}
\]

is an atlas on \( \text{Gr}_k(V) \).

The Grassmannian has a rich geometric structure. It is more accessible in the case of the projective space \( \mathbb{P}V = \text{Gr}_1(V) \). Any \( X_1 \in \text{Gr}_{n-1}(V) \) determines a partition

\[
\mathbb{P}V = A_{X_1}^{(n-1)} \sqcup \mathbb{P}X_1.
\]

The superscript is the dimension of the affine space. Now choose \( X_2 \in \text{Gr}_{n-2}(X_1) \) to obtain

\[
\mathbb{P}V = A_{X_1}^{(n-1)} \sqcup A_{X_2}^{(n-2)} \sqcup \mathbb{P}X_2.
\]

Continuing we partition the projective space \( \mathbb{P}V \) into a disjoint union of affine spaces of dimensions \( 0, 1, \ldots, n - 1 \). Note that \( A_{X_1}^{(n-1)} \) is dense, so projective space is a compactification of affine space.

**Smooth maps**

Any \( C^\infty \) concept on open subsets of affine space transports to smooth manifolds using an atlas. One need only define/check in a single chart since the charts in an atlas are \( C^\infty \) related. We apply this principle to define a smooth function between smooth manifolds.

**Definition 3.20.** Let \( M, N \) be smooth manifolds and \( f: M \rightarrow N \). Fix \( p \in M \). Then \( f \) is smooth at \( p \) if there exists a chart \((U_\alpha, x_\alpha)\) about \( p \) and a chart \((V_\beta, y_\beta)\) about \( f(p) \) so that the composite function \( y_\beta \circ f \circ x_\alpha^{-1} \) is \( C^\infty \) at \( x_\alpha(p) \in x(U) \).

If the charts take values in affine spaces \( A, B, \) respectively, then

\[
y_\beta \circ f \circ x_\alpha^{-1} : x_\alpha(U_\alpha \cap f^{-1}(V_\beta)) \rightarrow B,
\]

which is a function from an open subset of the affine space \( A \) to the affine space \( B \), and its smoothness is defined using standard multivariable calculus. It suffices to check smoothness in one pair of charts, since the answer is the same no matter which pair is chosen, as in the following.
Lemma 3.22. If in Definition 3.20 we choose different charts $(U'_\alpha, x'_{\alpha'})$ and $(V'_\beta, y'_{\beta'})$, then the local representation of $f$ with respect to these charts is smooth at $x'_{\alpha'}(p)$ iff the local representation (3.21) is smooth at $x'_{\alpha'}$. See Figure 4 for an illustration.

Proof. Observe

\begin{align*}
y'_{\beta'} \circ f \circ x'^{-1}_{\alpha'} &= (y'_{\beta'} \circ y'^{-1}_{\beta'}) \circ (y'_{\beta} \circ f \circ x'^{-1}_{\alpha}) \circ (x'_{\alpha} \circ x'^{-1}_{\alpha'}),
\end{align*}

and the overlap functions $y'_{\beta'} \circ y'^{-1}_{\beta'}$ and $x'_{\alpha} \circ x'^{-1}_{\alpha'}$ are smooth. Now apply the chain rule. □

Example 3.24. Consider the antipodal map $f : S^2 \to S^2$. Regarding $S^2 \subset \mathbb{R}^3_{x,y,z}$ as usual, $f$ is the restriction of the automorphism $(x, y, z) \mapsto (-x, -y, -z)$ of $\mathbb{R}^3$. Let $p = (1/\sqrt{2}, 1/\sqrt{2}, 0)$ and choose the charts $U_\alpha = \{x > 0\}$, $V_\beta = \{y < 0\}$ of the type in Example 3.3. Use coordinates $u, v$ and $u', v'$ in the charts, which are then given by the maps

\begin{align*}
x_\alpha : \begin{cases} u = y \\ v = z \end{cases} & \quad y_\beta : \begin{cases} u' = x \\ v' = z \end{cases}
\end{align*}

So the local expression $y_\beta \circ f \circ x^{-1}_{\alpha}$ is

\begin{align*}
u' &= -\sqrt{1 - u^2 - v^2} \\
v' &= -v
\end{align*}

which is a smooth function.
Tangent space

There is a separate handout on tangent spaces, so I’ll only recall the salient point here. Suppose $M$ is a smooth manifold with an atlas $\mathcal{A} = \{(U_\alpha, x_\alpha)\}_{\alpha \in \mathcal{A}}$. The tangent space $T_pM$ is a vector space attached to each $p \in M$. If $p \in U_\alpha$ is in the domain of a chart with coordinate function $x_\alpha: U_\alpha \to A_\alpha$ valued in an affine space $A_\alpha$ with underlying vector space $V_\alpha$ of translations, then $x_\alpha$ induces an isomorphism

\begin{equation}
\theta_\alpha: T_pM \xrightarrow{\cong} V_\alpha.
\end{equation}

If also $p \in U_{\alpha'}$ is in the domain of another chart, then the two isomorphisms (3.27) compare by the differential of the overlap function:

\begin{equation}
\theta_{\alpha'} \circ \theta_\alpha^{-1} = d(x_{\alpha'} \circ x_\alpha^{-1}).
\end{equation}

If $f: M \to N$ is a smooth map at $p \in M$, then the differential of $f$ at $p$ is a linear map

\begin{equation}
df_p: T_pM \to T_{f(p)}N.
\end{equation}

It is defined by choosing local charts $(U_\alpha, x_\alpha), (V_\beta, y_\beta)$ as in Definition 3.20. Then it is represented by $\{\text{Need to fix conflicting notation of } V_\beta \text{ and } V_\alpha\}$

\begin{equation}
d(y_\beta \circ f \circ x_\alpha^{-1}): V_\alpha \to W_\beta,
\end{equation}

where $y_\beta$ takes values in an affine space over $W_\beta$. The relationship between (3.29) and (3.30) is expressed in the commutative diagram

\begin{equation}
\begin{array}{ccc}
T_pM & \xrightarrow{df_p} & T_{f(p)}N \\
\cong & & \cong \\
V_\alpha & \xrightarrow{d(y_\beta \circ f \circ x_\alpha^{-1})} & W_\beta
\end{array}
\end{equation}

**Lecture 4: More on tangent vectors and differentials**

The material in the notes on Multivariable Analysis, specifically pp. 14–15, 32–35, 66–67 are relevant to this lecture.
Linear algebra preliminaries

Let $V, W$ be finite dimensional real vector spaces. The space of linear maps

$$\text{Hom}(V, W) = \{ T : V \rightarrow W : T \text{ is linear} \}$$

is a vector space: $(T_1 + T_2)(\xi) = T_1\xi + T_2\xi$ for all $T_1, T_2 \in \text{Hom}(V, W)$ and $\xi \in V$. The (abstract) dual space is a special case:

$$V^* = \text{Hom}(V, \mathbb{R}).$$

Remark 4.3. Suppose $V, V'$ are finite dimensional real vector spaces and

$$B : V \times V' \rightarrow \mathbb{R}$$

is a bilinear map. We say $B$ is nondegenerate if the linear maps

$$V \rightarrow (V')^*$$

$$\xi \mapsto (\xi' \mapsto B(\xi, \xi'))$$

and

$$V' \rightarrow V^*$$

$$\xi' \mapsto (\xi \mapsto B(\xi, \xi'))$$

are isomorphisms. Then $B$ exhibits a duality between $V$ and $V'$.

Suppose $\dim V = n$ and $e_1, \ldots, e_n$ is a basis of $V$. The dual basis $e^1, \ldots, e^n$ of $V^*$ is defined by

$$e^i(e_j) = \delta^i_j = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

On each side of the equality the indices $i, j$ do not repeat, so there is no sum. The compact notation in (4.7) represents $n^2$ equations, one for each choice of a pair $i, j \in \{1, \ldots, n\}$. In the context of Remark 4.3, the dual basis $e'_1, \ldots, e'_n$ of $V'$ is defined by

$$B(e_i, e'_j) = \delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

For the standard vector space $\mathbb{R}^n$ a vector $\xi = (\xi^1, \ldots, \xi^n)$ is represented by a column vector. The dual space $(\mathbb{R}^n)^*$ consists of linear functionals $\omega = (\omega_1, \ldots, \omega_n)$ which are represented as row vectors. (Recall that a collection of real numbers $A^i_j$ with one superscript and one subscript are
organized into a matrix in which the upper index is the row number and the lower index the column number.) The pairing $\omega(\xi) \in \mathbb{R}$ is computed as the product of a row vector and a column vector:

$$\omega(\xi) = \omega \xi^i = \left( \omega_1 \cdots \omega_n \right) \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^n \end{pmatrix}.$$  

Let $W$ be another vector space and $T: V \to W$ a linear map. There is an induced dual or pullback or transpose linear map $T^*: W \to V$ defined by

$$T^*(w^*)(v) = w^*(Tv), \quad v \in V, \quad w^* \in W^*.$$  

Each side of (4.10) is a real number.

**Tangent vectors as motion germs**

Recall first the situation in affine space. Let $A$ be an affine space over a vector space $V$. A local motion or local parametrized curve in $A$ is, for $\delta > 0$ a smooth function $\hat{\gamma}: (-\delta, \delta) \to A$. Its initial position is the point $\hat{p} = \hat{\gamma}(0) \in A$ and its initial velocity is the vector

$$\hat{\xi} = \hat{\gamma}'(0) = \lim_{h \to 0} \frac{\hat{\gamma}(h) - \hat{p}}{h} \in V.$$  

Define two local motions in $A$ to be equivalent if their initial positions and initial velocities agree. An equivalence class is a motion germ with well-defined position and velocity.³

**Remark 4.12.** There is a distinguished affine motion $t \mapsto \hat{p} + t \hat{\xi}$ in each equivalence class.

Let $U \subset A$ be an open set, $B$ is an affine space, and $f: U \to B$ a smooth function. Then if $\hat{p} \in U$, $\hat{\xi} \in V$, and $\hat{\gamma}: (-\delta, \delta) \to U$ represents a motion germ with position $\hat{p}$ and velocity $\hat{\xi}$,

$$\hat{\xi} f(\hat{p}) = df_{\hat{p}}(\hat{\xi}) = \left. \frac{d}{dt} \right|_{t=0} f(\hat{\gamma}(t))$$  

by the chain rule.

Let $M$ be a smooth manifold. Define a local motion $\gamma: (-\delta, \delta) \to M$ and its initial position $p = \gamma(0)$ as in affine space. To define an equivalence relation on local motions with initial position $p$, choose a chart $(U, x)$ about $p$ with values in an affine space $A$, and transport the equivalence relation on the corresponding $A$-valued local motions $\hat{\gamma} = x \circ \gamma$. Smoothness of overlap functions shows, via the chain rule, that the equivalence relation is independent of the chart. The velocity of a motion germ represented by $\gamma$ is an element of $T_pM$; it is the transport of the velocity of $\hat{\gamma} = x \circ \gamma$ using the isomorphism (3.27). Furthermore, the set of motion germs at $p$ is $T_pM$. We write

$$\xi = \hat{\gamma}'(0) = \left. \frac{d}{dt} \right|_{t=0} \gamma(t) = d\gamma_0(\left. \frac{d}{dt} \right) \in T_pM.$$
Remark 4.15. In our development we first define the tangent space and differential of a smooth map as in the handout. Then (4.14) is a theorem, not a definition.

If \( f : M \to N \) is a smooth map, then from this point of view its differential on tangent vectors is the induced map on motion germs; see Figure 5. The chain rule follows easily using motion germs; see Figure 6.

The cotangent space and the ring of \( C^\infty \) functions

Let \( M \) be a smooth manifold and \( p \in M \).

Definition 4.16. The cotangent space to \( M \) at \( p \) is \( T^*_p M = (T_p M)^* \), the dual to the tangent space.

If \( f : M \to \mathbb{R} \) is a smooth function, then its differential at \( p \) is a linear map \( df_p : T_p M \to \mathbb{R} \). In other words, the differential \( df_p \in T^*_p M \) is an element of the cotangent space.

Let \( C^\infty(M) \) denote the ring of smooth real-valued functions on \( M \). Addition and multiplication are inherited from the ring structure of \( \mathbb{R} \). We have not yet proved that \( C^\infty(M) \) contains more than locally constant functions; we will do so when we study partitions of unity. Then we will prove that about each point \( p \in M \) there exist functions \( x^1, \ldots, x^n \in C^\infty(M) \) whose differentials at \( p \) form a

---

3 More properly it is a \( C^1 \) motion germ. There are higher order \( C^k \) and \( C^\infty \) motion germs.
basis $dx_p^1, \ldots, dx_p^n$ of the cotangent space $T_p^* M$. This leads to a more natural definition of a smooth structure on a topological manifold in terms of the ring of smooth functions. We remark that from this point of view the cotangent space is more fundamental and the tangent space is defined to be the dual space to the cotangent space. That world order is reflected in the next section.

**Local computations in coordinate charts**

Let $(U, \phi)$ be a standard chart on a smooth manifold $M$, so $\phi: U \to \mathbb{A}^n$ for some open set $U \subset M$ and a positive integer $n$. Fix $p \in U$. Denote the coordinate functions as $x^i: U \to \mathbb{R}$.

**Lemma 4.17.** The differentials $dx_p^1, \ldots, dx_p^n$ form a basis of $T_p^* M$.

**Proof.** It suffices to check in any chart at $p$, so we check in the chart $(U, \phi)$. But in that chart the function $x^i$ is the standard affine coordinate functions with differential $(0 \cdots 1 \cdots 0) \in (\mathbb{R}^n)^*$. These form the standard basis. □

**Example 4.19 (sample computation).** We compute the Gauss map of a torus $M$ embedded in Euclidean 3-space $\mathbb{E}^3$. It is a map $f: M \to S^2$ which takes each point of $M$ to a unit normal vector to $M$ at that point. The computation is illustrated in Figure 8. Let $x, y, z$ be the standard affine coordinates on $\mathbb{E}^3$. Fix positive real numbers $R > r$. Define $M$ as the surface obtained by revolving...
a circle of radius $r$ about the $z$-axis, assuming its center to be at distance $R$ from the axis. Then $M$ is the image of the map $S^1 \times S^1 \to \mathbb{E}^3$ defined by

$$x = (R + r \cos \theta) \cos \phi$$

$$y = (R + r \cos \theta) \sin \phi$$

$$z = r \sin \theta$$

(4.20)

Restrict $0 < \theta, \phi < 2\pi$ to obtain a homeomorphism onto the image $U$, which is the complement of the union of two circles in $M$, and invert to obtain a chart with coordinate functions $\theta, \phi$.

**Figure 8.** Local computation of the Gauss map of a torus in Euclidean 3-space

The sphere $S^2$ is the unit sphere in the vector space $\mathbb{R}^3$, which we take to have standard coordinates $\xi^1, \xi^2, \xi^3$. In these coordinates the Gauss map $S^1 \times S^1 \to S^2$ is

$$\xi^1 = \cos \theta \cos \phi$$

$$\xi^2 = \cos \theta \sin \phi$$

$$\xi^3 = \sin \theta$$

(4.21)

as can be deduced by differentiating (4.20) with respect to $r$. (The motion with parameter $r$ has initial velocity the unit normal to $M$.) Now take a chart on $S^2$ to be the open subset where $\xi^1 > 0$ and use $\xi^2, \xi^3$ as coordinate functions on that chart. Then the local representation of the Gauss map from coordinates $\theta, \phi$ to coordinates $\xi^2, \xi^3$ is

$$\xi^2 = \cos \theta \sin \phi$$

$$\xi^3 = \sin \theta$$

(4.22)

The dual (4.10) to the differential $df_p: T_p M \to T_{f(p)} S^2$ is what we compute at any point $p \in U$ by differentiating the equations (4.22) which define $f$:

$$d\xi^2 = -\sin \phi \sin \theta \, d\theta + \cos \phi \cos \theta \, d\phi$$

$$d\xi^3 = \cos \theta \, d\theta$$

(4.23)
These can be transposed to compute the map on tangent vectors, i.e., the differential:

\[
\begin{align*}
\frac{\partial}{\partial \xi^2} &= -\sin \phi \sin \theta \frac{\partial}{\partial \theta} + \cos \phi \cos \theta \frac{\partial}{\partial \phi} \\
\frac{\partial}{\partial \xi^3} &= \cos \theta \frac{\partial}{\partial \theta}
\end{align*}
\]

One way to derive (4.24) is to evaluate (4.23) on \( \partial/\partial \theta \) and \( \partial/\partial \phi \).

Lecture 5: The inverse function theorem

Extensive notes for this lecture are in the text on Multivariable Analysis, especially pp. 16–17, 24–26, 30, 58–63*, 65.

Lecture 6: Normal forms for maximal rank maps

We begin in this lecture with the maximal rank condition in linear algebra. We prove that the space of maximal rank linear maps is open in the vector space of linear maps. This fits the slogan "invertibility is an open condition"; see Example 3.11. A maximal rank linear map has a simple normal form. We then go on to the case of a map between smooth manifolds, where the maximal rank condition makes sense on the differential. The circle of ideas around the inverse function theorem (Lecture 5) is used to pass from this first-order infinitesimal condition to a local normal form. We end with some global conditions on smooth maps.

Maximal rank linear maps

The ground field is arbitrary for the linear algebra of this section. In our application to smooth manifolds, we use the field of real numbers.

Definition 6.1. Let \( V,W \) be finite dimensional vector spaces and \( T: V \rightarrow W \) a linear map.

(i) The rank of \( T \) is the dimension of its image:

\[
\text{rank } T = \dim T(V) \leq \min(\dim V, \dim W).
\]

(ii) \( T \) has maximal rank if there is equality in (6.2).

A maximal rank map is injective/bijective/surjective if \( \dim V \leq \dim W \), respectively.

Lemma 6.3. Let \( V,W \) be finite dimensional vector spaces.
The space of maximal rank linear maps \( \text{MaxRank}(V, W) \subset \text{Hom}(V, W) \) is open.

If \( T: V \to W \) has maximal rank, then there exist bases \( e_1, \ldots, e_m \) of \( V \) and \( f_1, \ldots, f_n \) of \( W \) such that

\[
T(e_j) = f_j, \quad j = 1, \ldots, m, \quad \text{if } \dim V \leq \dim W,
\]

and

\[
T(e_j) = \begin{cases} f_j, & j = 1, \ldots, n; \\ 0, & j = n + 1, \ldots, m, \end{cases} \quad \text{if } \dim V \geq \dim W.
\]

**Proof.** We already gave the argument for (1) in case \( \dim V = \dim W \): then the subset of isomorphisms \( \text{Iso}(V, W) \subset \text{Hom}(V, W) \) is open.

If \( \dim V < \dim W \) and \( T_0: V \to W \) has maximal rank, choose \( W_0 \subset W \) complementary to \( T_0(V) \). Equivalently, \( W = T_0(V) \oplus W_0 \). Note that \( \dim V = \dim W/W_0 \). Let \( \pi: W \to W/W_0 \) be projection onto the quotient. Define

\[
p: \text{Hom}(V, W) \to \text{Hom}(V/W_0, W) \\
T \mapsto \pi \circ T
\]

Then \( p^{-1}(\text{Iso}(V/W_0, W)) \subset \text{MaxRank}(V, W) \subset \text{Hom}(V, W) \) is an open subset containing \( T_0 \), since \( \text{Iso}(V/W_0, W) \subset \text{Hom}(V/W_0, W) \) is open, and this proves that \( \text{MaxRank}(V, W) \) is open.

The argument for \( \dim V > \dim W \) is similar. Given \( T_0: V \to W \) of maximal rank, choose \( V_0 \subset V \) so that \( T_0|_{V_0}: V_0 \to W \) is an isomorphism. Let \( \iota: V_0 \to V \) be the inclusion. Define

\[
r: \text{Hom}(V, W) \to \text{Hom}(V_0, W) \\
T \mapsto T \circ \iota
\]

to be restriction to \( V_0 \). Then \( r^{-1}(\text{Iso}(V_0, W)) \subset \text{MaxRank}(V, W) \subset \text{Hom}(V, W) \) is open and contains \( T_0 \), which proves \( \text{MaxRank}(V, W) \subset \text{Hom}(V, W) \) is open.

For (2), if \( \dim V \leq \dim W \) choose an arbitrary basis \( e_1, \ldots, e_m \) of \( V \), define \( f_j = T(e_j), \ j = 1, \ldots, m \), and fill out the linearly independent set \( f_1, \ldots, f_m \) to a basis of \( W \). Similarly, if \( \dim V \geq \dim W \), choose an arbitrary basis \( f_1, \ldots, f_n \) of \( W \), use surjectivity to find vectors \( e_1, \ldots, e_n \) in \( V \) with \( T(e_j) = f_j, \ j = 1, \ldots, n \), and choose \( e_{n+1}, \ldots, e_m \) to be a basis of ker \( T \). \( \square \)

**The maximal rank condition for smooth maps of manifolds**

We introduce special terminology for the maximal rank condition.

**Definition 6.8.** Let \( M, N \) be smooth manifolds and \( f: M \to N \) a smooth map. Fix \( p \in M \) and set \( q = f(p) \). The differential of \( f \) at \( p \) is a linear map \( df_p: T_pM \to T_qN \).

(i) If \( df_p \) is injective, then \( f \) is an immersion at \( p \).
(ii) If $df_p$ is surjective, then $f$ is a submersion at $p$. We also say $f$ is regular at $p$, or $p$ is a regular point of $f$.

(iii) If $df_p$ is not surjective, then $p$ is a critical point of $f$.

(iv) If all $p \in f^{-1}(q)$ are regular points, then $q$ is a regular value of $f$.

(v) If there exists $p \in f^{-1}(q)$ a critical point, then $q$ is a critical value of $f$.

Be cognizant that regular and critical points lie in the domain and regular and critical values lie in the codomain. Repeat: points/domain, values/codomain.

**Remark 6.9.** If $q \not\in \text{image of } f$, then $q$ is a regular value, since the condition in (iv) is trivially satisfied.

A fundamental result, Sard’s Theorem, asserts that the set of critical values of any smooth function has measure zero in the codomain. (Part of that circle of ideas is defining measure zero.) As a corollary, the set of regular values is dense, and in particular is nonempty. On the other hand, the set of critical values can be empty, as for the identity map $\text{id}_M : M \to M$ on any $M$.

The proper notion of isomorphism for smooth manifolds is the following.

**Definition 6.10.** Let $M,N$ be smooth manifolds and $f : M \to N$ a smooth map. Then $f$ is a diffeomorphism if $f$ is bijective and $f^{-1}$ is smooth.

**Remark 6.11.**

1. If $f$ is a diffeomorphism, differentiate the equality $f^{-1} \circ f = \text{id}_M$ at $p \in M$ to find

$$ (df^{-1})_{f(p)} = (df_p)^{-1}, $$

as we already observed when proving the inverse function theorem.

2. Compositions of diffeomorphisms are diffeomorphisms.

3. If $U \subset M$ is open and $x : U \to A$ is a smooth map to an affine space $A$, then $(U,x)$ is a chart on $M$ iff $x$ is a diffeomorphism onto its image $x(U) \subset A$. (This is a homework problem.)

The following is a corollary of the inverse function theorem in affine space; it is the inverse function theorem for smooth manifolds.

**Theorem 6.13.** Let $M,N$ be smooth manifolds, $f : M \to N$ a smooth map, and suppose $df_p : T_pM \to T_{f(p)}N$ is an isomorphism for some $p \in M$. Then there exist open subsets $U \subset M$ containing $p$ and $V \subset N$ containing $f(p)$ such that $f|_U : U \to V$ is a diffeomorphism.

The converse statement follows from Remark 6.11(i). The missing item between (i) and (ii) in Definition 6.8 can now be restored: If $df_p$ is bijective, then $f$ is a local diffeomorphism at $p$.

**Proof.** Choose an $A$-valued chart $(\tilde{U}, x)$ about $p$ and a $B$-valued chart $(\tilde{V}, y)$ about $f(p)$, for some affine spaces $A,B$. Apply the inverse function theorem in affine space to

$$ y \circ f \circ x^{-1} : x(\tilde{U} \cap f^{-1}(\tilde{V})) \to B $$

using the fact that $d(y \circ f \circ x^{-1})_x = dy_{f(p)} \circ df_p \circ (dx^{-1})_{x(p)}$ is an isomorphism. Thus on a subset of its domain, which transports by $x^{-1}$ to a subset $U \subset \tilde{U}$, the map (6.14) is a diffeomorphism onto its image. The theorem follows. □
Now we turn to special coordinate systems and local forms.

**Proposition 6.15.** Let $M$ be a smooth manifold, $p \in M$ a point, $n = \dim_p M$ the dimension of $M$ at $p$, and $U \subset M$ an open set containing $p$.

1. Let $x^1, \ldots, x^n: U \to \mathbb{R}$ be smooth functions whose differentials $dx^1_p, \ldots, dx^n_p$ at $p$ are a basis of $T^*_p M$. Then there exists an open subset $U' \subset U$ such that $(U'; x^1, \ldots, x^n)$ is a chart.

2. Let $x^1, \ldots, x^k: U \to \mathbb{R}$, $k < n$, be smooth functions whose differentials $dx^1_p, \ldots, dx^k_p$ at $p$ are linearly independent in $T^*_p M$. Then there exists an open subset $U' \subset U$ and functions $x^{k+1}, \ldots, x^n: U' \to \mathbb{R}$ such that $(U'; x^1, \ldots, x^n)$ is a chart.

3. Let $x^1, \ldots, x^\ell: U \to \mathbb{R}$, $\ell > n$, be smooth functions whose differentials $dx^1_p, \ldots, dx^\ell_p$ at $p$ span $T^*_p M$. Then there exists an open subset $U' \subset U$ and a subset $\{i_1, \ldots, i_n\} \subset \{1, \ldots, \ell\}$ such that $(U'; x^{i_1}, \ldots, x^{i_n})$ is a chart.

**Proof.** Assertion (1) is a direct consequence of Theorem 6.13 and Remark 6.11(3).

For (2), choose a chart $(V, y)$ about $p$, and suppose $y: V \to \mathbb{R}^n$ takes values in the vector space $\mathbb{R}^n$ and $y(p) = 0$. We may also assume $V \subset U$. Then the ordered $k$-tuple $(x^1 \circ y^{-1}, \ldots, x^k \circ y^{-1})$ defines a map $g: y(V) \to \mathbb{R}^k$ whose differential at 0 is surjective. By Lemma 6.3 we can find a linear automorphism $S: \mathbb{R}^n \to \mathbb{R}^n$ so that $dg_0 \circ S: \mathbb{R}^n \to \mathbb{R}^k$ is projection onto the first $k$ components. Let $x^{k+1}, \ldots, x^n: V \to \mathbb{R}$ be the last $(n-k)$ coordinates of the chart $(V, S^{-1} \circ y)$. Then the $n$ functions $x^1, \ldots, x^n: V \to \mathbb{R}$ satisfy the hypothesis of (1), as we can check in the chart $(V, S^{-1} \circ y)$. Now apply the conclusion of (1) to prove (2).

For (3), choose a subset $\{i_1, \ldots, i_n\} \subset \{1, \ldots, \ell\}$ such that $dx^{i_1}_p, \ldots, dx^{i_n}_p$ is a basis of $T^*_p M$ and then apply (1). \hfill $\square$

We apply Proposition 6.15 to prove the analog of the normal form theorem Lemma 6.3(2) on smooth manifolds.

**Theorem 6.16.** Let $f: M \to N$ be a map of smooth manifolds. Fix $p \in M$ and suppose $\dim_p M = m$ and $\dim_{f(p)} N = n$. Assume $df_p: T_p M \to T_{f(p)} N$ has maximal rank. Then there exist charts $(U, x)$
about $p$ and $(V, y)$ about $f(p)$ such that $y \circ f \circ x^{-1}$ takes the form

\begin{equation}
    y^i = x^i, \quad i = 1, \ldots, m, \quad \text{if } m \leq n,
\end{equation}

and

\begin{equation}
    y^i = \begin{cases} 
    x^i, & i = 1, \ldots, n; \\
    0, & i = n + 1, \ldots, m, \quad \text{if } m \geq n.
\end{cases}
\end{equation}

Proof. Choose an arbitrary chart $(V; y^1, \ldots, y^n)$ on $N$ about $f(p)$. If $m \geq n$, consider the $n$ functions $x^1 := y^1 \circ f, \ldots, x^n := y^n \circ f$, defined on the neighborhood $f^{-1}(V)$ of $p$ in $M$. Their differentials at $p$ are linearly independent, so by Proposition 6.15(2) they can be completed to a chart with domain $U \subset f^{-1}(V)$. In this way we obtain the normal form (6.18). In case $m \leq n$, the differentials of these $n$ functions at $p$ span $T_p^* M$, so by Proposition 6.15(3) we can choose $m$ of them which form a basis. Renumbering so these are the first $m$, we obtain the normal form (6.17).

Global properties of smooth maps; submanifolds

We now turn to global properties.

**Definition 6.19.** Let $f: M \to N$ be a smooth map of smooth manifolds. Then $f$ is an **embedding** if it is an injective immersion which is a homeomorphism onto its image.

There are three conditions on $f$ in the definition: the local condition that $f$ be an immersion, i.e., its differential $df_p$ is injective for all $p \in M$; the global condition that $f$ be injective; and the global condition that the inverse of $f$ (with domain $f(M) \subset N$) be continuous. In Figure 10 we depict three immersions $f: \mathbb{R} \to \mathbb{A}^2$, illustrating these two global properties.

![Figure 10. Types of map with injective differential](image_url)
Just as a smooth manifold is defined by a local normal form—locally a smooth manifold is
diffeomorphic to affine space—so too is a submanifold defined by a local normal form: locally a
submanifold is diffeomorphic to an affine subspace of an affine space.

**Definition 6.20.** Let $M$ be a smooth manifold and $Q \subset N$ a subset. Then $Q$ is a *submanifold* of $N$ if for all $q \in Q$ there exists $\ell \in \{0, \ldots, n\}$ and a chart $(V, y)$ of $N$ about $q$ such that

$$y(Q \cap V) = \{(y^1, \ldots, y^n) \in \mathbb{A}^n : y^\ell+1 = \cdots = y^n = 0\} \cap y(V).$$

We illustrate a *submanifold chart* in Figure 11. The integer $\ell$ is the *codimension* of $Q$ in $N$ at the
point $q$.

**Remark 6.22.** A submanifold is a manifold in its own right. Namely, if $\mathcal{A} = \{(V_\alpha, y_\alpha)\}_{\alpha \in \mathcal{A}}$ is a
covering of $Q$ by submanifold charts, then $\mathcal{A} = \{(V_\alpha \cap Q; y_\alpha^1, \ldots, y_\alpha^\ell)\}$ is an atlas of $Q$. (This assumes constant codimension $\ell$; the reader needs merely to change the notation otherwise.)

**Example 6.23** (skew line on a torus). Consider the 2-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$. Fix $(x_0, y_0) \in \mathbb{R}^2$
so that $x_0 \neq 0$ and $y_0/x_0$ is irrational. Then the map

$$f : \mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$$

$$t \mapsto (tx_0, ty_0) \pmod{\mathbb{Z}^2}$$

is an injective immersion. The image $f(\mathbb{R}) \subset \mathbb{R}^2/\mathbb{Z}^2$ is dense in the torus. About every $t \in \mathbb{R}$ we
can choose special charts as in Theorem 6.16 so that (6.17) is satisfied, but the chart on the torus
is not a submanifold chart since (6.21) is not satisfied.
Lecture 7: Submanifolds, embeddings, and regular values; a counting invariant

Recall that there are three basic ways to associate a “shape” to a function \( f: M \to N \). We can take the image \( f(M) \subset N \); the preimage \( f^{-1}(q) \subset M \) of a point of \( N \), or more generally the preimage \( f^{-1}(Q) \subset M \) of a subset of \( N \); and the graph \( \Gamma(f) \subset M \times N \) of \( f \). If \( M, N \) are smooth manifolds and \( f \) a smooth function, then the graph \( \Gamma(f) \) is always a submanifold of \( M \times N \), and it is diffeomorphic to the domain \( M \). The first theorem in this lecture gives a sufficient condition on \( f \) for its image \( f(M) \) to be a submanifold of the codomain \( N \), namely that \( f \) be an embedding; then \( f(M) \) is diffeomorphic to \( M \). The second theorem gives a sufficient condition on \( q \in N \) for the inverse image \( f^{-1}(q) \) to be a submanifold of the domain \( M \), namely that \( q \) be a regular value. In both cases the condition is not necessary: a constant map has image a submanifold, and the inverse image of a critical value can “accidentally” be a manifold. We will soon study the condition—transversality—for the inverse image \( f^{-1}(Q) \subset M \) of a submanifold \( Q \subset N \) to be a submanifold.

In the last part of the lecture we construct our first topological invariant and use it to prove the fundamental theorem of algebra.

Embeddings and submanifolds

**Theorem 7.1.** Let \( f: M \to N \) be an embedding. Then \( f(M) \subset N \) is a submanifold.

![Constructing a submanifold chart](image)

**Proof.** Fix \( q \in Q \); we must construct a submanifold chart about \( q \). Let \( p \in M \) be the unique point so that \( f(p) = q \). Since \( f \) is immersive, by Theorem 6.16 there exist charts \((U, x)\) about \( p \) and \((V, y)\) about \( q \) so that \( y \circ f \circ x^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0) \); see Figure 12. We claim that there exists an open subset \( V' \subset V \) so that the restricted chart \((V', y)\) is a submanifold chart. If (6.21) fails, then there exists a sequence \( \{p_k\}_{k=1}^{\infty} \subset M \setminus U \) so that \( \lim_{k \to \infty} y^j(f(q_k)) = 0 \) for \( j = m+1, \ldots, n \). Hence the sequence \( \{f(q_k)\} \subset V \) converges to a point of \( f(U) \), and since \( f \) is a homeomorphism onto its image we conclude that \( \{p_k\} \subset M \setminus U \) converges to a point of \( U \), which is absurd since \( M \setminus U \) is a closed subset of \( M \). \( \square \)
Regular values and submanifolds

We first introduce some terminology which will recur throughout the course.

**Definition 7.2.** A sequence

\[ V \xrightarrow{T} W \xrightarrow{S} X \]

of linear maps of vector spaces is *exact* if \( S \circ T = 0 \) and \( \ker S = T(V) \) as subspaces of \( W \). A *long exact sequence*

\[ \cdots \to V^i \to V^{i+1} \to V^{i+2} \to \cdots \]

is a sequence of linear maps in which every two consecutive maps forms an exact sequence. A *short exact sequence* is a long exact sequence of the form

\[ 0 \to V' \xrightarrow{T} V \xrightarrow{S} V'' \to 0. \]

In (7.5) the linear map \( T: V' \to V \) is injective with cokernel (isomorphic to) \( V'' \), and the linear map \( S: V \to V'' \) is surjective with kernel (isomorphic to) \( V' \). Furthermore, if \( V', V, V'' \) are finite dimensional, then

\[ \dim V = \dim V' + \dim V''. \]

**Definition 7.7.** Let \( P \subset M \) be a submanifold and \( p \in P \).

1. The *codimension* of \( P \) in \( M \) at \( p \) is

\[ \text{codim}_p(P \subset M) = \dim_p M - \dim_p P = \dim(T_pM/T_pP). \]

2. The quotient space \( T_pM/T_pP \) is the *normal (space) to \( P \) at \( p \).*

We sometimes use the notation ‘\( \nu_p \)’ for the normal space at \( p \). Observe that there is a short exact sequence

\[ 0 \to T_pP \to T_pM \to \nu_p \to 0. \]

**Theorem 7.10.** Let \( f: M \to N \) be a smooth map of smooth manifolds and \( q \in N \) a regular value. Then \( P := f^{-1}(q) \subset M \) is a submanifold of codimension equal to \( \dim_q N \). Furthermore, if \( p \in P \),

\[ T_pP = \ker(df_p: T_pM \to T_pN). \]
We express (7.11) as the short exact sequence

\[
0 \rightarrow T_pP \rightarrow T_pM \xrightarrow{df} T_qN \rightarrow 0,
\]

illustrated in Figure 13. In general, the codimension of a submanifold \( P \subset M \) is a locally constant function \( \text{codim}: P \rightarrow \mathbb{Z}_{\geq 0} \). Theorem 7.10 asserts that if \( P \) is cut out by a single function, then codim is a constant function.

Remark 7.13. Not every submanifold is cut out by a single function. Compare (7.9) and (7.12) to conclude that in the situation of Theorem 7.10, the differential \( df \) identifies each normal space \( \nu_p \) with the fixed vector space \( T_qN \). It is not true that the normal spaces to every submanifold admit such a smoothly varying identification.\(^4\)

Proof of Theorem 7.10. Fix \( p \in P \) and choose charts \( (U, x) \) about \( p \) and \( (V, y) \) about \( q \) as in Theorem 6.16, so that \( y \circ f \circ x^{-1}(x^1, \ldots, x^k, \ldots, x^m) = (x^1, \ldots, x^k) \) and \( x^i(p) = 0, i = 1, \ldots, m \). Then \( (U, x) \) is a submanifold chart: \( x(P \cap U) = \{(x^1, \ldots, x^m) \in x(U) : x^1 = \cdots = x^k = 0\} \). The codimension is \( k \), and the exact sequence (7.12) is immediate in these charts. \( \square \)

Example 7.14 (the 2-sphere redux). The 2-sphere \( S^2 \subset \mathbb{A}^3_{x,y,z} \) is cut out by the single function

\[
(7.15)\quad f: \mathbb{A}^3 \longrightarrow \mathbb{R}
\]

\[
(x, y, z) \longmapsto x^2 + y^2 + z^2
\]

Namely, \( S^2 = f^{-1}(1) \). To verify that 1 \( \in \mathbb{R} \) is a regular value of \( f \), it suffices to observe that the differential \( df = 2x \, dx + 2y \, dy + 2z \, dz \) does not vanish at any point of \( f^{-1}(1) \). Note that \( df_{(0,0,0)} = 0 \); so \( (0,0,0) \in \mathbb{A}^3 \) is a critical point, 0 \( \in \mathbb{R} \) is a critical value, and yet \( f^{-1}(0) \subset \mathbb{A}^3 \) is a submanifold (though not of the expected codimension \( \dim \mathbb{R} \)).

Example 7.16 (the orthogonal group). Recall that \( M_n\mathbb{R} \) is the \( n^2 \)-dimensional vector space of \( n \times n \) matrices. Let \( ^t A \) denote the transpose of the matrix \( A \). The orthogonal group \( O_n \subset M_n\mathbb{R} \) is defined by the single condition \( A^t A = I \), where \( I \) is the identity matrix. To re-express this condition

\(^4\)We will soon study fiber bundles, and express this as the trivializability of the normal bundle to a submanifold cut out by a global function.
as the inverse image of a regular value of a function, we must note that the matrix \( S = A^tA \) is symmetric: \( tS = S \). Let \( S_n \mathbb{R} \subseteq M_n \mathbb{R} \) denote the vector subspace of symmetric matrices. Define

\[
(7.17) \quad f: M_n \mathbb{R} \rightarrow S_n \mathbb{R} \\
A \mapsto A^tA
\]

Then \( O_n = f^{-1}(I) \). To prove that \( O_n \subseteq M_n \mathbb{R} \) is a submanifold, we show that \( I \) is a regular value of \( f \) and apply Theorem 7.10. For any \( A, \hat{A} \in M_n \mathbb{R} \) we compute

\[
(7.18) \quad df_A(\hat{A}) = A^t\hat{A} + \hat{A}^tA.
\]

For \( A \in O_n \) and \( S \in S_n \mathbb{R} \) we must prove that the equation

\[
(7.19) \quad A^t\hat{A} + \hat{A}^tA = S
\]

has a solution \( \hat{A} \in M_n \mathbb{R} \), which it does: \( \hat{A} = \frac{1}{2}SA \).

The orthogonal group \( O_n \) is an example of a Lie group.

**Definition 7.20.** Let \( G \) be a set endowed with both a group structure and a smooth manifold structure. Suppose these structures are compatible in the sense that multiplication \( G \times G \rightarrow G \) and inversion \( G \rightarrow G \) are both smooth maps. Then \( G \) is a Lie group.

To verify that multiplication on \( O_n \) is smooth, we first observe that matrix multiplication \( M_n \mathbb{R} \times M_n \mathbb{R} \rightarrow M_n \mathbb{R} \) is a polynomial map, hence is smooth. Since \( O_n \subseteq M_n \mathbb{R} \) is a submanifold, so too is \( O_n \times O_n \subseteq M_n \mathbb{R} \times M_n \mathbb{R} \), and hence the restriction of multiplication to a map

\[
(7.21) \quad O_n \times O_n \rightarrow M_n \mathbb{R}
\]

is smooth. Furthermore, (7.21) factors through a map with codomain \( O_n \). The smoothness of the factored map follows from a general result, as does the smoothness of inversion on \( O_n \).

**Proposition 7.22.** Suppose \( f: M \rightarrow N \) is a smooth map of smooth manifolds, \( M' \subseteq M \) and \( N' \subseteq N \) are submanifolds, and the restriction of \( f \) to \( M' \) factors through a map \( f': M' \rightarrow N' \). Then \( f' \) is smooth.

**Proof.** Let \( p' \in M' \) and choose submanifold charts \((U, x)\) about \( p' \) and \((V, y)\) about \( f(p') \) such that \( f(U) \subseteq V \). If \( m' = \dim_{p'} M' \), \( m = \dim_p M \), \( n' = \dim_{f(p')} N' \), and \( n = \dim_{f(p)} N \), and if the smooth functions \( y^i = y^i(x^1, \ldots, x^m), i = 1, \ldots, n \), are the expression of \( y \circ f \circ x^{-1} \), then the smooth functions \( y^i = y^i(x^1, \ldots, x^m, 0, \ldots, 0), i = 1, \ldots, n' \), are the expression of \( y' \circ f' \circ (x')^{-1} \) in the charts on \( M', N' \) induced from the corresponding charts on \( M, N \).

\( \square \)

**Proposition 7.23.** The orthogonal group \( O_n \) is a Lie group.
A counting invariant; the fundamental theorem of algebra

We use the inverse function theorem to construct our first topological invariant. It illustrates a main theme of the class: we use calculus—local control from the infinitesimal hypothesis of maximal rank—to set up global invariants.

**Theorem 7.24.** Let $M$ be a compact smooth manifold, $N$ a smooth manifold with $\dim N = \dim N$, and $f : M \to N$ a smooth function. Set $N_{\text{reg}} \subset N$ the subset of regular values. Then the function

$$\# : N_{\text{reg}} \to \mathbb{Z}_{\geq 0}$$

$$q \mapsto \# f^{-1}(q)$$

is well-defined and locally constant.

The conclusion is that for any regular value $q \in N$ the subset $f^{-1}(q) \subset M$ is finite and its cardinality is a locally constant function of the regular value.

**Proof.** Fix $q \in N_{\text{reg}}$. Theorem 7.10 implies that $f^{-1}(q)$ is a 0-dimensional submanifold of $M$, so a finite or countable set of isolated points, and since $f^{-1}(q) \subset M$ is closed and $M$ is compact it follows that $f^{-1}(q)$ is a finite set. Hence (7.25) is well-defined. Suppose $\#(q) = N$ and write $f^{-1}(q) = \{p_1, \ldots, p_N\}$. The Inverse Function Theorem 6.13 implies that $f$ is a local diffeomorphism at each $p_i$; choose $U_i \subset M$ open so that $f : U_i \to f(U_i)$ is a diffeomorphism. Set

$$V = \bigcap_{i=1}^{N} f(U_i) \setminus f(M \setminus \bigcup_{i=1}^{N} U_i).$$

Then $V \subset N_{\text{reg}}$ is open, $q \in V$, and $\#|_V$ is constant. 

In the next lecture we prove that $N_{\text{reg}}$ is nonempty; in fact, it is a dense subset of $N$. In general we cannot control its connectivity. But for polynomial functions we have good control.

**Lemma 7.27.** Let $Q : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d$. Then $Q$ has at most $d$ roots.

**Proof.** Argue by induction: if $Q(z_0) = 0$, then $Q(z) = (z - z_0)Q_1(z)$ for a polynomial $Q_1$ of degree $d - 1$. 

**Theorem 7.28** (fundamental theorem of algebra). Any polynomial $P : \mathbb{C} \to \mathbb{C}$ has a root.

The one-point compactification of $\mathbb{C}$ is the complex projective line $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$; see (3.19). It is defined as the projectivization $\mathbb{P}(\mathbb{C}^2)$ of the standard 2-dimensional complex vector space, which we take to have components $z, w$. Thus a point of $\mathbb{CP}^1$ is an equivalence class of ordered pairs $[z, w]$ with at least one of $z, w$ nonzero and $[z, w] \sim [\lambda z, \lambda w]$ for all $\lambda \neq 0$. The complex line $\mathbb{C}$ embeds in $\mathbb{CP}^1$ as $z \mapsto [z, 1]$; the point $\infty$ is $[1, 0]$.

**Proof.** Define a smooth map $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ by

$$f([z, 1]) = [P(z), 1]$$

$$f([1, 0]) = [1, 0]$$
The critical points of \( f \) outside \( \infty = [1,0] \) are the roots of \( P' \), hence by Lemma 7.27 there are finitely many. Therefore, if \( N = \mathbb{C}P^1 \) is the codomain of \( f \), then \( N_{\text{reg}} \subset N \) is the complement of a finite set, so is connected. It follows that the locally constant function (7.25) is constant. It is clearly nonzero. If \( 0 \in N_{\text{reg}}, \) then \( f^{-1}(0) \subset \mathbb{C} \subset \mathbb{C}P^1 \) is nonempty, so \( P \) has a root. If \( 0 \in N \) is a critical value, then in particular it is a value of \( f \), so again \( P \) has a root.

Lecture 8: Measure zero, Sard’s theorem, introduction to fiber bundles

The main topic of this lecture, Sard’s theorem, is proved in an appendix in Guillemin-Pollack, and I will not duplicate the proof here. That appendix also lays out the theory of measure zero on a smooth manifold, which I will expand on in these notes. In the last part of the lecture the important notion of a fiber bundle is introduced, together with some examples.

Sard’s Theorem

This theorem has a long history, which includes Morse (1939), Sard (1942), Brown (1935), Dubrovickii (1953), and Thom (1954). The main result goes simply by the name of Sard.

**Theorem 8.1.** Let \( X, Y \) be \( C^\infty \) manifolds and \( f : X \to Y \) a \( C^\infty \) map. Denote by \( C \subset X \) the subset of critical points of \( f \). Then \( f(C) \subset Y \) has measure zero.

The proof implies a stronger result for \( C^k \) maps, where \( k \) is sufficiently large depending on the dimensions of \( X \) and \( Y \). Recall that \( f(C) \subset Y \) is the subset of critical values. Its complement is the set of regular values.

**Corollary 8.2.** The subset \( Y\setminus f(C) \) is dense.

This follows from the fact that sets of measure zero have nonempty interior. We often apply a weaker result: the set of regular values is nonempty. Another measure zero fact, that a finite or countable union of sets of measure zero has measure zero, implies the next result.

**Corollary 8.3.** Let \( \{X_i\}_{i \in I} \) be a collection of \( C^\infty \) manifolds, where \( I \) is finite or infinite. Let \( Y \) be a \( C^\infty \) manifold and \( f_i : X_i \to Y, \ i \in I, \) a \( C^\infty \) map. Then the set of simultaneous regular values of \( f_i \) is a dense subset of \( Y \).

Consider now the special case in which the domain has smaller dimension than the codomain. Every point of the domain is critical, since the differential cannot be surjective.

**Corollary 8.4.** Suppose \( X, Y \) are \( C^\infty \) manifolds with \( \dim X < \dim Y \) and \( f : X \to Y \) is a \( C^\infty \) map. Then \( f(X) \subset Y \) has measure zero.

As we will see, the proof of Corollary 8.4 can be given independently of Sard’s Theorem 8.1, and also in a more elementary fashion. As a particular application we prove the following.

**Corollary 8.5.** Any smooth map \( f : S^n \to S^m \) is homotopically trivial if \( n < m \).
Proof. By Corollary 8.4 there exists a point \( q \in S^m \) not in the image of \( f \), so \( f \) factors through a map \( f': S^n \to S^m \setminus \{q\} \). Stereographic projection is a diffeomorphism \( \varphi: S^m \setminus \{q\} \to \mathbb{R}^m \). Define the family of homotheties

\[
h_t: \mathbb{R}^m \to \mathbb{R}^m
\]
\[
\xi \mapsto (1 - t)\xi
\]

Let \( t: \mathbb{R}^m \to S^m \) denote the inclusion. Then \( t \circ h_t \circ \varphi \circ f' : S^n \to \mathbb{R}^m \) is a null homotopy of \( f' \). \( \square \)

Measure zero in affine space

We define the measure, or volume, of some standard subsets of \( \mathbb{A}^n \) and use them to define when an arbitrary subset \( E \subset \mathbb{A}^n \) has measure zero.

Definition 8.7.

(i) A standard box defined by real numbers \( a^1, \ldots, a^n, b^1, \ldots, b^n \) with \( a^i < b^i \), \( i = 1, \ldots, n \), is the set

\[
S = S(a^1, b^1; \ldots; a^n, b^n) = \{(x^1, \ldots, x^n) \in \mathbb{A}^n : a^i < x^i < b^i \text{ for all } i = 1, \ldots, n\}.
\]

If \( \lambda = b^i - a^i \) is independent of \( i \), then we call \( S \) a standard cube of side length \( \lambda \).

(ii) The volume of the standard box (8.8) is

\[
\mu(S) = \prod_{i=1}^n (b^i - a^i).
\]

(iii) A set \( E \subset \mathbb{A}^n \) has \((n\text{-dimensional})\ measure zero\) if for all \( \epsilon > 0 \) there exists a covering \( \{S_i\}_{i \in I} \) of \( E \) with \( I \) finite or countable such that \( \sum_{i \in I} \mu(S_i) < \epsilon \).

The definition (iii) of measure zero does depend on the ambient dimension: a nonempty open interval in \( \mathbb{A}^1 \) does not have 1-dimensional measure zero, but if we regard \( \mathbb{A}^1 \subset \mathbb{A}^n \) for \( n > 1 \), then it has \( n \)-dimensional measure zero. We use ‘measure zero’ if the dimension is clear from context.

We prove some basic properties of sets of measure zero.

Proposition 8.10.

(1) Let \( E \subset \mathbb{A}^n \) be a set of measure zero and \( E' \subset E \) a subset. Then \( E' \) has measure zero.

(2) Let \( \{E_i\}_{i \in I} \) be a finite or countable collection of measure zero subsets of \( \mathbb{A}^n \). Then \( \bigcup_{i \in I} E_i \) has measure zero.

(3) The affine subspace \( \mathbb{A}^m \subset \mathbb{A}^n \) has \( n \)-dimensional measure zero if \( m < n \).

(4) Let \( U \subset \mathbb{A}^n \) be an open subset, \( E \subset U \) a set of measure zero, and \( f: U \to \mathbb{A}^n \) a \( C^1 \) map. Then \( f(E) \subset \mathbb{A}^n \) has measure zero.

(5) A standard box does not have measure zero.

(6) If \( F \subset \mathbb{A}^n \) has nonempty interior, then \( F \) does not have measure zero.
(7) Let $E \subset \mathbb{A}^n$ be a closed subset. Suppose that for all $c \in \mathbb{R}$ the set $E \cap \{(c) \times \mathbb{A}^{n-1}\} \subset \mathbb{A}^{n-1}$ has $(n-1)$-dimensional measure zero. Then $E$ has $n$-dimensional measure zero.

A special case of (4) is that the image of a set of measure zero under a $C^\infty$ diffeomorphism has measure zero. In other words, measure zero is a $C^\infty$ concept.

Proof. Assertion (1) is immediate since a cover of $E$ by standard boxes of total volume $< \epsilon$ is a fortiori such a cover of $E$.

For (2), write $I = \{1,2,\ldots,N\}$ or $I = \mathbb{Z}^{>0}$. Then given $\epsilon > 0$, for each $i \in I$ let $\{S_{i,j}\}_{j \in I_i}$ be a cover of $E_i$ by at most countably many standard boxes of total volume $< \epsilon/2^i$. Then $\{S_{i,j}\}_{i \in I, j \in I_i}$ is an at most countable cover of $\bigcup_{i \in I} E_i$ by standard boxes of volume $< \epsilon$.

For (3), let $\{p_i\}_{i \in \mathbb{Z}^{>0}}$ be a countable dense subset of $\mathbb{A}^m$, for example the set of points with rational coordinates. Given $\epsilon > 0$, for each $i \in \mathbb{Z}^{>0}$ let $S_i$ be the standard box in $\mathbb{A}^n$ with center $p_i$ and side lengths $d_1 = \cdots = d_m = 1$, and $d_{m+1} = \cdots = d_n = (\epsilon/2^i)^{1/(n-m)}$. Then $\{S_i\}_{i \in \mathbb{Z}^{>0}}$ covers $\mathbb{A}^m \subset \mathbb{A}^n$ and has total $n$-dimensional volume $\epsilon$.

For (4), fix $p \in E$ and let $B_p \subset U$ be a ball whose closure lies in $U$. We prove that $f(E \cap B_p)$ has measure zero. Since $df$ is continuous, we can choose $C > 0$ so that $\|df\| \leq C$ on the compact set $\overline{B_p}$.

It follows that

$$d(f(x), f(x')) \leq C d(x, x'), \quad x, x' \in \overline{B_p}. \tag{8.11}$$

Hence if $S \subset \overline{B_p}$ is a standard cube of side length $\lambda$, then $f(S)$ is contained in a standard cube of side length $C\sqrt{n} \lambda$. Given $\epsilon > 0$, cover $E \cap B_p$ by at most countably many standard cubes of total volume $< \epsilon/(C\sqrt{n})^n$. It follows that $f(E \cap B_p)$ is covered by at most countably many standard cubes of total volume $< \epsilon$. This prove $f(E \cap B_p)$ has measure zero. Since $f(E)$ is covered by countably many sets of this form, (4) is proved.

The proof of (5) is a beautiful argument which Guillemin-Pollack credit to von Neumann. Let $S$ be a standard box and $\{S_i\}_{i \in I}$ an at most countable covering of $\overline{S}$ by standard boxes. Then we prove

$$\sum_{i \in I} \mu(S_i) \geq \mu(S). \tag{8.12}$$

It follows immediately that $S$ cannot have measure zero. To prove (8.12), for any standard box $T$ let $I(T)$ denote the number of points $(x^1,\ldots,x^n) \in T$ such that $x^j \in \mathbb{Z}$ for all $j = 1,\ldots,n$. Let the side lengths of $S$ be $d_1,\ldots,d_n$. Assume for the moment that each $d_j > 1$. Then

$$\prod_{j=1}^n (d_j - 1) < I(S) < \prod_{j=1}^n (d_j + 1). \tag{8.13}$$

Since $\overline{S}$ is compact, there is a finite subcover of $\{S_i\}_{i \in I}$, whose elements we denote $S_1,\ldots,S_N$. Let $S_i, i = 1,\ldots,N$, have side lengths $d_1(i),\ldots,d_n(i)$. Then since $I(S) \leq \sum_{i=1}^N I(S_i)$, we deduce

$$\prod_{j=1}^n (d_j - 1) < \sum_{i=1}^N \prod_{j=1}^n (d_j(i) + 1). \tag{8.14}$$
For any $c > 1$ the standard boxes $cS_1, \ldots, cS_N$ cover $cS$, and we can apply \((8.14)\) to conclude\(^5\)

\[(8.15)\]
\[
\prod_{j=1}^{n} (c d_j - 1) < \sum_{i=1}^{N} \prod_{j=1}^{n} (c d_j(i) + 1)
\]

which, after dividing by $c^n$, is

\[(8.16)\]
\[
\prod_{j=1}^{n} \left( d_j - \frac{1}{c} \right) < \sum_{i=1}^{N} \prod_{j=1}^{n} (d_j(i) + \frac{1}{c})
\]

Now take $c \to \infty$ to deduce the first inequality in

\[(8.17)\]
\[
\mu(S) = \prod_{j=1}^{n} d_j \leq \sum_{i=1}^{N} \prod_{j=1}^{n} d_j(i) = \sum_{i=1}^{N} \mu(S_i) \leq \sum_{i \in I} \mu(S_i),
\]

which is \((8.12)\).

Assertion (6) now follows quickly: if $F \subset \mathbb{A}^n$ has nonempty interior, it properly contains a standard box, which by \((5)\) does not have measure zero. Now apply \((1)\).

It\(^6\) suffices to assume in \((7)\) that $E$ is compact, since any closed $E \subset \mathbb{A}^n$ is the (countable) union of the compact sets $E \cap B_n(x)$, $n \in \mathbb{Z} > 0$, where $B_n(x)$ is the ball of radius $n$ about a fixed point $x \in \mathbb{A}^n$. Then $E \subset (a, b) \times \mathbb{A}^{n-1}$ for some finite closed interval $[a, b] \subset \mathbb{R}$. For any subset $J \in [a, b]$, let

\[(8.18)\]
\[
E_J = E \cap (J \times \mathbb{A}^{n-1}).
\]

Let $\epsilon > 0$ be given. For each $c \in [a, b]$ let $S_1(c), \ldots, S_{N_c}(c)$ be a covering of the compact set $E_c \subset \mathbb{A}^{n-1}$ by $(n-1)$-dimensional standard boxes of total volume $< \epsilon / 2(b - a)$. By compactness of $E$ there is an open interval $J(c) \subset [a, b]$ about $c$ so that $\{J(c) \times S_i(c)\}_{i=1}^{N_c}$ covers $E_J(c)$. The intervals $\{J(c)\}_{c \in [a, b]}$ cover $[a, b]$, so there exists a finite subcover. It is not difficult then to find a finite collection of open intervals $J_1, \ldots, J_M$ which cover $[a, b]$; each $J_j \subset J(c_j)$ for some $c_j$ such that $J(c_j)$ is in the finite subcover; and the total length of $J_1, \ldots, J_M$ is $< 2(b - a)$. The $n$-dimensional standard boxes $\{J_j \times S_i(c_j)\}_{j=1, i=1}^{M, N_c}$ cover $E$ and have total $n$-dimensional volume $< \epsilon$. \(\square\)

**Measure zero on smooth manifolds**

**Definition 8.19.** Let $Y$ be a smooth manifold. A subset $E \subset Y$ has **measure zero** if for all $\mathbb{A}^n$-valued charts $(V, y) \subset Y$, the set $y(E \cap Y) \subset \mathbb{A}^n$ has measure zero.

The choice of $n$ may vary on components of $Y$. Definition 8.19 would be impractical if we had to check all charts in a maximal atlas, but Proposition 8.10(4) guarantees the following.

---

\(^5\)Even if $d_j < 1$ for some $j$, for sufficiently large $c$ the following is valid.

\(^6\)Guillemin-Pollack, Appendix A, has more detail for the proof in this paragraph.
Proposition 8.20. A subset $E \subset Y$ has measure zero if the condition of Definition 8.19 holds for a set of charts of $Y$ which cover $E$.

The complement of a set of measure zero on a smooth manifold is nonempty. The following much stronger assertion holds.

Proposition 8.21. Let $E \subset Y$ have measure zero. Then $Y \setminus E$ is dense.

Proof. $(Y \setminus E)^c \subset E$ is open and has measure zero (Proposition 8.10(1)), so by Proposition 8.10(6) must be empty. Therefore, $Y \setminus E = Y$ as claimed. $\square$

Now we are in a position to prove a special version of Sard’s Theorem.

Proof of Corollary 8.4. By second countability it suffices to check locally on $X$. With respect to local charts we represent $f$ by a smooth map $g: U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^n$ is open. Define

$$G: U \times \mathbb{R}^{m-n} \to \mathbb{R}^m,$$

$$(x, y) \mapsto g(x)$$

Then since $U \times \{0\} \subset \mathbb{R}^n \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^{m-n} = \mathbb{R}^m$ has $m$-dimensional measure zero (Proposition 8.10(3)), it follows from Proposition 8.10(4) that $G(U \times \{0\}) = g(U)$ has $m$-dimensional measure zero. $\square$

Proof of Sard’s Theorem

I defer to the appendix in Guillemin-Pollack.

Introduction to fiber bundles

(8.23) Special types of maps: review. In previous lectures we introduced conditions on a smooth map $f: X \to Y$. The infinitesimal condition that the rank of the differential be maximal, which we can impose at a point of the domain $X$ or at all points of $X$, leads to a local normal form for $f$ (Theorem 6.16). We also imposed global conditions. For example, an injective immersion which is a homeomorphism onto its image is an embedding, and the image of an embedding is a submanifold (Theorem 7.1). A submersion is a map whose differential is surjective everywhere, and in that case the fibers (inverse images of points) are submanifolds (Theorem 7.10). Now we introduce a special type of submersion in which the fibers do not jump. The key idea of local triviality was singled out by Steenrod in his classic 1951 text The Topology of Fibre Bundles, though the notion had already been around for more than a decade at that point.

Definition 8.24. Let $\pi: E \to M$ be a smooth map of smooth manifolds. We say $\pi$ is a fiber bundle if for all $p \in M$ there exists an open neighborhood $U \subset M$ about $p$ and a diffeomorphism
\[ \varphi : U \times \pi^{-1}(p) \to \pi^{-1}(U) \] such that the diagram

\[
\begin{align*}
U \times \pi^{-1}(p) & \xrightarrow{\varphi} \pi^{-1}(U) \\
& \downarrow \quad \downarrow \pi \\
U & \xrightarrow{\text{pr}_1} \pi
\end{align*}
\]

(8.25)

commutes. The domain \( E \) of \( \pi \) is the \textit{total space} and the codomain \( M \) is the \textit{base} of the fiber bundle \( \pi \).

**Figure 14.** A fiber bundle with local trivialization about \( p \)

Here \( \text{pr}_1 \) is projection onto the first factor, followed by the inclusion \( U \hookrightarrow M \). The \textit{condition} that pairs \( U, \varphi \) exist about every point \( p \in M \) is called \textit{local triviality}. It implies that \( \pi \) is a submersion, hence the fibers are submanifolds.

**Remark 8.26.** Some authors require that \( \pi \) be surjective. We allow the fibers \( \pi^{-1}(p) \) to be empty.

**Remark 8.27.** There is also a notion of a fiber bundle with fixed fiber \( F \); we introduce it in the next lecture.

**Remark 8.28.** Local triviality (8.25) provides a diffeomorphism \( \pi^{-1}(p') \xrightarrow{\cong} \pi^{-1}(p) \) of the fiber over any \( p' \in U \) with the fixed fiber \( \pi^{-1}(p) \). If we, heuristically, rewrite \( \pi \) as a map from \( M \) to sets, then local triviality expresses the local constancy of this map.

**Example 8.29** (trivial fiber bundle). Let \( M, F \) be smooth manifolds. Then projection \( \text{pr}_1 : M \times F \to M \) is a fiber bundle. To verify the condition of local triviality, for any \( p \in M \) we can choose \( U = M \) and \( \varphi = \text{id} \). This is the trivial fiber bundle over \( M \) with fiber \( F \). Any fiber bundle is locally isomorphic to a trivial fiber bundle. (We introduce the appropriate notion of isomorphism in the next lecture.)
Example 8.30 (a nontrivial example). The map

\[ \pi: O_n \rightarrow S^{n-1} \]
\[ A \mapsto A\xi_0 \]

is a fiber bundle, where, say, \( \xi_0 = (1, 0, \ldots, 0) \). It is nontrivial (meaning not isomorphic to a trivial bundle) if \( n \geq 3 \). We will not prove that statement now.

In the next lecture we give more examples and focus on the cases of interest in this course: the tangent bundle to a smooth manifold and the normal bundle to a submanifold.

(8.32) **Perspective.** Any map (of sets) \( \pi: E \rightarrow M \) induces a partition of the domain into its fibers. Fiber bundles induce “regular” partitions in that the fibers are locally diffeomorphic to each other. In this way fiber bundles provide useful decompositions of smooth manifolds, and that can be a great tool to study global properties. In a different direction, fiber bundles can encode the geometry of the base manifold. The tangent bundle is an example, and there are many associated bundles that also come into play. Since the total space is a smooth manifold in its own right, we can apply the tools of manifold theory to it and often learn about the base in the process.

### Lecture 9: Fiber bundles and vector bundles

**Recollection of fiber bundles**

We resume our discussion of fiber bundles. Recall that a fiber bundle \( \pi: E \rightarrow M \) is a special kind of map between smooth manifolds. It satisfies the local triviality condition specified in Definition 8.24. Note here that locality is in the codomain, which is called the base of the fiber bundle. In that version we ask that the fibers be “locally constant” functions of the base: nearby fibers are diffeomorphic in a smooth way, which is the content of the map \( \varphi \) in (8.25). A closely related notion fixes a model manifold for the fiber.

**Definition 9.1.** Let \( \pi: E \rightarrow M \) be a smooth map of smooth manifolds, and let \( F \) be a (nonempty) smooth manifold. We say \( \pi \) is a fiber bundle with fiber \( F \) if for all \( p \in M \) there exists an open neighborhood \( U \subset M \) about \( p \) and a diffeomorphism \( \varphi: U \times F \rightarrow \pi^{-1}(U) \) such that the diagram

\[ \begin{array}{ccc}
U \times F & \xrightarrow{\varphi} & \pi^{-1}(U) \\
\downarrow{\text{pr}_1} & \nearrow{\pi} & \downarrow{\pi} \\
M & & M
\end{array} \]

commutes.
(9.3) Notation. If $\pi : E \to M$ is a fiber bundle, then for $p \in M$ we set $E_p = \pi^{-1}(p)$ to be the fiber over the point $p$ in the base.

(9.4) Maps of fiber bundles. The parametrized version of a smooth map of manifolds is a map of fiber bundles. Let $\pi' : E' \to M$ and $\pi : E \to M$ be fiber bundles over the same base. Then a map of fiber bundles is a smooth map $\varphi : E' \to E$ which fits into the commutative diagram

\[
\begin{array}{ccc}
E' & \xrightarrow{\varphi} & E \\
\downarrow{\pi'} & & \downarrow{\pi} \\
M & & 
\end{array}
\]

(9.5)

It is a smooth family of smooth maps $\varphi_p : E'_p \to E_p$ parametrized by $p \in M$. When the fiber bundles $\pi', \pi$ have extra structure—such as bundles of affine spaces, vector spaces, Lie groups, etc.—then we may require that each $\varphi_p$ preserve that structure. Below we encounter linear maps of vector bundles.

Examples of fiber bundles

There are several other examples on your homework.

Example 9.6 (covering spaces). A (smooth) covering space $\pi : E \to M$ is a fiber bundle. To see this, recall the definition: every point $p \in M$ has an open neighborhood $U \subset M$ which is evenly covered, i.e., there is a discrete set $S$ and a homeomorphism

\[
\varphi : U \times S \longrightarrow \pi^{-1}(U)
\]

(9.7) which commutes with projection to $U$. This is precisely the local trivialization condition. So a fiber bundle with discrete fibers is a covering space.

Example 9.8 (affine lines in a plane). Let $V$ be a 2-dimensional real vector space, and let $A$ be an affine space over $V$. Let $E$ be the 2-dimensional manifold of affine lines in $A$. Each affine line determines a line in $V$—a 1-dimensional subspace—namely its tangent line. The assignment of a tangent line is a smooth map

\[
\pi : E \longrightarrow \mathbb{P}V.
\]

(9.9) We claim that $\pi$ is a fiber bundle. Fix $K \in \mathbb{P}V$ and $p \in A$. We produce a local trivialization of $\pi$ on $U = \mathbb{P}V \setminus \{K\}$. First, observe that $p$ determines a section $s_p : \mathbb{P}V \to E$ of (9.9) which assigns to each $L \in \mathbb{P}V$ the unique affine line through $p$ with tangent line $L$. As depicted in Figure 15, define

\[
\varphi : U \times K \longrightarrow \pi^{-1}(U)
\]

\[
L, \xi \longmapsto s_p(L) + \xi
\]

(9.10)
Remark 9.11. The section \( s_p \) is an example of a “smoothly varying” family of affine lines. The fiber bundle (9.9) gives meaning to ‘smoothly varying’.

Remark 9.12. The fibers of (9.9) have more structure: they are affine spaces. More precisely, \( \pi^{-1}(L) \) is affine over the quotient vector space \( V/L \). (If the affine line \( \ell \subset A \) has tangent line \( L \subset V \), and \( \xi \in V \), then the affine line \( \ell + \xi \) only depends on \( \xi \) (mod \( L \)) since translation by vectors in \( L \) preserves \( \ell \).) In fact, there is a vector bundle \( Q \to \mathbb{P}V \) whose fiber at \( L \in \mathbb{P}V \) is the vector space \( V/L \), and (9.9) is a bundle of affine spaces over \( Q \to \mathbb{P}V \), a parametrized version of a single affine space over a single vector space. This illustrates the idea that fiber bundles can have more structure, in which case we require that local trivializations preserve that structure. Note that in this case the local trivialization (9.10) \textit{is} an affine map on each fiber.

Example 9.13 (a surjective submersion which is not a fiber bundle). We work in \( \mathbb{A}^3 \) with coordinates \( (x, y, z) \). Define

\[
E = \{(x, y, z) \in \mathbb{A}^3 : y^2 + z^2 = 1 \} \setminus \{(0, 0, +1), (0, 0, -1)\}.
\]

This is a cylinder with two points \( n, s \) deleted. Let \( P \) denote the space of affine planes in \( \mathbb{A}^3 \) which contain the \( z \)-axis; then \( P \) is diffeomorphic to \( \mathbb{RP}^1 \), the space of lines through the origin in the \( x, y \)-plane. Define

\[
\pi : E \longrightarrow P
\]
the map which takes \( p \in E \) to the plane containing the distinct non-collinear points \( n, s, p \), as depicted in Figure 16. Then \( \pi \) is surjective and a submersion. (Proof of the latter: a motion germ in \( P \) is represented by a curve \( \Pi_t \) of planes through the \( z \)-axis. Intersect with the affine line \( x = 1, z = 0 \) to lift to a motion \( p_t \) in \( E \) such that \( \pi(p_t) = \Pi_t \).) However, \( \pi \) is not a fiber bundle. The typical fiber of \( \pi \) is an ellipse minus the points \( n, s \), whereas the fiber over the \( x, z \)-plane \( \Pi_{x,z} \) is the union of two affine lines minus \( n, s \), which is not diffeomorphic to the other fibers. Hence \( \pi \) cannot be locally trivial over \( \Pi_{x,z} \).

**Vector bundles**

(9.16) *Fiber product.* As a preliminary, we introduce the fiber product of fiber bundles. It is the parametrized version of the Cartesian product of manifolds. Let \( M \) be a smooth manifold and \( \pi_i: E_i \to M, i = 1, 2 \), fiber bundles over \( M \). Define

\[
E_1 \times_M E_2 = \{(e_1, e_2) \in E_1 \times E_2 : \pi_1(e_1) = \pi_2(e_2)\}.
\]

Then \( E_1 \times_M E_2 \subset E_1 \times E_2 \) is a submanifold. This is easily proved once we introduce transversality, though it is not difficult to do directly. The maps \( \pi_1, \pi_2 \) agree and determine a map

\[
\pi: E_1 \times_M E_2 \to M.
\]

Then (9.18) is a fiber bundle. Namely, local trivializations \( \varphi_1, \varphi_2 \) of \( E_1, E_2 \) over open neighborhoods \( U_1, U_2 \) of a point \( p \in M \) combine to a local trivialization of (9.18) over \( U_1 \cap U_2 \). The construction generalizes to the fiber product of a finite set of fiber bundles over a common base.

(9.19) *Vector space.* Recall the definition of a vector space. It consists of data \((V, 0, +, \times)\) where \( V \) is a set; \( 0 \in V \) is a distinguished element, the zero vector; \( +: V \times V \to V \) is called vector addition; and \( \times: \mathbb{R} \times V \to V \) is called scalar multiplication. There are many axioms which tell that \((V, 0, +)\) is an abelian group, scalar multiplication distributes over vector addition, etc.

(9.20) *Vector bundle.* Just as the fiber product (9.16) is a parametrized version of Cartesian product, a vector bundle is a parametrized version of a vector space.

**Definition 9.21.** A *vector bundle* \((\pi, 0, +, \times)\) consists of a fiber bundle \( \pi: E \to M \); a section \( 0: M \to E \) of \( \pi \), called the *zero section*; a smooth map \(+: E \times_M E \to E\) such that

\[
\begin{array}{ccc}
E \times_M E & \xrightarrow{+} & E \\
\downarrow & & \downarrow \\
M & & \pi
\end{array}
\]

(9.22)
commutes; and a smooth map \( \times : \mathbb{R} \times E \to E \) such that

\[
\begin{CD}
\mathbb{R} \times E @>\times>> E \\
@V\pi VV @VV\pi V \\
M @. E
\end{CD}
\]  

commutes. We require the vector space axioms and also that local trivializations for \( \pi \) be linear maps on fibers.

The vector space axioms tell that each fiber \( E_p, p \in M \), of \( \pi : E \to M \) is a vector space. Thus \( \pi \) exhibits a family of vector spaces parametrized by \( M \). The last condition, that local trivializations be linear on fibers, requires this be a locally trivial family of vector spaces. Explicitly, it asserts that about each \( p \in M \) there exists an open neighborhood \( U \subset M \) and a diffeomorphism \( \varphi \) in the diagram

\[
\begin{CD}
U \times E_p @>\varphi>> \pi^{-1}(U) \\
@V\pi_1 VV @VV\pi V \\
U @. \pi^{-1}(U)
\end{CD}
\]  

such that \( \varphi|_{p' \times E_p} : E_p \to E_{p'} \) is a linear isomorphism for all \( p' \in U \).

Analogous to Definition 9.1 is the definition of a vector bundle with fiber a fixed vector space.

**Constructions of vector bundles**

\((9.25)\) *Transition functions.* Let \( V \) be a finite dimensional real vector space and suppose \( \pi : E \to M \) is a vector bundle with fiber \( V \). Let \( U_1, U_2 \subset M \) be open sets equipped with local trivializations

\[
\begin{CD}
U_i \times V @>\varphi_i>> \pi^{-1}(U_i) \\
@V\pi_1 VV @VV\pi V \\
U_i @. \pi^{-1}(U_i)
\end{CD}
\]  

Over the intersection \( U_1 \cap U_2 \) the ratio of the trivializations is the *transition function*

\[
g_{21} : U_1 \cap U_2 \to \text{Aut}(V) \\
p \mapsto (\xi \mapsto (\text{pr}_2 \circ \varphi_2^{-1} \circ \varphi_1)(p, \xi))
\]  

which compares them. Note that the order matters: \( g_{21} \) is the transition from trivialization 1 to trivialization 2. The transition functions satisfy

\[
(9.28) \quad g_{12} \circ g_{21} = \text{id}_V \quad \text{on } U_1 \cap U_2,
\]

\[
(9.29) \quad g_{32} \circ g_{21} = g_{31} \quad \text{on } U_1 \cap U_2 \cap U_3.
\]
Equation (9.29) is called the cocycle condition. It can be written

\[(9.30) \quad g_{23} \circ g_{13}^{-1} \circ g_{12} = \text{id}_V,\]

whose three constituents on the left hand side are obtained by striking out a single digit from 123 and alternating the “signs” as we go across: \[123 \circ 123^{-1} \circ 123.\]

**Theorem 9.32.** Let \(M\) be a smooth manifold and \(\{U_\alpha\}_{\alpha \in A}\) an open cover. Let \(V\) be a finite dimensional real vector space, and suppose given smooth functions

\[(9.33) \quad g_{\alpha_2\alpha_1} : U_{\alpha_1} \cap U_{\alpha_2} \to \text{Aut}(V), \quad \alpha_1, \alpha_2 \in A\]

such that for all \(\alpha_1, \alpha_2, \alpha_3 \in A,

\[(9.34) \quad g_{\alpha_1\alpha_2} \circ g_{\alpha_2\alpha_1} = \text{id}_V \quad \text{on } U_{\alpha_1} \cap U_{\alpha_2}, \]

\[(9.35) \quad g_{\alpha_3\alpha_2} \circ g_{\alpha_2\alpha_1} = g_{\alpha_3\alpha_1} \quad \text{on } U_{\alpha_1} \cap U_{\alpha_2} \cap U_{\alpha_3}.\]

Then there is a canonical vector bundle \(\pi : E \to M\) equipped with local trivializations over the sets \(U_\alpha\) such that the transition functions equal (9.33).

I will not spell out the precise meaning of ‘canonical’; heuristically, it means there are no choices made beyond the given data.

**Proof.** Define an equivalence relation \(\sim\) on the disjoint union \(\bigsqcup_{\alpha \in A} U_\alpha \times V\) as

\[(9.36) \quad (p_1, \xi_1) \sim (p_2, \xi_2) \quad \text{iff} \quad p_1 = p_2, \quad \xi_2 = g_{\alpha_2\alpha_1}(p)(\xi_1)\]

where \(p_i \in U_{\alpha_i}\) and \(\xi_i \in V, \ i = 1, 2.\) Define

\[(9.37) \quad E = \bigsqcup_{\alpha \in A} U_\alpha \times V / \sim,\]

the set of equivalence classes. Endow \(E\) with the quotient topology of the product topology. For each \(\alpha \in A\) the composition

\[(9.38) \quad U_\alpha \times V \to \bigsqcup_{\alpha \in A} U_\alpha \times V \to E\]

is a homeomorphism onto its image, and the images cover \(E.\) It follows that \(E\) is locally Euclidean. The projections \(U_\alpha \times V \to U_\alpha\) stitch to a surjective continuous map

\[(9.39) \quad \pi : E \to M.\]
Since $M$ is second countable and the fibers of $\pi$ are homeomorphic to the second countable space $V$, it follows that $E$ is second countable. Also, $M$ is Hausdorff, so if two points in $E$ project to distinct points in $M$ they can be separated by open sets. The fibers of $E$ are homeomorphic to $V$, so are Hausdorff. It follows that $E$ is Hausdorff. Therefore, $E$ is a topological manifold.

Choose an atlas of $M$ such that the domain $U$ of each chart $(U, x)$ is contained in some $U_\alpha$. The composition

$$\pi^{-1}(U) \to U \times V \xrightarrow{x \times \text{id}} \mathbb{A}^n \times V$$

is a homeomorphism onto its image, so is a chart on $E$, and these charts have $C^\infty$ overlaps. (The first map in (9.40) is the inverse of the restriction of (9.38) to $U \times V$, inverted on its image.) This endows $E$ with the structure of a smooth manifold, and the map (9.39) is smooth.

The vector space structure on $V$—the zero vector, vector addition, and scalar multiplication—induce a vector space structure on (9.39) since the equivalence relation (9.36) is defined by linear maps. The vector space axioms hold on each fiber of $\pi$ since they hold on $V$. The diffeomorphisms (9.38) are local trivializations which are linear on each fiber.

Example 9.41. Theorem 9.32 applies also to complex vector bundles. (A complex vector bundle is defined by replacing ‘$\mathbb{R}$’ in Definition 9.21 with ‘$\mathbb{C}$’. Write points of $\mathbb{C}P^1$ as equivalence classes $[z^0, z^1]$ of pairs of complex numbers, not both zero, under the equivalence $[z^0, z^1] = [\lambda z^0, \lambda z^1]$ for $\lambda \in \mathbb{C} \setminus \{0\}$. Cover $\mathbb{C}P^1$ by two (affine) open sets

$$U_1 = \{[z, 1] : z \in \mathbb{C}\}$$
$$U_2 = \{[1, w] : w \in \mathbb{C}\},$$

and for $k \in \mathbb{Z}$ define the transition function

$$g^{(k)}_{21} : U_1 \cap U_2 \to \text{Aut}(\mathbb{C}) = \mathbb{C}^\times$$
$$[z, 1] \mapsto z^k$$

where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. Note that $z \neq 0$ if $[z, 1] \in U_1 \cap U_2$. The preceding construction gives a complex line bundle $L^{(k)} \to \mathbb{C}P^1$ for each $k$.

Another construction. In practice, we often encounter a variation of Theorem 9.32 in which we have a family of vector spaces parametrized by a smooth manifold and we want to construct from it a vector bundle.

Theorem 9.45. Let $M$ be a smooth manifold and

$$\pi : E = \bigsqcup_{p \in M} E_p \to M$$
a set of vector spaces parametrized by \( M \). Suppose given an open cover \( \{U_\alpha\}_{\alpha \in A} \) together with a family of vector spaces \( \{V_\alpha\}_{\alpha \in A} \) and set isomorphisms \( \varphi_\alpha, \alpha \in A \), such that the diagram

\[
\begin{array}{ccc}
U_\alpha \times V_\alpha & \longrightarrow & \bigsqcup_{p \in U_\alpha} E_p \\
\downarrow \varphi_\alpha & & \downarrow \pi \\
U_\alpha & \longrightarrow & p \in U_\alpha
\end{array}
\]

(9.47)

commutes. There are induced transition functions

\[
g_{\alpha_2 \alpha_1} : U_{\alpha_1} \cap U_{\alpha_2} \longrightarrow \text{Iso}(V_{\alpha_1}, V_{\alpha_2}),
\]

(9.48)

and assume they satisfy the cocycle conditions

\[
g_{\alpha_1 \alpha_2} \circ g_{\alpha_2 \alpha_1} = \text{id}_{V_{\alpha_1}} \quad \text{on } U_{\alpha_1} \cap U_{\alpha_2},
\]

(9.49)

\[
g_{\alpha_2 \alpha_3} \circ g_{\alpha_3 \alpha_2} = g_{\alpha_2 \alpha_1} \quad \text{on } U_{\alpha_1} \cap U_{\alpha_2} \cap U_{\alpha_3}.
\]

(9.50)

Then there is a canonical manifold structure on \( E \) so that (9.46) is a vector bundle and the \( \varphi_\alpha \) are smooth local trivializations.

The proof is similar to that of Theorem 9.32. Use the disjoint union of the \( \varphi_\alpha \) to topologize \( E \) and make charts on \( E \) as before.

(9.51) **Dual bundle.** As an application of Theorem 9.45 we construct the dual bundle to a vector bundle. It is the parametrized version of the passage from a vector space to its linear dual. As a preliminary, recall that if \( T : V \rightarrow W \) is a linear map of vector spaces, then there is a dual map \( T^* : W^* \rightarrow V^* \) defined by

\[
\langle T^* w^*, v \rangle = \langle w^*, Tv \rangle, \quad v \in V, \quad w^* \in W^*,
\]

(9.52)

where the pairing on the left hand side is between \( V^* \) and \( V \), and the pairing on the right hand side is between \( W^* \) and \( W \). If \( T \) is invertible, then so is \( T^* \). For \( V \) finite dimensional, there is an isomorphism of Lie groups

\[
\begin{array}{ccc}
\text{Aut}(V) & \longrightarrow & \text{Aut}(V^*) \\
T & \longmapsto & (T^*)^{-1}
\end{array}
\]

(9.53)

Let \( \pi : E \rightarrow M \) be a vector bundle. Choose an open cover \( \{U_\alpha\}_{\alpha \in A} \) of \( M \) together with for each \( \alpha \in A \) a local trivialization

\[
\begin{array}{ccc}
U_\alpha \times V_\alpha & \longrightarrow & \pi^{-1}(U_\alpha) \\
\downarrow \varphi_\alpha & & \downarrow \pi \\
U_\alpha & \longrightarrow & \pi^{-1}(U_\alpha)
\end{array}
\]

(9.54)
over each $U_\alpha, \alpha \in A$. Set $E^* = \bigsqcup_{p \in M} E^*_p$ and for $\alpha \in A$ define

$$\tilde{\varphi}_\alpha: U_\alpha \times V_\alpha^* \rightarrow \bigsqcup_{p \in U_\alpha} E^*_p,$$

(9.55)

$$p \cdot \xi^* \mapsto (\varphi_\alpha(p)^*)^{-1}(\xi^*)$$

where $\varphi_\alpha(p) \in \text{Iso}(V_\alpha, E_p)$ and we apply its inverse dual to obtain an isomorphism $V_\alpha^* \rightarrow E^*_p$. Since (9.53) is a homomorphism, the cocycle conditions for the transition functions derived from the $\varphi_\alpha$ imply the cocycle conditions (9.49), (9.50) for the transition functions derived from the $\tilde{\varphi}$. Then Theorem 9.45 applies to construct a smooth vector bundle structure on $E^* \rightarrow M$.

Remark 9.56. A similar procedure allows us to carry over functorial maps in linear algebras to vector bundles. For example, if $E' \rightarrow M$ and $E \rightarrow M$ are vector bundles, then there is a vector bundle $\text{Hom}(E', E) \rightarrow M$ whose fiber at $p \in M$ is $\text{Hom}(E'_p, E_p)$.

(9.57) Quotient bundle. Let $M$ be a smooth manifold and $\pi': E' \rightarrow M$, $\pi: E \rightarrow M$ smooth vector bundles. Suppose $\iota: M \rightarrow \text{Hom}(E', E)$ is a section of $\text{Hom}(E', E) \rightarrow M$ which is injective on each fiber. Then we can replace $E'_p$ with $\iota(E'_p) \subset E_p$ and so assume that $\pi'$ is a subbundle of $\pi$. Define the family of vector spaces

$$E'' = \bigsqcup_{p \in M} E_p/E'_p \rightarrow M$$

(9.58)

Then (9.58) is a vector bundle in a natural way. Namely, choose an open cover $\{U_\alpha\}_{\alpha \in A}$ of $M$ together with a family $\{V_\alpha\}_{\alpha \in A}$ of vector spaces, a family $\{V'_\alpha\}_{\alpha \in A}$ of subspaces, and local trivializations $\varphi_\alpha: U_\alpha \times V_\alpha \rightarrow \pi^{-1}(U_\alpha)$ of $\pi$ which restrict to local trivializations $\varphi'_\alpha: U_\alpha \times V'_\alpha \rightarrow (\pi')^{-1}(U_\alpha)$ of $\pi'$. Define

$$\varphi''_\alpha: U_\alpha \times V_\alpha/V'_\alpha \rightarrow \bigsqcup_{p \in U_\alpha} E_p/E'_p,$$

(9.59)

$$p \cdot [\xi] \mapsto [\varphi_\alpha(\xi)]$$

Now apply Theorem 9.45.

Tangent, cotangent, and normal bundles

Let $M$ be a smooth manifold. Recall the construction of the tangent space $T_pM$ from Lecture 3 and the handout that goes with it.
(9.60) **Tangent bundle.** Define

$$\pi : TM = \bigsqcup_{p \in M} T_p M \rightarrow M$$

Let \( \{(U_\alpha, x_\alpha)\}_{\alpha \in A} \) be an atlas with \( x_\alpha : U_\alpha \rightarrow A_\alpha \), where \( A_\alpha \) is an affine space over a vector space \( V_\alpha \). Recall that for each \( p \in U_\alpha \) the chart gives an isomorphism \( T_p M \rightarrow V_\alpha \). Use its inverse to construct a set isomorphism

$$\varphi_\alpha : U_\alpha \times V_\alpha \rightarrow \bigsqcup_{p \in M} T_p M$$

Now apply Theorem 9.45 to produce the tangent bundle \( \pi \) as a smooth vector bundle.

**Definition 9.63.** A **vector field** is a section of the tangent bundle \( \pi : TM \rightarrow M \).

(9.64) **Charts on \( TM \).** Suppose \( (U; x^1, \ldots, x^n) \) is a standard chart on \( M \). We use it to construct a standard chart \( (\pi^{-1}U; \pi^* x^1, \ldots, \pi^* x^n, \dot{x}^1, \ldots, \dot{x}^n) \), where \( \pi^* x^i = x^i \circ \pi \) is the pullback function on \( \pi^{-1}U \). The functions \( \dot{x}^i : \pi^{-1}U \rightarrow \mathbb{R} \) are determined by

$$\dot{x}^i(\xi) = \frac{\partial}{\partial x^i} \pi(\xi), \quad \xi \in \pi^{-1}U.$$  

The overlap functions with a chart induced from a second chart \( (U; y^1, \ldots, y^n) \) of \( M \) with the same domain are computed by:

$$\xi = \dot{x}^i(\xi) \frac{\partial}{\partial x^i} \pi(\xi), \quad \xi \in \pi^{-1}U.$$  

which leads to the overlap functions

$$y^\alpha = y^\alpha(x^1, \ldots, x^n)$$

$$\dot{y}^\alpha = \dot{x}^i \frac{\partial y^\alpha}{\partial x^i},$$

where \( y^\alpha(x^1, \ldots, x^n) \) are the overlap functions on \( M \).

(9.68) **Cotangent bundle.** Apply (9.51) to construct the cotangent bundle

$$T^*M \rightarrow M$$

as the dual vector bundle to the tangent bundle.

**Definition 9.70.** A **1-form** is a section of the cotangent bundle \( T^*M \rightarrow M \).

Observe that if \( f : M \rightarrow \mathbb{R} \) is a smooth real-valued function on \( M \), then its differential \( df \) is a 1-form on \( M \).
Normal bundle. Suppose $M$ is a smooth manifold and $N \subset M$ is a smooth submanifold. Then for all $p \in N$, the tangent space to $N$ is a subspace $T_pN \subset T_pM$ of the tangent space to $M$. This leads to an inclusion of vector bundles $TN \subset TM|_N$ over $N$. Now apply (9.57) to construct the normal bundle $\nu \to N$ as the quotient bundle. There is a short exact sequence of vector bundles

$$0 \to TN \to TM|_N \to \nu \to 0$$

over $N$. It is a family of short exact sequences of vector spaces parametrized by $N$. The fiber at $p \in N$ is the quotient space

$$\nu_p = T_pM/T_pN.$$

### Lecture 10: Partitions of unity

This is the first of several make-up lectures which will be scattered throughout the rest of the semester. The video is available through Canvas. As with other lectures, what is in these notes does not follow the lecture completely: there are simplifications, complements, more details, etc.

#### Introduction and motivation

Let $M$ be a smooth manifold. Often we encounter a situation in which we have in hand a geometric object locally on $M$ and we want to patch together to a global object. As a concrete example, consider the tangent bundle $\pi : TM \to M$. We seek a smoothly varying inner product on the fibers of $\pi$. A chart $(U, x)$ on $M$ identifies $T_pM$, $p \in U$, with a fixed vector space $V$. So we can choose an inner product on $V$ and use the chart to transport it to a smoothly varying family of inner products on $T_pM$, $p \in U$. We can do so on each chart in an atlas, and so cover $M$ by open sets on which we have the desired inner products. What a partition of unity enables us to do is splice these together into a single global inner product (called a Riemannian metric). A partition of unity gives a weighted average a geometric quantity, such as an inner product, and the technique applies as long as the space of those quantities is convex, as is the space of positive definite inner products on a real vector space.

### Preliminary: some point-set topology

In lecture I proved that a locally compact, Hausdorff, second countable topological space is paracompact. Here I’ll simplify a bit and replace ‘locally compact’ by ‘locally Euclidean’, i.e., prove the theorem for topological manifolds.

---

7A topological space $M$ is *locally compact* if for all $p \in M$ there exists an open set $U$ and a compact set $C$ such that $p \in U \subset C$. If $M$ is locally compact Hausdorff, then we can choose $C = \overline{U}$. A topological manifold is locally compact Hausdorff: choose $U$ to be the inverse image of a ball in affine space under a local homeomorphism.
(10.2) Exhaustion of a manifold by compact subsets. An exhaustion of a space is a nested sequence of subsets whose union is the entire space. The following theorem proves that a topological manifold admits an exhaustion by compact subsets. To give a bit of space between them, each compact subset is conveniently expressed as the closure of an open set. Of course, if the manifold is compact there is nothing to prove.

**Theorem 10.3.** Let $M$ be a topological manifold. Then there exists $\{G_j\}_{j \in J}$ with $J = \{1, \ldots, N\}$ finite or $J = \mathbb{Z}^{>0}$ countable such that for all $j \in J$ we have

1. $\overline{G_j}$ is compact,
2. $\overline{G_j} \subset G_{j+1}$,
3. $M = \bigcup_{j \in J} G_j$.

**Proof.** Choose a countable basis for the topology of $M$ and throw out the basis sets whose closure is not compact. Then the remaining sets $B_1, B_2, \ldots$ form a basis. (If $U \subset M$ is open and $p \in U$, choose and open set $U' \subset U$ which contains $p$ and has compact closure. Then $U'$ is a union of sets in the original basis, but each of those sets has compact closure, since $\overline{U'}$ is compact, so these sets are in $\{B_i\}_{i=1,2,\ldots}$. Repeat for all $p \in U$.)

Set $G_1 = B_1$. Suppose $G_1, \ldots, G_j$ are defined. Inductively define $G_{j+1} = B_1 \cup B_2 \cup \cdots \cup B_k$, where $k$ is the smallest positive integer such that $\overline{G_j} \subset B_1 \cup B_2 \cup \cdots \cup B_k$. \(\square\)

(10.4) Topological manifolds are paracompact. We begin with open covers and refinements.

**Definition 10.5.** Let $M$ be a topological space. Let $\{U_\alpha\}_{\alpha \in A}$ and $\{V_\beta\}_{\beta \in B}$ be sets of open subsets of $M$.

1. $\{U_\alpha\}_{\alpha \in A}$ is an open cover of $M$ if $\bigcup_{\alpha \in A} U_\alpha = M$.
2. $\{V_\beta\}_{\beta \in B}$ is a subcover of $\{U_\alpha\}_{\alpha \in A}$ if there exists an injective function $r : B \to A$ such that $V_\beta = U_{r(\beta)}$ for all $\beta \in B$.
3. A refinement of $\{U_\alpha\}_{\alpha \in A}$ is an open cover $\{V_\beta\}_{\beta \in B}$ together with a function $r : B \to A$ such that $V_\beta \subset U_{r(\beta)}$ for all $\beta \in B$.
4. The collection $\{U_\alpha\}_{\alpha \in A}$ is locally finite if for all $p \in M$ there exists an open neighborhood $W \subset M$ such that $\{\alpha \in A : W \cap U_\alpha \neq \emptyset\}$ is finite.
5. $M$ is paracompact if every open cover of $M$ has an open locally finite refinement.

**Theorem 10.6.** Let $M$ be a topological manifold. Then $M$ is paracompact. In fact, every open cover has a countable open locally finite refinement.

**Proof.** Fix an exhaustion $\{G_1, G_2, \ldots\}$ of $M$ by open sets with compact closure as in Theorem 10.3. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $M$. For each $j \geq 3$ the compact set $\overline{G_j \setminus G_{j-1}}$ is a subset of the open set $G_{j+1} \setminus \overline{G_{j-2}}$. Choose a finite subset $F_j \subset A$ such that $U_j = \{U_\alpha \cap (G_{j+1} \setminus \overline{G_{j-2}})\}_{\alpha \in F_j}$ cover the compact set $\overline{G_j \setminus G_{j-1}}$. Also, choose a finite subset $F_2 \subset A$ such that $U_2 = \{U_\alpha \cap G_3\}_{\alpha \in F_2}$ cover the compact set $\overline{G_3}$ Then $U_2 \cup \bigcup_{j \geq 3} U_j$ is the desired locally finite refinement. \(\square\)
Partitions of unity

(10.7) Bump functions. On homework you proved the following.

Lemma 10.8. Let $n$ be a positive integer and let $\mathbb{A}^n$ be standard $n$-dimensional affine space with coordinates $x^1, \ldots, x^n$. Then there exists a smooth function $\rho: \mathbb{A}^n \to \mathbb{R}$ such that

$$\rho(x^1, \ldots, x^n) = \begin{cases} 1, & |x^i| \leq 1 \text{ for all } i \in \{1, \ldots, n\}; \\ 0, & |x^i| \geq 2 \text{ for some } i \in \{1, \ldots, n\}. \end{cases}$$

Introduce the notation

$$C(r) = \{(x^1, \ldots, x^n) \in \mathbb{A}^n : |x^i| < r \text{ for all } i \in \{1, \ldots, n\}\}.$$

Then (10.9) asserts that $\rho \equiv 1$ on $\overline{C(1)}$ and $\rho \equiv 0$ on $\mathbb{A}^n \setminus C(2)$.

Definition 10.11. Let $M$ be a smooth manifold.

(i) A partition of unity $\{\rho_i\}_{i \in I}$ is a set of $C^\infty$ functions $\rho_i: M \to \mathbb{R}$ such that
(a) \( \{\text{supp } \rho_i\}_{i \in I} \) is locally finite
(b) \( \rho_i \geq 0 \)
(c) \( \sum_{i \in I} \rho_i(p) = 1 \) for all \( p \in M \)

(ii) If \( \{U_\alpha\}_{\alpha \in A} \) is an open cover of \( M \), then \( \{\rho_i\}_{i \in I} \) is subordinate to \( \{U_\alpha\}_{\alpha \in A} \) if there exists a function \( r: I \to A \) such that \( \text{supp } \rho_i \subset U_{r(i)} \) for all \( i \in I \).

(iii) If \( I = A \) and \( r = \text{id}_A \), then we say \( \{\rho_i\}_{i \in I} \) is subordinate with the same index set.

**Theorem 10.12.** Let \( M \) be a smooth manifold equipped with an open cover \( \{U_\alpha\}_{\alpha \in A} \).

1. There exists a countable partition of unity \( \{\rho_i\}_{i \in I} \) subordinate to \( \{U_\alpha\}_{\alpha \in A} \) such that \( \text{supp } \rho_i \) is compact for all \( i \in I \).
2. There exists a partition of unity \( \{\varphi_\alpha\}_{\alpha \in A} \) subordinate to \( \{U_\alpha\}_{\alpha \in A} \) with the same index set such that at most countably many \( \varphi_\alpha \) are not identically zero.

![Figure 19. Construction of a partition of unity](image)

**Proof.** Fix an exhaustion \( \{G_j\}_{j \in J} \) of \( M \) by open sets with compact closure as in Theorem 10.3. For \( p \in M \) let \( j_p \) be the largest positive integer such that \( p \in M \setminus \overline{G_{j_p}} \). Choose \( \alpha_p \in A \) such that \( p \in U_{\alpha_p} \), and then choose a standard chart \( (V_p, x_p) \) such that: (i) \( x_p(p) = 0 \), (ii) \( V_p \subset U_{\alpha_p} \cap (G_{j_p+2} \setminus \overline{G_{j_p}}) \), and (iii) \( x_p(V_p) \) contains the closed cube \( C(2) \). Transport the bump function (10.9) via \( x_p \) to a smooth function \( \psi_p: M \to \mathbb{R} \), extending by zero on the complement of \( V_p \). Note that \( \psi_p \) has compact support. Define \( W_p = x_p^{-1}(C(1)) \); then \( \psi_p \equiv 1 \) on the open set \( W_p \). For each \( j \in J \) choose finitely many \( p \) with \( j_p = j \) such that the corresponding open sets \( W_p \) cover the compact set \( \overline{G_j \setminus G_{j-1}} \).

Enumerate the functions obtained (for all \( j \)) as \( \psi_1, \psi_2, \ldots \), indexed by a set \( I \) which is finite or countable. By construction \( \{\text{supp } \psi_i\}_{i \in I} \) is a locally finite cover of \( M \) by compact sets. Define

\[
(10.13) \quad \rho_i = \frac{\psi_i}{\sum_{i \in I} \psi_i}, \quad i \in I.
\]

The denominator is a finite sum in a neighborhood of each \( p \in M \), so is a smooth function, and it is positive everywhere. Then \( \text{supp } \rho_i = \text{supp } \psi_i \) is a compact subset of \( U_{\alpha_p} \) if the function \( \psi_i \) corresponds to the point \( p \). It follows that \( \{\rho_i\}_{i \in I} \) is the desired partition of unity in (1).

Let \( r: I \to A \) be the refinement function determined by \( \text{supp } \rho_i \subset U_{r(i)} \) (where \( r(i) = \alpha_p \) a few lines up). Define

\[
(10.14) \quad \varphi_\alpha = \sum_{i \in r^{-1}(\alpha)} \rho_i, \quad \alpha \in A.
\]
Then \( \{ \varphi_\alpha \}_{\alpha \in A} \) is a partition of unity which satisfies the conditions in (2).

**Example 10.15.** Consider \( M = \mathbb{R} \) with open cover \( \{M\} \) consisting of a single set. The partition of unity in Theorem 10.12(2) is the single constant function with value one. By contrast, the construction of Theorem 10.12(1) gives a partition of unity consisting of countably many functions with compact support.

**Applications**

(10.16) *Bump functions on a manifold.* The following construction is often useful.

**Corollary 10.17.** Let \( M \) be a smooth manifold with subsets \( C \subseteq U \subseteq M \) such that \( C \) is closed and \( U \) is open. Then there exists a smooth function \( f : M \to \mathbb{R} \) such that

(i) \( 0 \leq f \leq 1 \),

(ii) \( f|_C = 1 \),

(iii) \( \text{supp } f \subseteq U \).

**Proof.** Choose a partition of unity \( \{ \varphi_U, \varphi_{M \setminus C} \} \) subordinate to the open cover \( \{U, M \setminus C\} \) of \( M \). Then \( f = \varphi_U \) is the desired function. \( \square \)

(10.18) *Proper functions.* Recall that a continuous function between topological spaces is proper if the inverse image of every compact set is compact. On a compact space, a constant function is proper. The following result shows that noncompact manifolds also admit proper functions.

**Corollary 10.19.** Let \( M \) be a smooth manifold. Then there exists a proper function \( f : M \to \mathbb{R} \).

**Proof.** Let \( \{U_\alpha\}_{\alpha \in A} \) be the set of open subsets of \( M \) with compact closure; it is an open cover of \( M \). Choose a partition of unity \( \{ \rho_i \}_{i \in I} \) subordinate to this cover as in Theorem 10.12(1): each \( \text{supp } \rho_i \) is compact and \( I \) is at most countable. Choose \( I = \{1, \ldots, N\} \) for some \( N \in \mathbb{Z}^>0 \) or \( I = \mathbb{Z}^>0 \). Define

(10.20)
\[
    f = \sum_{i \in I} i \rho_i.
\]

Then \( f \) is a positive function. If \( p \in M \) satisfies \( f(p) \leq j \), then since \( \sum_{i \in I} \rho_i(p) = 1 \), it follows from (10.20) that \( \rho_i(p) \neq 0 \) for some \( i \leq j \). Hence

(10.21)
\[
    f^{-1}([-j, j]) \subseteq \bigcup_{i=1}^{j} \text{supp } \rho_i.
\]

Since every compact subset of \( \mathbb{R} \) is contained in some interval \([−j, j]\), it follows that \( f \) is proper. \( \square \)
Lecture 11: Whitney embedding theorem; transversality

Most of this lecture concerns the Whitney embedding theorem, which states that every abstract smooth manifold can be realized as a submanifold of affine space. More precisely, it tells that if the manifold has dimension \( n \), then it embeds in an affine space of dimension \( 2n \). That result, the “hard” Whitney theorem, is not proved here. Rather, we prove the “easy” Whitney theorem that an \( n \)-manifold embeds in an affine space of dimension \( 2^n - 1 \). The proof proceeds in stages. First we demonstrate that a compact manifold embeds in some affine space. The proof uses \( C^\infty \) cutoff functions. Then we prove that an \( n \)-dimensional submanifold of affine space, compact or not, embeds into \( \mathbb{A}^{2n+1} \). Finally, we use these results to prove the theorem for general (noncompact) manifolds.

In the last part of the lecture we introduce transversality, a central concept for the next part of the course.

Embeddings of compact manifolds

(11.1) Connectivity. Observe that a function \( M \to \mathbb{A}^N \) is an ordered \( N \)-tuple of real-valued functions on \( M \). If it defines an embedding, then by adjoining more real-valued functions to the \( N \)-tuple we obtain an embedding into a higher dimensional affine space. Now suppose we prove that every compact connected manifold of a given dimension \( n \) embeds in some affine space. If \( M \) is an arbitrary compact \( n \)-manifold, then it has a finite set of components \( \{ M_1, \ldots, M_k \} \). Construct embeddings \( M_i \to \mathbb{A}^{N_i} \), and extend by constant functions to obtain a map \( f: M \to \mathbb{A}^N \) for \( N = \max_i N_i \). Then \( f \) is an immersion but may not be injective. Adjoin one more real-valued function \( \rho \) to \( f \), namely the locally constant function with value \( j \) on \( M_j \). Then \( (f, \rho): M \to \mathbb{A}^{N+1} \) is an injective immersion, hence an embedding since \( M \) is compact.

Remark 11.2. The proof of Theorem 11.11 below applies directly to compact manifolds which need not be connected.

(11.3) Examples. We report on the embedding question in specific cases.

Example 11.4 (dimension one). A compact connected 1-manifold is diffeomorphic to \( S^1 \), a fact we will prove in a subsequent lecture. There is no embedding \( S^1 \to \mathbb{A}^1 \): a smooth real-valued function on \( S^1 \) achieves a maximum, and the map fails to be an immersion where the maximum is achieved. Of course, \( S^1 \) embeds in \( \mathbb{A}^2 \).

![Figure 20. A family of compact connected 2-manifolds](image-url)
Example 11.5 (dimension two). The classification of surfaces, which we do not prove in this course, states that every compact connected 2-manifold is diffeomorphic to a manifold in one of two families. The first is depicted in Figure 20, where for each $g \in \mathbb{Z}^{\geq 0}$ there is a surface $\Sigma_g$, the “2-sphere with $g$ holes”. There exist embeddings $\Sigma_g \hookrightarrow \mathbb{A}^3$, but not embeddings into an affine space of lower dimension. There is a second infinite family whose first member is $\mathbb{RP}^2$, the real projective plane. (The Klein bottle is also in this family.) These manifolds do not admit embeddings into $\mathbb{A}^3$, though again we will not prove this assertion. On the other hand, the function

$$f: \mathbb{R}P^2 \to \mathbb{A}^4$$

$$(11.6)$$

$$(x, y, z) \mapsto \frac{1}{x^2 + y^2 + z^2} (x^2, xy, yz, xz)$$

is an embedding. (A point of $\mathbb{R}P^2$ is an equivalence class of ordered triples of real numbers, not all zero, under the equivalence relation $[x, y, z] = [\lambda x, \lambda y, \lambda z]$ for $\lambda \in \mathbb{R}^\times = \mathbb{R}^{\neq 0}$.) Observe too that the function

$$f: \mathbb{R}P^2 \to \mathbb{A}^3$$

$$(11.7)$$

$$(x, y, z) \mapsto \frac{1}{x^2 + y^2 + z^2} (xy, yz, xz)$$

is an immersion (which is not injective).

Given a manifold $M$ we can ask for the minimal $N$ such that $M$ embeds (or immerses) into $\mathbb{A}^N$. The Whitney theorem gives an upper bound to $N$. The actual minimum is difficult to determine.

Example 11.8 (real projective spaces). The question of a minimal $N$ such that an embedding $\mathbb{RP}^n \hookrightarrow \mathbb{A}^N$ exists has been intensively studied. Here are the results in low dimensions:

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(11.9)

(See https://www.lehigh.edu/~dmd1/immtable for an extensive table.) As you see, the hard Whitney upper bound $N = 2n$ is beat in several cases.

(11.10) Injective immersions and embeddings. Recall (Definition 6.19) that an embedding $f: M \to N$ of manifolds is an injective immersion which is a homeomorphism onto its image $f(M) \subset N$ (with the induced topology). Suppose that $f$ is an injective immersion. If $M$ is compact, and $C \subset M$ is a closed subset, then $C$ is compact, so too is $f(C)$, and so $f(C) \subset f(M)$ is closed. (A compact subset of a Hausdorff topological space is closed.) Therefore, to produce an embedding of a compact manifold it suffices to produce an injective immersion.
Theorem 11.11. Let $M$ be a compact smooth manifold. Then for some $N \in \mathbb{Z}^{>0}$ there exists an embedding $f : M \to \mathbb{A}^N$.

Proof. Assume without loss of generality that $M$ has constant dimension $n$. (See (11.1).) For $r \in \mathbb{R}^{>0}$, let $B(r) \subset \mathbb{A}^n$ denote the open ball of radius $r$ about 0. Apply Corollary 10.17 to produce a $C^\infty$ function $\chi : \mathbb{A}^n \to \mathbb{R}$ such that

\begin{align}
0 \leq \chi \leq 1 & \quad \text{on } \mathbb{A}^n \\
\chi = 1 & \quad \text{on } B(1) \\
\chi = 0 & \quad \text{on } \mathbb{A}^n \setminus B(2)
\end{align}

Cover $M$ by a finite set $\{(U_\alpha, x_\alpha)\}_{\alpha \in A}$ of standard charts (in other words, $x_\alpha : U_\alpha \to \mathbb{A}^n$) such that $B(2) \subset x_\alpha(U_\alpha) \subset \mathbb{A}^n$ for all $\alpha \in A$ and $\{x_\alpha^{-1}(B(1))\}_{\alpha \in A}$ is an open cover of $M$. (Composition of an arbitrary standard chart $(U, x)$ with a translation and homothety yields a chart $(U, x')$ which satisfies $B(2) \subset x'(U) \subset \mathbb{A}^n$. A compact manifold can be covered with finitely many such charts.) For each $\alpha \in A$, $i \in \{1, \ldots, n\}$, use the cutoff function $\chi$ to define global functions $\tilde{x}_\alpha^i, \rho_\alpha : M \to \mathbb{R}$:

\begin{align}
\tilde{x}_\alpha^i = \begin{cases} 
(\chi \circ x_\alpha)^{-1} x_\alpha^i, & \text{on } U_\alpha; \\
0, & \text{on } M \setminus x_\alpha^{-1}(B(2)). 
\end{cases}
\end{align}

\begin{align}
\rho_\alpha = \begin{cases} 
\chi \circ x_\alpha, & \text{on } U_\alpha; \\
0, & \text{on } M \setminus x_\alpha^{-1}(B(2)). 
\end{cases}
\end{align}

These are smooth functions on $M$, each constructed as a pair of smooth functions defined on open subsets of $M$ such that the functions agree on the intersection. Assemble these into a single function $f : M \to \mathbb{A}^{(n+1)\#A}$:

\begin{align}
f = \{ (\rho_\alpha, \tilde{x}_\alpha^1, \ldots, \tilde{x}_\alpha^n) \}_{\alpha \in A}.
\end{align}

We claim that $f$ is an injective immersion, from which it follows that $f$ is an embedding, since $M$ is compact. For $\alpha \in A$, set $B_\alpha = \rho_\alpha^{-1}(1)$. Since $x_\alpha^{-1}(B(1)) \subset \rho_\alpha^{-1}(1)$, we conclude that $\{B_\alpha\}_{\alpha \in A}$ is an open cover of $M$. Now if $p, q \in B_\alpha$, then $d\tilde{x}_\alpha^1(p), \ldots, d\tilde{x}_\alpha^n(p)$ are linearly independent. Hence $f$ is an immersion. If $p, q \in B_\alpha$, then $\tilde{x}_\alpha^1, \ldots, \tilde{x}_\alpha^n$ separate $p$ and $q$. If $p \in B_\alpha$ and $a \notin B_\alpha$, then $\rho_\alpha(p) = 1$ and $\rho_\alpha(q) \neq 1$. Hence $f$ is injective. \hfill \square

Cutting down the dimension

In this subsection, $M$ need not be compact. We prove that a submanifold $M$ of an affine space $A$ can be projected to a submanifold of a quotient affine space of smaller dimension as long as $2 \dim M + 1 < \dim A$. 

(11.15) **Quotient affine spaces.** Let \( A \) be an affine space over a vector space \( V \), and suppose \( W \subset V \) is a linear subspace. Then \( W \) acts freely on \( A \) by translation, and as usual we denote the quotient—the set of orbits of the action—as \( A/W \). The translation action \( V \times A \to A \) descends to an action \( V/W \times A/W \to A/W \) of \( V \) on the quotient. The subspace \( W \) acts freely, and so we obtain an action \( V\hat{\times}A\hat{\times}W \to A\hat{\times}W \). It is easy to verify that this action is free, so \( A/W \) has the structure of an affine space over \( V/W \).

**Theorem 11.16.** Let \( M \) be an \( n \)-dimensional manifold which is embedded into a finite dimensional affine space. Then

1. \( M \) admits an immersion into \( \mathbb{A}^{2n} \), and
2. \( M \) admits an embedding into \( \mathbb{A}^{2n+1} \).

\[ \text{Figure 21. Reducing the dimension of an embedding} \]

**Proof.** Let \( f : M \to A \) be an embedding into an affine space \( A \) over a finite dimensional real vector space \( V \). For \( \xi \in V^{\neq 0} \) let

\[ (11.17) \quad \pi^{(\xi)} : A \to A/\mathbb{R} \cdot \xi \]

be the affine projection. Its differential is constant on \( A \); it is the linear projection \( V \to V/\mathbb{R} \cdot \xi \) with kernel \( \mathbb{R} \cdot \xi \). Hence the composition

\[ (11.18) \quad \pi^{(\xi)} \circ f : M \to A/\mathbb{R} \cdot \xi \]

is an immersion—its differential \( d(\pi^{(\xi)} \circ f) = d\pi^{(\xi)} \circ df \) is injective—if and only if \( \mathbb{R} \cdot \xi \) is not in the image of the composition

\[ (11.19) \quad F_1 : TM \xrightarrow{df} A \times V \xrightarrow{pr_2} V. \]

By Sard’s theorem the map \( F_1 \) has a dense set of regular values, and if \( 2n < \dim V \), then any regular value is not in the image. Choose \( \xi \in V^{\neq 0} \) not in the image, so that \((11.18)\) is an immersion. Repeat until we arrive at an embedding into an affine space of dimension \( 2n \), thus proving (1).
For the composite (11.18) to be injective, we must choose \( \xi \) so that for all \( p_0, p_1 \in M \) the displacement vector \( f(p_1) - f(p_0) \) is not in the linear span of \( \xi \). Define

\[
F_2 : M \times M \times \mathbb{R} \to V \\
p_0, p_1, t \mapsto t[f(p_1) - f(p_0)]
\]

(11.20)

As long as \( 2n + 1 < \text{dim} \ V \) we can choose \( \xi \in V \neq 0 \) not in the image of \( F_2 \), and by Corollary 8.3 we can find a nonzero vector \( \xi \) which is not in the image of either \( F_1 \) or \( F_2 \). In that case, (11.18) is an injective immersion.

If \( M \) is compact, then (2) follows. If not, the preceding gives an injective immersion \( f : M \to \mathbb{A}^{2n+1} \). Compose \( f \) with a translation and homothety centered at \( 0 \in \mathbb{A}^{2n+1} \) so that \( f(M) \subset B(1) \). Choose a proper function \( \rho : M \to \mathbb{R} \) (see Corollary 10.19). Set

\[
g = (f, \rho) : M \to \mathbb{A}^{2n+2}.
\]

(11.21)

The argument above justifies the choice of \( \xi \in \mathbb{R}^{2n+2} \) so that \( \pi(\xi) \circ g : M \to \mathbb{A}^{2n+1} \) is an injective immersion and \( \xi \in \mathbb{R} \cdot (0, \ldots, 0, 1) \). We claim that for all \( r > 0 \) there exists \( s > 0 \) such that

\[
(\pi(\xi) \circ g)^{-1}(B(r)) \subset \rho^{-1}(B(s)).
\]

(11.22)

Since \( \rho \) is proper, \( \rho^{-1}(B(s)) \) is compact, and hence the closed subset in (11.22) is also compact. It follows that \( \pi(\xi) \circ g \) is a proper injective immersion, hence an embedding (as you proved on a homework assignment), which is the statement of (2). It remains to prove the claim.

If the claim is false, choose a sequence \( \{p_i\} \subset M \) such that \( (\pi(\xi) \circ g)(p_i) \in B(r) \) for all \( i \) and \( \rho(p_i) \to \infty \). Consider

\[
\eta_i = \frac{1}{\rho(p_i)}[g(p_i) - \pi(\xi)g(p_i)] \in \mathbb{R} \cdot \xi.
\]

Identifying \( \mathbb{A}^{2n+k} \) with \( \mathbb{R}^{2n+k} \), \( k = 1, 2 \), we write

\[
\frac{1}{\rho(p_i)} g(p_i) = \left( \frac{1}{\rho(p_i)} f(p_i); 1 \right) \to (0; 1)
\]

(11.24)
as \( i \to \infty \). Also, \( (1/\rho(p_i))\pi(\xi)g(p_i) \to 0 \) as \( i \to \infty \). Hence \( \eta_i \to (0; 1) \), and since \( \mathbb{R} \cdot \xi \subset \mathbb{A}^{2n+2} \) is closed we must have \( (0; 1) \in \mathbb{R} \cdot \xi \). But that contradicts our choice of \( \xi \).

\[\square\]

**Noncompact manifolds**

We complete the proof of the easy Whitney embedding theorem with the following.

**Theorem 11.25.** Let \( M \) be a smooth manifold. Then there exists an embedding of \( M \) into a finite dimensional real affine space.
Once this is proved, apply Theorem 11.16(2).

**Lemma 11.26.** Let $M$ be a smooth manifold which admits a finite atlas. Then there exists an embedding of $M$ into a finite dimensional real affine space.

**Proof.** Follow the proof of Theorem 11.11 to construct an injective immersion (11.14). Now adjoin a proper function, as in (11.21) to obtain a proper injective immersion, i.e., an embedding. □

**Proof of Theorem 11.25.** Construct an exhaustion of $M$ by a sequence $\{G_j\}_{j \in J}$ of nested open sets with compact closure, as in Theorem 10.3. Define

\[
M_1 = G_2 \\
M_2 = G_3 \\
M_j = G_{j+1} \setminus \overline{G_{j-2}}, \quad j \geq 3;
\]

then each $M_j$ is an open submanifold of $M$. Note $M_j \cap M_{j+3} = \emptyset$ for all $j \in J$. Also, $M_j$ admits a finite atlas. Namely, for each $p \in M_j = \overline{G_{j+1}} \setminus G_{j-2}$ choose a chart on an open neighborhood $U_p \subset M$ of $p$. For a finite subset $F_j \subset M_j$ the collection $\{U_p\}_{p \in F_j}$ covers the compact set $\overline{M_j}$. Thus we obtain a finite atlas of $M_j$ of charts with domain $U_p \cap M_j, p \in F_j$. Now apply Theorem 11.25 and Theorem 11.16 to construct an embedding $f_j: M_j \to \mathbb{A}^{2n+1}$ for each $j \in J$. Choose a partition of unity $\{\rho_j, \chi_j\}$ on $M$ subordinate to the open cover $\{M_j, M \setminus (\overline{G_j \setminus G_{j-1}})\}$. Then $M \setminus \text{supp} \chi_j$ is an open set which contains $\overline{G_j \setminus G_{j-1}}$, and $\rho_j \equiv 1$ on $M \setminus \text{supp} \chi_j$. Set $f_j = \rho_j f_j: M \to \mathbb{A}^{2n+1}$. Then $\tilde{f}_j = f_j$ on $M \setminus \text{supp} \chi_j$, hence $\tilde{f}_j$ restricts to an injective map on $\overline{G_j \setminus G_{j-1}}$, and $d(\tilde{f}_j)_p$ is injective for all $p \in \overline{G_j \setminus G_{j-1}}$.

**Figure 22.** The compact subset $\overline{G_j \setminus G_{j-1}}$ of $M_j = G_{j+1} \setminus \overline{G_{j-2}}$

Define

\[
M^{(0)} = \bigsqcup_{j \equiv 0 \pmod{3}} M_j \\
M^{(1)} = \bigsqcup_{j \equiv 1 \pmod{3}} M_j \\
M^{(2)} = \bigsqcup_{j \equiv 2 \pmod{3}} M_j
\]
For $i \in \{0, 1, 2\}$, let $f^{(i)} : M^{(i)} \to A^{2n+2}$ be the function whose first $2n + 1$ coordinates are given by the disjoint union of the $\tilde{f}_j$ and whose last coordinate is $j \rho_j$ on $M_j$. Then $f^{(i)}$ restricts to an injective map on each $\overline{G_j \setminus G_{j-1}}$ and its differential is injective at each point of $\overline{G_j \setminus G_{j-1}}$. Since $\rho_j$ has compact support in $M_j$, the function $\tilde{f}_j$ has value 0 outside a compact subset of $M_j$, so extends to a global function $M \to A^{2n+2}$. Hence $f^{(i)}$ also extends to a global function $\tilde{f}^{(i)} : M \to A^{2n+2}$. Let $\rho : M \to \mathbb{R}$ be a proper function (Corollary 10.19). Finally, set

$$f = (f^{(0)}, f^{(1)}, f^{(2)}, \rho) : M \to A^{6n+7}.$$ 

Then $f$ is a proper injective immersion, hence is an embedding. □

Transversality

(11.30) Transversality for linear maps. Let $T : V \to W$ be a linear map between vector spaces and $U \subset W$ a subspace. Then we say $T$ is transverse to $U$, written $T \not\!\perp U$, if and only if the subspaces $T(V)$ and $U$ span $W$:

$$W = T(V) + U.$$ 

This is equivalent to the condition that the composition

$$V \xrightarrow{T} W \xrightarrow{} W/U$$

be surjective, where the second map is projection onto the quotient.

(11.33) Nonlinear maps. Transversality is defined via linearization; it is a local condition.

Definition 11.34. Let $X, Y$ be smooth manifolds, $Z \subset Y$ a submanifold, $f : X \to Y$ a smooth map, and $p \in X$ such that $f(p) \in Z$. Then $f$ is transverse to $Z$ at $p$, written $f \not\!\perp_p Z$ if

$$T_{f(p)}Y = df_p(T_pX) + T_{f(p)}Z.$$ 

We say $f$ is transverse to $Z$, written $f \not\!\perp Z$, if $f \not\!\perp_p Z$ for all $p \in X$ such that $f(p) \in Z$.

The nonlinear transversality condition (11.35) is the linear transversality condition $df_p \not\!\perp T_{f(p)}Z$.

Remark 11.36.

1. For $q \in Y$ we have $f \not\!\perp \{q\}$ if and only if $q$ is a regular value of $f$.
2. Any map $f$ satisfies $f \not\!\perp Y$.
3. If $\dim X + \dim Z < \dim Y$, then $f \not\!\perp Z$ if and only if $f(X) \cap Z = \emptyset$.
4. If $Z_1, Z_2 \subset Y$ are submanifolds, and $f_1 : Z_1 \to Y$ is the inclusion map, then we say $Z_1 \not\!\perp Z_2$ if and only if $f_1 \not\!\perp Z_2$. This is a symmetric relation: $f_1 \not\!\perp Z_2$ if and only if $f_2 \not\!\perp Z_1$. 

Theorem 11.38. Let \( X, Y \) be smooth manifolds, \( Z \subset Y \) a submanifold, and \( f : X \to Y \) a smooth map. Assume \( f \not\sim Z \). Then \( W := f^{-1}(Z) \subset X \) is a submanifold. Furthermore, if \( p \in X \) satisfies \( f(p) \in Z \), then

1. \( T_pW = df_p^{-1}(T_{f(p)}Z) \).
2. \( df_p \) induces an isomorphism of normal spaces \( \nu_p(W \subset X) \to \nu_{f(p)}(Z \subset Y) \).
3. \( \text{codim}_p(W \subset X) = \text{codim}_{f(p)}(Z \subset Y) \).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure23.pdf}
\caption{Local reduction: transversality $\rightarrow$ regular value}
\end{figure}

Proof. Each statement is local. Choose a submanifold chart \((W, y)\) on \( Y \) with \( f(p) \in W \), and suppose \( y : W \to A \), where \( A \) is an affine space over a vector space \( V \). Furthermore, let \( A' \subset A \) be an affine subspace so that \( y^{-1}(A') = W \cap Z \). Suppose \( V' \subset V \) is the subspace of translations which preserve \( A' \). Let \( \pi : A \to A/V' \) be projection onto the quotient affine space, and let \( q \in A/V' \) be the image of \( A' \subset A \) under \( \pi \). Since \( f \not\sim_p Z \) it follows that \( d(\pi \circ y \circ f)_p \) is surjective. Since surjectivity is an open condition, choose an open neighborhood \( U \subset X \) of \( p \) so that \( \pi \circ y \circ f \big|_U : U \to A/V' \) is a submersion; in particular, \( q \in A/V' \) is a regular value and \( (\pi \circ y \circ f \big|_U)^{-1}(q) = W \cap U \). Now apply Theorem 7.10. \qed

Lecture 12: Stability under deformations; manifolds with boundary

Stability

(12.1) Introduction. Recall that we produced a local normal form for a smooth function whose differential has maximal rank (Theorem 6.16). We also proved in Lemma 6.3 that maximal rank is an open condition for linear maps between fixed vector spaces. Now we consider families of maps between manifolds, and we can allow the manifolds to vary as well. We prove that not only are these local conditions—and the local condition of transversality to a submanifold—open conditions, but
so too are related global conditions. This openness, or stability under deformation, is an important feature when constructing topological invariants, which we do in the next few lectures.

(12.2) *Smooth families of manifolds and maps.* A parametrized family of geometric objects lives over a parameter space, which in smooth geometry is a smooth manifold $S$. The nicest smooth families of manifolds are fiber bundles: they are locally trivial (Definition 8.24). The tangent spaces then form a locally trivial family of vector spaces. Thus suppose $\pi_X : X \to S$ is a fiber bundle. Since $\pi_X$ has surjective differential, the kernels of the differential form a vector bundle, hence over the total space $X$ we obtain a short exact sequence of vector bundles

$$0 \longrightarrow \ker d\pi_X \longrightarrow TX \longrightarrow \pi_X^*TS \longrightarrow 0$$

The kernel of the differential—which consists of “vertical tangent vectors”—is the tangent bundle along the fibers, denoted

$$T(X/S) = \ker d\pi_X.$$ 

![Figure 24. The tangent bundle along the fibers](image)

A smooth family of maps, then, is a smooth map between the total spaces of two fiber bundles which commutes with projection to the base. So it is a commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{F} & Y \\
\downarrow{\pi_X} & & \downarrow{\pi_Y} \\
S & \xrightarrow{\pi_S} & S
\end{array}$$

in which $\pi_X$ and $\pi_Y$ are fiber bundles and $F$ is a smooth map. For each $s \in S$ the restriction

$$F_s : X_s \to Y_s$$

of $F$ to $X_s = \pi_X^{-1}(s)$ is the map at the parameter value $s$; see Figure 25.

*Remark* 12.7. If $\pi_X, \pi_Y$ are product fiber bundles with fibers $X, Y$, then (12.5) is replaced with a smooth map $S \times X \to Y$. 


The differential as a family of linear maps. A special case of (12.5) is a family of linear maps between locally trivial families of vector spaces, i.e., between vector bundles. For example, suppose $f: X \to Y$ is a smooth map between smooth manifolds. At each $x \in X$ the differential is a linear map $df_x: T_x X \to T_{f(x)} Y$, and these linear maps fit together into a map of vector bundles over $X$:

$$df: TX \to f^* TY.$$  

In the case of a family (12.5) of smooth maps over a base $S$, the differential $dF: TX \to F^* TY$ restricts to the vertical tangent bundles to give a family of vector bundle maps $TX_s \to TY_s$ parametrized by $S$, i.e., a family of linear maps $T(X/S)_x \to T(Y/S)_{f(x)}$ parametrized by $X$:

$$dF_{\text{vert}}: T(X/S) \to F^* T(Y/S).$$  

Maximal rank condition with variable vector spaces. Recall from Lemma 6.3 that maximal rank maps between fixed vector spaces are an open subset of all linear maps.

**Lemma 12.12.** Let

$$E' \xrightarrow{T} E \xleftarrow{\pi'} M \xrightarrow{\pi} E$$

be a family of linear maps between vector bundles. Then

$$\{m \in M : T_m: E'_m \to E_m \text{ has maximal rank}\}$$

is an open subset of $M$.  

**Figure 25.** A smooth family of maps
Proof. Let \( m \in M \) be a point at which \( T_m \) has maximal rank. Choose an open neighborhood \( U \subset M \) of \( m \) together with local trivializations of \( \pi' \) and \( \pi \) over \( U \). We obtain a square of fiberwise linear maps over \( U \): \[ \begin{matrix} U \times E'_m & \to & U \times E_m \\ \downarrow \varphi & & \downarrow \psi \\ E'_|U & \to & E|U \end{matrix} \] (12.15)

Then \( \psi^{-1} \circ T \circ \varphi \) defines a smooth map \( i: U \to \operatorname{Hom}(E'_m, E_m) \), and \( T_{m'}, m' \in U \) has maximal rank if and only if \( i(m') \) does. Since maximal rank maps in \( \operatorname{Hom}(E'_m, E_m) \) are an open set (Lemma 6.3), it follows that (12.14) contains a neighborhood of \( m \). \( \square \)

**Stability theorem.** We prove the stability theorem for maps of fiber bundles; the special case with fixed fibers (as in Remark 12.7) occurs often.

**Theorem 12.17.** Let \( S \) be a smooth manifold; \( \pi_X: X \to S \) and \( \pi_Y: Y \to S \) smooth fiber bundles; \( \pi_Z: Z \to S \) a smooth sub-fiber bundle of \( \pi_Y \); and \( F: X \to Y \) a smooth map of fiber bundles, i.e., (12.5) commutes. Assume \( \pi_X \) is proper. Suppose for some \( s_0 \in S \) the map \( F_{s_0}: X_{s_0} \to Y_{s_0} \) is one of

(i) a local diffeomorphism  
(ii) an immersion  
(iii) a submersion  
(iv) transverse to \( Z_{s_0} \)  
(v) an injective immersion  
(vi) an embedding  
(vii) a diffeomorphism

Then there exists an open neighborhood \( W \subset S \) of \( s_0 \) such that \( F_s \) satisfies the same condition for all \( s \in W \).

A few explanations are in order. First, in case (iv) we mean that \( F_s \mid Z \) is a submanifold for \( s \in W \). Also, the statement that \( \pi_Z \) is a sub-fiber bundle of \( \pi_Y \) means: \( Z \subset Y \) is a submanifold, \( \pi_Z \) is the restriction of \( \pi_Y \) to \( Z \), and there is a cover of \( U \) by simultaneous local trivializations of \( \pi_Z \) and \( \pi_Y \) to a constant submanifold:

\[ \begin{matrix} U \times (\pi_Z^{-1}(p) \subset \pi_Y^{-1}(p)) & \to & (\pi_Z^{-1}(U) \subset \pi_Y^{-1}(U)) \\ \downarrow \varphi & & \downarrow \pi_Y \\ U & \to & \pi_Y \\ \downarrow \text{pr}_1 & & \end{matrix} \] (12.18)

Finally, the properness assumption implies in particular that each \( X_s \) is a compact manifold. That compactness implies openness\(^8\) (here in parameter space) is a general principle by the first argument what follows.\(^9\)

\(^8\)Recall the proof that the complement of a compact subset of a Hausdorff space—e.g. a metric space—is open.  
\(^9\)The first part of the proof is a bit more complicated in the case of a fiber bundle than for fixed manifolds, as in Remark 12.7. In that case, we are interested in the partial differential of \( F \) in the \( X \)-direction, which at a fixed
Proof. We prove (i)–(iii) simultaneously. Choose simultaneous local trivializations of \( \pi_X \) and \( \pi_Y \) about \( s_0 \). For each \( x \in \mathcal{X}_{s_0} \) the vertical differential \( dF^\text{vert}: T_x(\mathcal{X}/S) \to T_{F(x)}(Y/S) \) has maximal rank. Since maximal rank is an open condition, choose a product open set \( W_x \times U_x \), relative to the local trivialization of \( \pi_X \), so that \( dF^\text{vert} \) has maximal rank in \( W_x \times U_x \). By compactness choose a finite subset \( F \subset \mathcal{X}_{s_0} \) so that \( \bigcup_{x \in F} U_x = \mathcal{X}_{s_0} \). Then \( W = \bigcap_{x \in F} W_x \subset S \) is the desired open set.

As in the proof of Theorem 11.38, we can reduce (iv) to (iii).

For (v), assume \( W \subset S \) is a neighborhood of \( s_0 \) such that \( F_s \) is an immersion for \( s \in W \). We claim \( F_s \) is injective. If not, choose a sequence \( \{s_n\} \subset S \) such that \( s_n \to s_0 \) as \( n \to \infty \), and choose sequences \( \{x_n\}, \{x'_n\} \subset \mathcal{X} \) such that \( \pi_X(x_n) = \pi_X(x'_n) = s_n \) and \( F(x_n) = F(x'_n) \) for all \( n \). Since the fibers of \( \pi_X \) are compact, we can and do choose convergent subsequences \( \{x_{n_k}\} \subset \{x_n\} \) and \( \{x'_{n_k}\} \subset \{x'_n\} \) (defined by the same function \( n: \mathbb{Z}^\geq 0 \to \mathbb{Z}^\geq 0, k \mapsto n_k \) for all \( n \). Since \( F_{s_0} \) is injective, \( x_0 = x'_0 \). We claim that \( dF_{x_0}: T_{x_0} \mathcal{X} \to T_{F(x_0)}Y \) is injective. Namely, relative to the short exact sequence (12.3) we have\(^{10}\)

\[
\begin{array}{cccccc}
0 & \longrightarrow & T_{x_0}(\mathcal{X}/S) & \longrightarrow & T_{x_0}\mathcal{X} & \longrightarrow & T_{s_0}S & \longrightarrow & 0 \\
& & \downarrow{dF^{\text{vert}}_{x_0}} & & \downarrow{dF_{x_0}} & & \downarrow{\text{id}} & & \\
0 & \longrightarrow & T_{F(x_0)}(Y/S) & \longrightarrow & T_{F(x_0)}Y & \longrightarrow & T_{s_0}S & \longrightarrow & 0
\end{array}
\]

(12.19)

The vertical map on the subspaces is injective, since \( F_{s_0} \) is an immersion. It follows from (12.19) that \( dF_{x_0} \) is also injective. Now the local normal form Theorem 6.16 shows that \( F \) is injective in a neighborhood of \( x_0 \), which contradicts the existence of the sequences \( \{x_{n_k}\}, \{x'_{n_k}\} \).

Part (vi) follows from (v) since a proper injective immersion is an embedding.

For (vii), choose \( W \) connected so that \( F_s \) is an injective local diffeomorphism for all \( s \in W \). (The existence of \( W \) follows from (i) and (v).) Shrink \( W \) if necessary to choose simultaneous local

---

\(^{10}\)If we choose a splitting, then this is a triangular decomposition of a linear transformation. A diagram chase shows that relative to such a decomposition, a linear transformation is maximal rank if it is maximal rank on both the sub and the quotient.
trivializations of $\pi_X, \pi_Y$, so we may assume constant fibers $X, Y$. Fix a component $Y_0 \subset Y$ and choose the corresponding component $X_0 \subset X$ with $F_{s_0}(X_0) = Y_0$. Now any local diffeomorphism is an open map, so $F_s(X_0) \subset Y$ is open for all $s \in W$. Furthermore, $X_0$ is compact so $F_s(X_0) \subset Y$ is closed. Given $s \in S$, choose a path $t \mapsto s_t$ from $s_0$ to $s$ in $W$. Then for any $x \in X_0$, that path $t \mapsto F_{s_t}(x)$ connects $F_{s_0}(x) \in Y_0$ to $F_s(x) \in Y$. It follows that $F_s(X_0) \subset Y_0$. Since $Y_0$ is a component, and any open and closed subset is a union of components, we conclude $F_s(X_0) = Y_0$. Therefore, $F_s$ is surjective.\[ □ \]

Remark 12.20. For fixed manifolds $X, Y$ we can form the set $C^\omega(X, Y)$ of smooth maps $X \to Y$. Just as there are different topologies on the set of continuous maps between topological spaces, so too are there different (Whitney) topologies on the space of $C^\omega$ maps. With the correct topology, Theorem 12.17 asserts that if $X$ is compact, then the various subsets are open. In fact, the formulation in terms of finite dimensional families of maps is a convenient technique to avoid working directly with infinite dimensional function spaces. (We work instead functorially with a variable base $S$.) The slogan stability=openness has a flipside: approximation=density. We will discuss some approximation theorems in future lectures.

Calculus on closed affine half-spaces

(12.21) Motivation. We offer three motivations to generalize the concept of a manifold to a manifold with boundary. First, we will shortly construct topological invariants and will prove that they are invariant under homotopy. A smooth homotopy of maps $X \to Y$ is a smooth map $[0, 1] \times X \to Y$. But $[0, 1] \subset \mathbb{A}^1$ is not locally Euclidean, so is not a manifold, and neither is $[0, 1] \times X$. Both are manifolds with boundary. Second, we often do calculus on subsets of affine space which are manifolds with boundary, for example closed intervals in $\mathbb{R}$ or closed disks in $\mathbb{A}^2$. Those should be included in our calculus on curved spaces. Finally, in homology theory one studies cycles $a$ in a topological space and the notion of a homology between cycles $a_0$ and $a_1$: a homology is a chain $b$ such that $\partial b = a_1 - a_0$. Note the theory is vacuous if the space is a single point. There is a similar smooth notion called bordism in which a topological space is replaced by a smooth manifold $M$, a cycle by a smooth map $Y \to M$ in which $Y$ is a compact manifold, and a bordism from $Y_0 \to M$ to $Y_1 \to M$ is a smooth map $X \to M$ where $X$ is a compact manifold with boundary $\partial X = Y_0 \cup Y_1$ and the map restricts on the boundary to the given maps. There the theory is non-trivial even if $M = \text{pt}$. For example, there does not exist a compact 3-manifold with boundary $X$ such that $\partial X = \mathbb{R}P^2$.

Remark 12.22. It is tempting to use ‘manifold-with-boundary’ rather than ‘manifold with boundary’ to emphasize that ‘manifold with boundary’ is a single concept; ‘with boundary’ is not a modifier to ‘manifold’. A manifold is in particular a manifold with boundary, so the manifold with boundary concept generalizes the manifold concept we have been studying heretofore. If manifolds with boundary are around, then sometimes ‘manifold without boundary’ is used for ‘manifold’. (The

\[ \text{In this argument we use the fact that manifolds, which are locally Euclidean, are also locally path connected. Therefore, the partition of a manifold into components is the same as its partition into path components.} \]
hyphens are not standard, and besides an $n$-manifold is a manifold of constant dimension $n$, which would lead to ‘$n$-manifold-with-boundary’ which transgresses some laws of acceptability.)

**Remark 12.23.** When manifolds with boundary are admitted into the game, we use the term *closed manifold* for a compact manifold without boundary. (The term *open manifold* is less common. One possible meaning is a manifold without boundary, each of whose components is not compact.)

(12.24) *Local model.* Recall that the local model for a topological manifold is a finite dimensional affine space: topological manifolds are locally affine. For manifolds with boundary the local model is a closed half of an affine space.

![Figure 27. Local model for a manifold with boundary](image)

**Definition 12.25.** Let $A$ be a finite dimensional affine space, $H \in A$ an affine hyperplane, and $A^- \subset A$ the closure of one component of $A \setminus H$. Then $A^-$ is a *closed affine half-space*.

An affine hyperplane $H$ is the zero set of a nonconstant affine function $f: A \to \mathbb{R}$, and $A^- = \{ a \in A : f(a) \leq 0 \}$. So the data in Definition 12.25 can be taken to be the pair $(A, f)$. Of course, $A^-$ inherits a topology by dint of being a subspace of $A$. Then $\partial A^- = H$, and $A^-$ partitions as

\[
A^- = \operatorname{Int} A^- \sqcup \partial A^-.
\]

(12.27) *Standard local model.* For each $n \in \mathbb{Z}^>0$ there is a standard model: $A = \mathbb{A}^n$ with standard affine coordinates $x^1, \ldots, x^n$; the hyperplane is $H = \{ x^1 = 0 \}$; and we choose

\[
A^- = (\mathbb{A}^n)^- = \{ x^1 \leq 0 \}.
\]

The choice of the first (as opposed to last) coordinate, and of the nonpositive (as opposed to nonnegative) half-space is deliberate, as we explain in the next lecture.

(12.29) *Smooth functions and diffeomorphisms on closed affine half-spaces.* Use $A^-$ as a shorthand for the triple of data $(A, H, A^-)$ in Definition 12.25. The setting for calculus is two closed affine half-spaces $A^-, B^-$, an open subset $U \subset A^-$, and a function $f: U \to B^-$. Let the vector space of translations of $A, B$ be $V, W$, and let $V', W'$ be the subspaces of translations which preserve the hyperplanes $H \subset A$ and $K \subset B$. 
Definition 12.30. Let $p \in U$. We say $f$ is $C^\infty$ at $p$ if there exists an open neighborhood $\tilde{U} \subset A$ of $p$ and a $C^\infty$ function $\tilde{f}: \tilde{U} \to B$ such that $\tilde{f}|_{U \cap \tilde{U}}$ equals the composition

\[(12.31)\quad U \cap \tilde{U} \xrightarrow{f|_{U \cap \tilde{U}}} B^- \to B.\]

Lemma 12.32. The linear transformation $d\tilde{f}_p: V \to W$ is independent of the extension $\tilde{f}$ of $f$.

Proof. If $p \in U \cap \text{Int } A^-$ then $df_p: V \to W$ is defined and equals $d\tilde{f}_p$ for any extension. If $p \in U \cap H$, then since the differential is continuous,

\[(12.33)\quad d\tilde{f}_p = \lim_{p' \to p} d\tilde{f}_{p'} = \lim_{p' \to p} df_p.\]

Remark 12.34. Lemma 12.32 extends to all higher derivatives: the infinite jet of a $C^\infty$ function $f$ is well-defined, even at points of the boundary $H$.

The other statement we need from calculus is that a diffeomorphism preserves the partition (12.26). We continue with the setup of this subsection (12.29).

Lemma 12.35. Suppose $f: U \to B^-$ is a diffeomorphism onto its image. Then

\[(12.36)\quad f(U \cap H) \subset K,\]
\[(12.37)\quad f(U \cap \text{Int } A^-) \subset \text{Int } B^-.

Proof. Suppose $p \in U \cap H$ and $\tilde{f}: \tilde{U} \to B$ is an extension of $f$ near $p$, where $\tilde{U} \subset A$ is an open neighborhood of $p$. Since $d\tilde{f}_p$ is invertible, the inverse function theorem implies that possibly after reducing $\tilde{U}$ to a smaller open neighborhood, the map $\tilde{f}: \tilde{U} \to B$ is a diffeomorphism onto its image $\tilde{f}(\tilde{U}) \subset B$. Assume $f(p) \notin K$, and reduce $\tilde{U}$ further so that $\tilde{f}(\tilde{U}) \subset \text{Int } B^-$. Furthermore, since $f$ is a diffeomorphism of $U \subset A^-$ onto $f(U) \subset B^-$, there is an open neighborhood of $f(p)$ in $\text{Int } B^- \subset B^- \subset B$ contained in $f(U)$. Upon reducing $\tilde{U}$ even further, then, we may assume...
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\[ \tilde{f}(\tilde{U}) \subset f(U). \] But then \( \tilde{f}^{-1}: \tilde{f}(\tilde{U}) \to A \) carries an open neighborhood of \( f(p) \) in \( B \) to an open neighborhood of \( p \) in \( A \), from which we deduce that there exists an open neighborhood of \( p \) in \( A \) that is contained in \( U \). However, every open neighborhood of \( p \) in \( A \) contains points of \( A \setminus A^- \), which contradicts \( U \subset A^- \).

Assertion (12.37) follows from (12.36).

\[ \square \]

Lecture 13: Manifolds with boundary

In this lecture we modify some of the basic theorems we have proved for manifolds to manifolds with boundary.

Basic definitions

A manifold with boundary is defined using the local model (12.24) and its standard variant (12.27).

Definition 13.1.

1. Let \( X \) be a topological space. Then \( X \) is a topological manifold with boundary if it is Hausdorff, second countable, and locally homeomorphic to a closed affine half-space. A chart on \( X \) is a pair \((U, x)\) consisting of an open subset \( U \subset X \) and a homeomorphism \( x: U \to A^- \) into a closed affine half-space.

2. Let \( X \) be a topological manifold with boundary. An atlas is a covering of \( X \) by charts with \( C^\infty \) overlaps, as in Definition 2.8.

3. A smooth manifold with boundary is a topological manifold with boundary equipped with a maximal atlas of standard charts.

Smoothness of the overlaps is defined in (12.29).
**Partition of a manifold with boundary.** Recall from (12.26) that a closed affine half-space is partitioned into its interior and its boundary. Lemma 12.35 implies that the overlap functions on a smooth manifold $X$ with boundary preserve this partition. Therefore, there is a global partition

$$X = \text{Int} \ X \sqcup \partial X.$$ 

**Proposition 13.4.**

1. $\text{Int} \ X$ is a smooth manifold.
2. $\partial X$ is a smooth manifold.

If (a component of) $X$ has dimension $n$, then $\dim \text{Int} \ X = n$ and $\dim \partial X = n - 1$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{chart_on_X}
\caption{A chart on $X$ induces charts on $\text{Int} \ X$ and $\partial X$}
\end{figure}

**Proof.** If $x: U \to A^-$ is a chart on an open subset $U \subset X$ with values in a closed affine half-space $A^-$, set $U' = U \cap \text{Int} \ X$ and

$$x': U' \xrightarrow{x|_{U'}} \text{Int} \ A \to A$$

Then $(U', x')$ is a chart on $\text{Int} \ X$. Execute this construction on each chart of an atlas on $X$ to produce an atlas on $\text{Int} \ X$ and so prove (1). Similarly, set $U'' = U \cap \partial X$. By the definition of the partition (13.3), the restriction of $x$ to $U'' \subset U$ factors to a map

$$x'': U'' \xrightarrow{x|_{U''}} H,$$

where $H = \partial A$. Then $(U'', x'')$ is a chart on $\partial X$. The overlaps between charts are smooth by Lemma 12.35, so we obtain an atlas on $\partial X$, which proves (2).

**Example 13.7.** A closed ball is a manifold with boundary. Namely, for any $n \in \mathbb{Z}_{>0}$ set

$$D^n = \{(x^1, \ldots, x^n) \in \mathbb{A}^n : (x^1)^2 + \cdots + (x^n)^2 \leq 1\}.$$  

Then

$$\text{Int} \ D^n = B^n = \{(x^1, \ldots, x^n) \in \mathbb{A}^n : (x^1)^2 + \cdots + (x^n)^2 < 1\},$$  

$$\partial D^n = S^{n-1} = \{(x^1, \ldots, x^n) \in \mathbb{A}^n : (x^1)^2 + \cdots + (x^n)^2 = 1\}.$$
**13.10** Cartesian products. Let $X$ be a manifold with boundary and $Y$ a manifold (no boundary). Then $X \times Y$ is a manifold with boundary, and $\partial(X \times Y) = \partial X \times Y$. An atlas for $X \times Y$ can be constructed using Cartesian products of charts of $X$ and charts of $Y$ over atlases for $X$ and $Y$.

**The tangent space**

The tangent space $T_pX$ to a manifold with boundary $X$ at a point $p \in X$ is defined exactly as in the case with no boundary (Lecture 3). If $(U, x)$ is a chart about $p$ with values in a closed affine half-space $A^-$, then there is an isomorphism

$$(13.11) \quad T_pX \overset{(U, x)}{\cong} V,$$

where $V$ is the vector space of translations of the affine space $A$; see (3.27). The tangent spaces glue to a vector bundle $TX \to X$.

**13.12** Tangent space at a boundary point. The definition of $T_pX$ holds for $p \in \partial X$; there is no drop in dimension of the tangent space at a boundary point. However, there is a canonical subspace of $T_pX$ of codimension one, namely the tangent space $T_p(\partial X)$ to the boundary. In a boundary chart $(U, x)$, as depicted in Figure 30, the isomorphism (13.11) extends to a commutative diagram

$$(13.13) \quad \begin{array}{ccc}
T_p(\partial X) & \longrightarrow & V' \\
\downarrow & & \downarrow \\
T_pX & \longrightarrow & V
\end{array}$$

where $V' \subset V$ is the subspace of translations that preserve the hyperplane $H \subset A$. In a standard chart—one in which $A = \mathbb{R}^n$ and $H = \{x^1 = 0\}$—the vector fields $\frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}$ span $T_p(\partial X)$, as depicted in Figure 31. The tangent bundle of the boundary is a subbundle of the tangent bundle restricted to the boundary:

$$(13.14) \quad T(\partial X) \hookrightarrow TX|_{\partial X}$$
\textbf{(13.15) Normal space at the boundary.} At \( p \in \partial X \) we define the \textit{normal line} to the boundary as

\[ \nu_p = \nu_p(\partial X \subset X) = T_pX/T_p(\partial X). \]

The normal line carries a canonical orientation.

\textbf{Definition 13.17.} Let \( L \) be a real line, i.e., a real one-dimensional vector space. An \textit{orientation} of \( L \) is a choice of component of \( L \setminus \{0\} \).

There are two components of \( L \setminus \{0\} \), hence two orientations. For the normal line \( \nu_p \) we choose the component of \textit{outward} normals, so tangents to smooth motions \( \gamma: (-\delta, 0] \rightarrow X \) such that \( \gamma(0) = p \) and \( \gamma(t) \in X \setminus \partial X \) for \( t < 0 \).

\textbf{Figure 32.} Orientation of the normal line

There is a short exact sequence of vector spaces

\[ 0 \longrightarrow T_p(\partial X) \longrightarrow T_pX \longrightarrow \nu_p \longrightarrow 0 \]

In a standard boundary chart with local coordinates \( x^1, \ldots, x^n \), the basis of \( T_pX \) is

\[ \begin{array}{c}
\frac{\partial}{\partial x^1} \\
\vdots \\
\frac{\partial}{\partial x^n} \\
\end{array} \quad \text{basis of } \nu_p \\
\begin{array}{c}
\frac{\partial}{\partial x^1} \\
\vdots \\
\frac{\partial}{\partial x^n} \\
\end{array} \quad \text{basis of } T_p(\partial X) \]
**Remark 13.20.** The order of the basis is an example of a general principle: Quotient Before Sub. This rule will recur when we discuss orientations in more generality. It is a convention which experience shows is a good one.

**Remark 13.21.** Another relevant convention has the acronym ONF, which stands for Outward Normal First, something One Never Forgets.

**Submanifolds**

![Figure 33. Disallowed local models](image)

**(13.22) Non-examples.** Once we introduce manifolds with boundary there are several new possibilities for a local model of a submanifold. Recall (Definition 6.20) the local model of a submanifold of a manifold (no boundaries) is an affine subspace of an affine space. One could conceivably allow a manifold with boundary to be a submanifold of a manifold, as in (i) in Figure 33, but we do not allow it. (Nonetheless, that configuration occurs; we simply do not use the term ‘submanifold’ to describe it.) In the second drawing, we do not allow (ii) or (iii), which are analogous to (i). Nor do we allow (iv), in that case because the subset is not transverse to the boundary.

![Figure 34. Local model of a submanifold of a manifold with boundary](image)
**Local model.** The local model (Figure 34) we use is the following data: \( A \) is an affine space, \( H \subset A \) is a codimension one affine subspace, \( A^{-} = \text{closure of a chosen component of } A \setminus H \), and \( S \subset A \) is an affine subspace such that \( S \nparallel H \). Define \( S^{-} = S \cap A^{-} \). If \( V \) is the vector space of translations of \( A \), and \( V', V'' \subset V \) the subspaces of translations which preserve \( H, S \), respectively, then the transversality condition is \( V = V' + V'' \). For the standard local model, choose \( A = \mathbb{A}^n \), \( A^{-} = (\mathbb{A}^n)^{-} = \{ x^1 \leq 0 \} \), and \( S = \{ x^{n-k+1} = \cdots = x^n = 0 \} \) where \( k \) is the codimension.

![Figure 35. A submanifold chart of a manifold with boundary](image)

**Definition 13.24.** Let \( X \) be a manifold with boundary and \( W \subset X \) a subset. Then \( W \) is a submanifold if for each \( p \in W \) there exists a chart \( (U, x) \) of \( X \) about \( p \) with codomain a local model \( (A^{-}, S) \) such that \( x(W \cap U) \subset S^{-} \) and \( x(\partial W \cap U) \subset S^{-} \cap H \).

A submanifold chart is depicted in Figure 35. The local model enforces

\[
\partial W = W \cap \partial X \\
W \nparallel \partial X
\]

(13.25)

**Remark 13.26.** Sometimes what we are calling ‘submanifold’ is called a ‘neat submanifold’. Since this is the only notion of submanifold we use for a manifold with boundary, we omit ‘neat’.

**Submanifolds of a manifold with boundary via pullback**

One convenient method to construct submanifolds of manifolds (no boundary) is via inverse image of a regular value (Theorem 7.10) or transverse pullback of a submanifold (Theorem 11.38). Similar results hold for a submanifold with boundary, as we prove in this section. The first theorem is closely related to these results.

**Theorem 13.27.** Let \( X \) be a manifold. Suppose \( f: X \to \mathbb{R} \) is a smooth function, and \( c \in \mathbb{R} \) is a regular value of \( f \). Then \( f^{-1}(\mathbb{R} \leq c) \) is a manifold with boundary \( f^{-1}(c) \).

Notice that \( f^{-1}(\mathbb{R} \geq c) \subset X \) is not a submanifold; the local model is (i) in Figure 33.

**Remark 13.28.** In Figure 36 the red exes depict critical points and critical values. In Morse Theory one studies how the inverse image of a regular value changes as we cross a critical value. Here—the poster child of elementary Morse theory—as the regular value increases from \(-\infty\) to \(+\infty\), the inverse image undergoes a sequence of surgeries:

\[
\emptyset \longrightarrow S^1 \longrightarrow S^1 \cup S^1 \longrightarrow S^1 \longrightarrow \emptyset.
\]

(13.29)
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Also, if $c_0 < c_1$ are regular values, then $f^{-1}(c_0, c_1)$ is a bordism from $f^{-1}(c_0)$ to $f^{-1}(c_1)$.

Example 13.30. The closed ball in Example 13.7 can be constructed via the function

$$f: \mathbb{A}^n \rightarrow \mathbb{R}$$

$$(x^1, \ldots, x^n) \mapsto (x^1)^2 + \cdots + (x^n)^2$$

using the regular value $c = 1$.

Proof of Theorem 13.27. The interior $f^{-1}(\mathbb{R}^<c) \subset X$ is an open subset, so a submanifold. The boundary $f^{-1}(c) \subset X$ is a submanifold since $c$ is a regular value. But it remains to find boundary charts for $p \in f^{-1}(c)$. Apply Proposition 6.15(2) to construct a chart of $X$ on an open neighborhood $U \subset X$ of $p$ with local coordinate functions $x^1 = f - c$, $x^2, \ldots, x^n$. The restriction to $U \cap \{x^1 \leq 0\}$ is a standard boundary chart for $f^{-1}(\mathbb{R}^<c)$; the overlaps are smooth since they are restrictions of overlaps of charts of $X$. ☐

(13.32) Notation. Let $X$ be a manifold with boundary, $Y$ a manifold, and $f: X \rightarrow Y$ a function. We denote the restriction of $f$ to $\partial X$ as $\partial f: \partial X \rightarrow Y$.

Proposition 13.33. Let $X$ be a manifold with boundary, $Y$ a manifold, and $f: X \rightarrow Y$ a smooth function. Then the set of simultaneous regular values of $f, \partial f$ is dense in $Y$.

Proof. A regular point $p \in \partial X$ of $\partial f$ is also a regular point of $f$: if $d(\partial f)_p$ is surjective in

$$T_p(\partial X) \rightarrow T_{f(p)}Y$$

(13.34)
then so too is $df_p$. Now use Corollary 8.3 to deduce that the set of simultaneous regular values of

$$f|_{\text{Int} X}: \text{Int} X \rightarrow Y$$

$$\partial f: \partial X \rightarrow Y$$

is dense. □

![Figure 37. A submanifold with boundary cut out by a regular value](image)

**Theorem 13.36.** Let $X$ be a manifold with boundary, $Y$ a manifold, $f: X \rightarrow Y$ a smooth function, and suppose $q \in Y$ is a regular value of $f$ and $\partial f$. Then $W := f^{-1}(q) \subset X$ is a submanifold. Furthermore, the differential of $f$ induces an isomorphism

$$\nu_p(W \subset X) \xrightarrow{df_p \cong} T_{f(p)}Y.$$  

In particular, $\text{codim}_p(W \subset X) = \dim_{f(p)} Y$.

**Proof.** For $p \in W \cap \text{Int} X$ use Theorem 7.10 to construct a submanifold chart. Suppose $p \in W \cap \partial X$. Choose a chart $(V; y^1, \ldots, y^n)$ about $q$, and suppose $y^1(p) = \cdots = y^n(p) = 0$. Then choose a boundary chart $(U; x^1, \ldots, x^m)$ about $p$ such that $f(U) \subset V$. Then we claim the differentials of $x^1, f^*y^1, \ldots, f^*y^n$ are linearly independent at $p$. For if $a dx^1_p + b_i d(f^*y^i)_p = 0$ is a linear relation, then evaluating on $T_p(\partial X)$ we conclude that $b_i d(f^*y^i)_p = 0$ restricted to $T_p(\partial X)$. Since $q$ is a regular value of $\partial f$, we conclude that this is the trivial linear relation: $b_i = 0$ for all $i$. Since $dx_p \neq 0$, we conclude $a = 0$ and the differentials are linearly independent as claimed. Now apply Proposition 6.15(2) to construct a chart $x^1, x^2, \ldots, x^{m-n}, f^*y^1, \ldots, f^*y^n$ of $X$ on an open subset of $U$. This is a standard submanifold chart; see Figure 38. □

The technique used in the proof of Theorem 11.38 applies to manifolds with boundary to prove the following.

**Theorem 13.38.** Let $X$ be a manifold with boundary, $Y$ a manifold, $Z \subset Y$ a submanifold, $f: X \rightarrow Y$ a smooth function, and suppose $f, \partial f \cap Z$. Then $W := f^{-1}(Z) \subset X$ is a submanifold. Furthermore, if $p \in X$ satisfies $f(p) \in Z$, then
Lecture 14: Classification of 1-manifolds with boundary

Introduction

In this lecture we prove the following important result.

**Theorem 14.1.** Let \( X \) be a connected one-dimensional manifold with boundary. Then

\[
X \approx S^1 \text{ or } [0, 1] \text{ or } \mathbb{R} \text{ or } [0, 1].
\]

(The ‘\( \approx \)’ symbol means ‘is diffeomorphic to’.)

**Corollary 14.3.** Let \( X \) be a compact one-dimensional manifold with boundary. Then \( \#\partial X \) is even.

**Proof.** The boundary \( \partial X \) is a compact 0-manifold, so a finite set of points. By Theorem 14.1 \( X \) is a finite union of circles and closed intervals, each of which has an even number of boundary points (zero and two, respectively). \( \square \)

**Remark 14.4.** We will construct counting invariants—degrees, intersection numbers, etc.—depending on choices. When we compare the count for different choices, we construct a bordism between the choices, and the counts appear as the boundary of a compact 1-manifold with boundary. Corollary 14.3 will be used to deduce that the count modulo two is independent of the choices. Later we introduce orientations to refine the mod two invariants to integer invariants.
Classification of low-dimensional manifolds. The classification result Theorem 14.1 uses the simplicity of dimension one. There are classification results in higher dimensions, but unsurprisingly they are more complicated. We give a flavor of some results, and restrict to compact connected manifolds.

In dimension two there is a complete classification of compact connected manifolds. They come in two families, which are enumerated in Example 1.22.

In dimension three the situation is much more complicated. If we add the hypothesis that the manifold be simply connected, as well as compact, then it has the homotopy type of $S^3$. The Poincaré Conjecture, a theorem proved by Grigori Perelman in 2002, asserts that a compact manifold with the homotopy type of $S^3$ is diffeomorphic to $S^3$. There are also structure theorems for compact 3-manifolds not homotopy equivalent to the 3-sphere. The most profound is Bill Thurston’s Geometrization Conjecture, again a theorem proved by Perelman.

In dimension four the classification of compact manifolds is intractable since any finitely presented group can be the fundamental group of such a manifold, and there is no algorithm to distinguish finitely presented groups. Even if we restrict to simply connected compact 4-manifolds, there is no easy classification. The measure of our ignorance is the fact that the 4-dimensional Poincaré Conjecture is open: it is not known if a closed 4-manifold homotopy equivalent to $S^4$ is diffeomorphic to $S^4$. On the other hand, for simply connected topological manifolds the situation is under control due to work of Mike Freedman in the early 1980s. He proved that the homeomorphism type is determined by the homotopy type and, in “half” of the cases, a mod 2 invariant due to Rob Kirby and Larry Siebenmann. In particular, he proved that a homotopy 4-sphere is homeomorphic to the 4-sphere, which is the topological version of the Poincaré conjecture. When we come to smooth structures on these simply connected compact topological manifolds, there are new phenomena not seen in lower dimensions. Namely, there exist such manifolds which admit no smooth structure, and there exist others for which there are multiple nondiffeomorphic smooth structures. (In dimensions one, two, and three every topological manifold admits a unique smooth structure.)

Local geometric structures in differential topology. The general technique we use to prove Theorem 14.1 is one that appears often: a local geometric structure is introduced, and it is used to prove a global result. Sometimes that involves solving a differential equation. Sometimes that involves proving that some quantity is independent of the choice of a particular geometric structure on the manifold. In the case at hand, we introduce a Riemannian metric—a notion of lengths of tangent vectors—on our given 1-manifold $X$ and then write an ordinary differential equation for a motion $f: \mathbb{R} \to X$ to have unit speed, i.e., velocity of length one at all times. The function $f$ is used to get hold of the diffeomorphism type of $X$.

Remark 14.7. An alternative approach is to introduce a Morse function $f: X \to \mathbb{R}$, i.e., a function with finitely many nondegenerate critical points. (Nondegeneracy means the second derivative of $f$ at a critical point is nonzero.) Then simple surgeries are used to eliminate most or all of the critical points, which again gives control of the global structure of $X$.

This general strategy is used to prove some of the aforementioned classification results. For example, one approach to the classification of compact 2-manifolds is via Morse Theory, as in
Remark 14.7. The approach pioneered by Richard Hamilton to prove the Poincaré conjecture in dimension three is to introduce a Riemannian metric and study the Ricci flow equation, a nonlinear evolution equation of heat type for the initial metric. Perelman was able to complete the proof by showing how to continue the evolution after singularities developed. The endpoint of the evolution is a simpler Riemannian metric which reveals the global structure. In dimension four one also introduces a Riemannian metric, but now that is fixed and one writes a nonlinear elliptic partial differential equation for extrinsic objects: connections. There is a \textit{moduli space} of solutions which is used to detect obstructions to smoothness or to construct invariants of the smooth structure (which must then be proved independent of the choice of Riemannian metric). This \textit{self-duality equation} arose in quantum field theory, and was explored in a geometric context by several mathematicians (Atiyah, Singer, Hitchin, Drinfeld, Manin, Ward, \ldots), leading to Simon Donaldson’s application to the topology of 4-manifolds.

\textbf{(14.8) Outline of this lecture.} Returning to the task at hand, we begin with two background topics: Riemannian metrics and ordinary differential equations (ODEs) on manifolds. Both are important in differential geometry and differential topology.\footnote{One might ask the relationship of ‘geometry’ and ‘topology’. There is no sharp divide, and in some sense this is one of many false dichotomies in mathematics. One viewpoint is that geometry is the general study of shapes, and topology is a branch of geometry which concerns more global properties. From another view—Klein’s \textit{Erlangen Program}—a type of geometry is specified by its symmetry group and that type of geometry is the study of properties invariant under the group. Very roughly, topology is the case when the symmetry group is infinite dimensional. But all of this is a bit fanciful.} After a little maneuver—passage to a double cover—we introduce the basic ODE. Then we use a maximal solution to gain control of the global structure of $X$ and complete the proof of Theorem 14.1.

\textbf{Riemannian metrics}

\textbf{(14.9) Inner products.} Let $V$ be a real vector space. An \textit{inner product} or \textit{metric} on $V$ is a function

\begin{equation}
  g : V \times V \to \mathbb{R}
\end{equation}

which for all $N \in \mathbb{Z}^{>0}$, $\xi, \eta, \xi_i, \eta_j \in V$, $1 \leq i, j \leq N$, and $a^i, b^j \in \mathbb{R}$ satisfies

\begin{align*}
  &g(a^i \xi_i, b^j \eta_j) = a^i b^j g(\xi_i, \eta_j) \quad \text{(14.11)} \\
  &g(\xi, \eta) = g(\eta, \xi) \quad \text{(14.12)} \\
  &g(\xi, \xi) > 0 \quad \text{if } \xi \neq 0. \quad \text{(14.13)}
\end{align*}

The first condition (14.11), \textit{bilinearity}, is linear in $g$ and cuts a linear subspace out of the space of all functions (14.10). The second condition (14.12), \textit{symmetry}, is also linear in $g$. The last condition (14.13) is not linear in $g$.\footnote{One might ask the relationship of ‘geometry’ and ‘topology’. There is no sharp divide, and in some sense this is one of many false dichotomies in mathematics. One viewpoint is that geometry is the general study of shapes, and topology is a branch of geometry which concerns more global properties. From another view—Klein’s \textit{Erlangen Program}—a type of geometry is specified by its symmetry group and that type of geometry is the study of properties invariant under the group. Very roughly, topology is the case when the symmetry group is infinite dimensional. But all of this is a bit fanciful.}
**Definition 14.14.** Let $A$ be an affine space. A subset $C \subset A$ is *convex* if for all $p_0, p_1 \in S$ the line segment $\{(1-t)p_0 + tp_1 : 0 \leq t \leq 1\}$ is contained in $S$. A subset $C \subset A$ is a *cone* if for all $p \in S$ the ray $\{tp : t > 0\}$ is contained in $S$.

Condition (14.13) implies that the space $\text{Inn} V$ of inner products on $V$ is a convex cone inside the vector space $\text{Sym}^2 V^*$ of symmetric bilinear forms; see Figure 39.

![Figure 39. The space of inner products on a real vector space](image)

An inner product on $V$ determines a norm on $V$:

$$
\|\xi\| = \sqrt{g(\xi, \xi)}, \quad \xi \in V.
$$

*Remark* 14.16. We have not assumed that $V$ is finite dimensional, but in the sequel we only consider inner products on finite dimensional vector spaces.

(14.17) *Inner products on a vector bundle.* We consider inner products, or metrics, on a smooth family of vector spaces.

**Definition 14.18.** Let $\pi : E \to S$ be a real vector bundle. A *metric* on $\pi$ is a smoothly varying family of inner products on the fibers.

This is wishy-washy until we explain the meaning of ‘smoothly varying’. The simplest statement is that for every local trivialization

$$
U \times V \xrightarrow{\varphi} \pi^{-1}(U)
$$

(14.19)

the inner products on $E_s$, $s \in U$, transport to a *smooth* function $U \to \text{Inn} V \subset \text{Sym}^2 V^*$.

*Remark* 14.20. Associated to $\pi$ is a fiber bundle $\text{Inn} E \to S$ whose fiber at $s \in S$ is the space of inner products on $E_s$. Then Definition 14.18 amounts to a smooth section of $\text{Inn} E \to S$.

**Proposition 14.21.** Let $\pi : E \to S$ be a real vector bundle. Then there exists a metric on $\pi$. 

Proof. Suppose \( \{U_\alpha\}_{\alpha \in A} \) is a cover of \( S \) by open sets equipped with a local trivialization

\[
U \times V_\alpha \xrightarrow{\varphi_\alpha} \pi^{-1}(U)
\]

(14.22)

For each \( \alpha \in A \) choose an inner product \( g_\alpha \) on \( V_\alpha \). Let \( \{\rho_\alpha\}_{\alpha \in A} \) be a partition of unity subordinate to \( \{U_\alpha\}_{\alpha \in A} \) with the same index set \( A \). Then \( g = \sum_{\alpha \in A} \rho_\alpha g_\alpha \) is a metric on \( \pi \).

\[\square\]

Remark 14.23.

1. The key point is that inner products are a convex set, so can be averaged against a partition of unity. The same argument proves the existence of sections of any fiber bundle whose fibers are convex subsets of a fiber bundle of affine spaces.

2. A bit more argument proves that the space of metrics is contractible.

(14.24) Riemannian metrics. The discussion in (14.17) applies in particular to the tangent bundle \( \pi : TX \to X \) to a smooth manifold \( X \).

Definition 14.25. Let \( X \) be a smooth manifold. A Riemannian metric on \( X \) is a metric on the tangent bundle \( \pi : TX \to X \).

Proposition 14.21 implies that any smooth manifold admits a Riemannian metric. There is a rich geometry of Riemannian manifolds, but in this class we only use Riemannian metrics as auxiliary devices, as discussed in (14.6).

Ordinary differential equations on manifolds

An application of the contraction mapping fixed point theorem guarantees the existence and uniqueness of local solutions to an ODE on affine space. That theorem is proved in Lecture 17 of the notes on multivariable analysis that I handed out at the beginning of the course. There is also a global theorem in Lecture 18. In this section I summarize those results in a geometric form on manifolds; Theorem 14.31 below is easily proved from the theorems on affine space.

(14.26) Integral curves. The following definition is illustrated in Figure 40.

Definition 14.27. Let \( X \) be a smooth manifold.

1. A vector field on \( X \) is a smooth section of the tangent bundle \( TX \to X \).

2. Let \( \xi \) be a vector field on \( X \) and \( J \subset \mathbb{R} \) an open interval. A motion \( \gamma : J \to X \) on \( X \) is an integral curve of \( \xi \) if

\[
\dot{\gamma}(t) = \xi_{\gamma(t)}; \quad t \in J.
\]

(14.28)
In the figure an initial time \( t_0 \in J \) and initial position \( p_0 \in X \) are depicted. Equation (14.28) is an ordinary differential equation.

**Remark 14.29.** There is a generalization to time-varying vector fields, but in fact that is a special case of a fixed vector field. Namely, consider the trivial fiber bundle \( \rho: J \times X \rightarrow J \). Then a time-varying vector field on \( X \) is a smooth section of the relative tangent bundle (12.4) whose fiber at \( (t, p) \in J \times X \) is \( T_pX \). Since the relative tangent bundle is a subbundle of \( T(J \times X) \), we can regard this as a vector field on \( J \times X \). An integral curve for this vector field is a solution of (14.28) where now the right hand side can also vary with \( t \in J \).

The “device” we use here—encode parameters in a space, reduce a more complicated situation to a familiar one on a more complicated space—is an important general technique.

(14.30) **Existence and uniqueness.** The main theorem asserts the existence and uniqueness of integral curves, both locally and globally.

**Theorem 14.31.** Let \( \xi \) be a smooth vector field on a smooth manifold \( X \). Fix \( t_0 \in \mathbb{R} \) and \( p_0 \in X \).

1. There exists an open interval \( J \subset \mathbb{R} \) containing \( t_0 \) and a smooth function \( \gamma: J \rightarrow X \) such that

\[
\dot{\gamma}(t) = \xi_{\gamma(t)}, \quad t \in J \\
\gamma(t_0) = p_0.
\]  

Furthermore, any two solutions agree on the intersection of their domains.

2. There exists an open interval \( J_{\max} \subset \mathbb{R} \) containing \( t_0 \) and a solution \( \gamma_{\max}: J_{\max} \rightarrow X \) to (14.32) such that any solution is contained in \( \gamma_{\max} \).

The containment in (2) makes sense if we identify a function with its graph, so for any solution \( \gamma: J \rightarrow X \) we have \( J \subset J_{\max} \) and \( \gamma = \gamma_{\max}|_J \).

**Unit speed parametrization**
We have one more preliminary to the proof of Theorem 14.1. Henceforth, $X$ is a connected 1-dimensional manifold with boundary. Apply Proposition 14.21 to introduce a Riemannian metric $g$ on $X$, which we fix for the remainder of the lecture.

\textbf{(14.33)} Setting up the ODE. We aim to construct a motion $f : J \to X$ on a maximal interval $J \subset \mathbb{R}$ which has unit speed: the norm of the velocity $\dot{f}(t)$ is one for all $t \in J$. It is instructive to write this condition with respect to a local coordinate $x : U \to \mathbb{R}$ on an open subset $U \subset X$. Assume $f(J) \subset U$, or cut down $J$ until it is so. Let $G : x(U) \to \mathbb{R}$ be the positive function $G(x) = \| \frac{\partial}{\partial x} \|^2$. Then the composite function $h : J \to \mathbb{R}$ satisfies the equation

\begin{equation}
\dot{h}(t)^2 = \frac{1}{G(x)},
\end{equation}

and so

\begin{equation}
\dot{h}(t) = \pm \frac{1}{\sqrt{G(x)}}.
\end{equation}

The sign ambiguity means we have two ODEs, not one. Geometrically, we have at each $p \in X$ two opposite tangent vectors $\xi \in T_pX$ with unit norm $\| \xi \| = 1$. To write a single ODE we must pick one out. Of course, we can make such a choice locally, but we would have to prove that there exists a global choice; it is possible that one local choice, continued to be continuous, can come back to the opposite choice. Rather than disentangle the global issue, we use another “device” that is often the better route: we construct a new space which encodes both choices.

\textbf{(14.36)} The orientation double cover. Consider the function

\begin{equation}
TX \to \mathbb{R}
\end{equation}

\begin{equation}
\xi \mapsto g(\xi, \xi)
\end{equation}

We claim that $1 \in \mathbb{R}$ is a regular value. Namely, if $\xi \in T_pX$ has unit norm, then

\begin{equation}
\left. \frac{d}{dt} \right|_{t=1} g(t\xi, t\xi) = 2 \neq 0,
\end{equation}

so the differential of (14.37) is surjective at $\xi$. Let $\tilde{X} \subset TX$ be the inverse image of $1$; it is the space of unit norm tangent vectors. The restriction

\begin{equation}
\pi : \tilde{X} \to X
\end{equation}

of the tangent bundle $TX \to X$ is a double cover: there are two vectors of unit norm in each tangent space.
Remark 14.40. Since $X$ is 1-dimensional, the tangent bundle $TX \to X$ is a real line bundle: the fibers are 1-dimensional real vector spaces. Recall Definition 13.17 that an orientation of a fiber $T_pX$ is a choice of component of nonzero vectors. A unit norm vector is nonzero, and there is a unique unit norm vector in each component. Hence the double cover (14.39) can be identified with the orientation double cover, the double cover of $X$ whose fiber are the two orientations of $T_pX$.

There is a tautological vector field $\xi$ on the manifold $\tilde{X}$. Namely, a point $\tilde{p} \in \tilde{X}$ is a point $p = \pi(\tilde{p})$ of $X$ together with a tangent vector $\eta \in T_pX$. The double cover $\pi$ induces an isomorphism of tangent spaces $d\pi_{\tilde{p}}: T_{\tilde{p}}\tilde{X} \to T_pX$, and the value of $\xi$ at $\tilde{p}$ is the vector in $T_{\tilde{p}}\tilde{X}$ which projects to $\eta$. (It is natural to conflate $\tilde{p}$ and $\eta$, since $\tilde{p} \in T_{\tilde{p}}\tilde{X}$.)

**Figure 41.** The integral curve $\tilde{f}$ and its projection $f$

Proof of Theorem 14.1

(14.41) Construction of $f$. Fix $\tilde{p}_0 \in \tilde{X}$ and let $p_0 = \pi(\tilde{p}_0) \in X$ be its projection. Apply Theorem 14.31 to construct an open interval $J_{\max} \subset \mathbb{R}$ and a smooth function $\tilde{f}: J_{\max} \to \tilde{X}$ which is an integral curve of the tautological vector field $\xi$ and satisfies the initial condition $\tilde{f}(0) = \tilde{p}_0$. Set $f = \pi \circ \tilde{f}: J_{\max} \to X$. Note that both $\tilde{f}$ and $f$ are local diffeomorphisms: the differential is an isomorphism since $\xi$ is nonvanishing. We use the map $f$ to gain global control over the diffeomorphism type of $X$. The proof breaks up into two main cases according to whether $J_{\max} = \mathbb{R}$ or $J_{\max}$ is a proper subinterval.

---

13 Alert! When you read an assertion like this, pause to work out a proof for yourself. Develop this habit of mind—it is important to being a mathematician. In this case if $L$ is a line with an inner product, and $\eta \in L$ is a nonzero vector, then $\eta$ is a basis and any other vector equals $t\eta$ for a unique $t \in \mathbb{R}$. The equation $1 = g(t\eta, t\eta) = t^2 g(\eta, \eta)$ has two solutions which are opposite nonzero vectors $\xi, -\xi$. If $\gamma: [0, 1] \to L$ is a continuous path which connects them, apply the intermediate value theorem to the continuous function $t \mapsto g(\gamma(t), \xi)$ to conclude that $\gamma(t) = 0$ for some $t \in [0, 1]$. Therefore, $\xi$ and $-\xi$ lie in opposite components of $L \setminus \{0\}$. The point of this footnote is not so much the proof, but the state of mind required to read and do mathematics.

14 I’m sure an alert went off and you are now constructing this double cover and proving the local triviality!
\[ J_{\text{max}} = \mathbb{R}. \] As a first step we prove surjectivity.

**Proposition 14.43.** Assume \( J_{\text{max}} = \mathbb{R} \). Then \( f \) is surjective.

**Proof.** Since \( f \) is a local diffeomorphism, it is an open map and so \( f(\mathbb{R}) \subset X \) is open. We claim \( f(\mathbb{R}) \subset X \) is also closed, and then since \( f(\mathbb{R}) \) is nonempty and \( X \) is connected, it follows that \( f(\mathbb{R}) = X \). Thus suppose \( \{t_n\} \subset \mathbb{R} \) is a sequence such that there exists \( q \in X \) with \( f(t_n) \to q \) as \( n \to \infty \). Choose a chart \((U, x)\) about \( q \) and let \( K \subset U \) be a compact subset which contains \( x(q) \) in its interior. After possibly omitting a finite number of terms of the sequence, we may assume that \( \{\gamma(t_n)\} \subset K \), where \( \gamma \) is the composition of \( x \) and a restriction of \( f \) to an open interval \( I \) which contains \( \{t_n\} \). Define \( G: K \to \mathbb{R} \) by \( G(x) = \|\frac{d}{dx}\|_x^2 \). Since \( K \) is compact, we can choose \( C > 0 \) such that \( G(x) \leq C \). Hence

\[
(14.44) \quad \dot{\gamma}(t) = \frac{1}{\sqrt{G(\gamma(t))}} \geq \frac{1}{\sqrt{C}}, \quad t \in I.
\]

Now for \( n, m \in \mathbb{Z} \) we have

\[
(14.45) \quad |\gamma(t_n) - \gamma(t_m)| = \left| \int_{t_m}^{t_n} dt \dot{\gamma}(t) \right| \geq \frac{1}{\sqrt{C}} |t_n - t_m|,
\]

which implies

\[
(14.46) \quad |t_n - t_m| \leq \sqrt{C} |\gamma(t_n) - \gamma(t_m)|.
\]

Since \( \gamma(t_n) \to x(q) \), the sequence \( \{\gamma(t_n)\} \) is Cauchy, so by (14.46) so to is the sequence \( \{t_n\} \). Hence there exists \( t_0 \in \mathbb{R} \) with \( t_n \to t_0 \), from which \( f(t_0) = q \). This proves the surjectivity of \( f \). \( \square \)

**Corollary 14.47.** If \( J_{\text{max}} = \mathbb{R} \) and \( f \) is injective, then \( X \approx \mathbb{R} \).

**Proof.** \( f \) is a bijective local diffeomorphism, hence a global diffeomorphism. \( \square \)

**Proposition 14.48.** If \( J_{\text{max}} = \mathbb{R} \) and \( f \) is not injective, then \( X \approx S^1 \).
Proof. Choose $t_0, t_1 \in \mathbb{R}$ such that $f(t_0) = f(t_1)$. Let $\sigma : \hat{X} \to \hat{X}$ be the involution (nonidentity deck transformation) of the double cover $\pi : \hat{X} \to X$. Then either $\hat{f}(t_0) = \hat{f}(t_1)$ or $\hat{f}(t_0) = \sigma \hat{f}(t_1)$. In the former case, the motion $t \mapsto \hat{f}(t + t_1 - t_0)$ is an integral curve of $\xi$ which maps $t_0 \mapsto \hat{f}(t_1) = \hat{f}(t_0)$, hence the uniqueness statement in Theorem 14.31 implies $\hat{f}(t) = \hat{f}(t + t_1 - t_0)$ for all $t \in \mathbb{R}$. It follows that $\hat{f}(0) = \hat{f}(t_1 - t_0)$, and so $f(0) = f(t_1 - t_0) = p_0$. Consider $f^{-1}(p_0) \subset \mathbb{R}$. It is a discrete subset, since $f$ is a local diffeomorphism. Let $T \in \mathbb{R}$ be the minimal positive element of $f^{-1}(p_0)$. The uniqueness argument implies $f(t + T) = f(t)$ for all $t \in \mathbb{R}$, from which the map $f$ factors:

\begin{equation}
\begin{array}{c}
\mathbb{R} \\
\downarrow f \\
\mathbb{R} / (\mathbb{Z} \cdot T) \\
\downarrow f
\end{array}
\end{equation}

Here $\mathbb{Z} \cdot T \subset \mathbb{R}$ is the subgroup generated by $T$, and $\mathbb{R} / (\mathbb{Z} \cdot T) \approx S^1$. Minimality of $T$ implies that $\hat{f}$ is injective. It is also surjective and a local diffeomorphism, hence a global diffeomorphism.

It remains to rule out $\hat{f}(t_0) = \hat{f}(t_1)$, a situation depicted in Figure 43. If so, then the motions $t \mapsto \hat{f}(t_0 + t)$ and $t \mapsto \sigma \hat{f}(t_1 - t)$ are integral curves of $\xi$ which agree at $t = 0$, hence are equal. Set $t = (t_1 - t_0)/2$ to deduce $\hat{f}(\frac{t_0 + t_1}{2}) = \sigma \hat{f}(\frac{t_0 + t_1}{2})$, which is absurd. \hfill \qed

(14.50) $J_{\text{max}} \neq \mathbb{R}$. If the maximal open interval of definition of the integral curve $\hat{f}$ in (14.41) is a proper subset of $\mathbb{R}$, then $J_{\text{max}} = (a, b)$ or $(a, \infty)$ or $(-\infty, b)$ for some $a, b \in \mathbb{R}$. We treat the three cases simultaneously.

Proposition 14.51. $f : J_{\text{max}} \to X$ extends to $\bar{f} : \overline{J_{\text{max}}} \to X$ and $\bar{f}(\overline{J_{\text{max}}} \setminus J_{\text{max}}) \subset \partial X$.

Proof. If $\tilde{p} \in \bar{f}(\overline{J_{\text{max}}} \setminus J_{\text{max}})$, then the proof of Proposition 14.43 shows that there exists a sequence $\{t_n\} \in J_{\text{max}}$ such that $t_n \to a$ or $t_n \to b$ and $f(t_n) = \pi(\tilde{p})$. Use this to define $\bar{f}$. If $\pi(\tilde{p}) \subset \text{Int} X$, the careful reader will note that we must prove that $\bar{f}$ is smooth. Do so by choosing a boundary chart of $\hat{X}$ at $\hat{p}$ (we prove next that $\hat{p} \in \partial \hat{X}$), smoothly extending the vector field $\xi$ in the chart, and so smoothly extending the integral curve.

Figure 43. Ruling out distinct lifts of $f(t_0) = f(t_1)$
Proposition 14.52.  \( f: \overline{J_{\text{max}}} \to X \) is a diffeomorphism. Therefore, \( X \approx [0,1] \) or \( X \approx [0,1) \).

Proof. It follows from Proposition 14.51 that \( f(J_{\text{max}}) \subset X \) is closed. The image is open, since \( f \) is a local diffeomorphism, and now since \( X \) is connected we deduce that \( f \) is surjective. Injectivity of \( f \) follows as in the proof of Proposition 14.48: if \( f \) is not injective, then it is periodic and the domain \( J_{\text{max}} \) can be extended to \( \mathbb{R} \). The bijective local diffeomorphism \( f \) is a global diffeomorphism. \( \square \)


Lecture 15: Brouwer fixed point theorem; mod 2 degree; transversality in families

After building many foundations we capitalize on our work and prove some theorems in topology. The first is a nonretraction theorem, which has as a corollary the Brouwer fixed point theorem. Next, we attempt to define the mod 2 degree of a map. We have most of the tools to do so, but fall short at a crucial stage. Telling this tale now motivates the work on transversality we begin at the end of this lecture.

Nonretraction; a fixed point theorem

(15.1) A nonretraction theorem. A retraction of a set \( X \) onto a subset \( A \subset X \) is a left inverse of the inclusion map \( i: A \to X \), i.e., a map \( r: X \to A \) such that \( r|_A = \text{id}_A \). The following result proves that a compact manifold with boundary does not retract smoothly to its boundary.

Theorem 15.2. Let \( X \) be a compact manifold with boundary. Then there does not exist a retraction \( r: X \to \partial X \).

Figure 44. The inverse image of a point under a putative retraction
Proof. Suppose \( r : X \to \partial X \) is a retraction. Use Sard’s theorem (Theorem 8.1) to choose a regular value \( q \in \partial X \). Then \( W := r^{-1}(q) \subset X \) is a 1-dimensional submanifold, by Theorem 13.36. In particular,

\[
\partial W = W \cap \partial X = \{ w \in \partial X : \partial r(w) = q \} = \{ q \},
\]

since \( \partial r = \text{id}_{\partial X} \). This contradicts Corollary 14.3. \( \square \)

Remark 15.4. Compactness is crucial. For example, if \( Y \) is a closed manifold, then \( X = (-1, 0] \times Y \) retracts onto its boundary \( \partial X = \{ 0 \} \times Y \).

(15.5) A fixed point theorem. Let \( f : X \to X \) be a map from a set to itself. A point \( p \in X \) with \( f(p) = p \) is a fixed point of \( f \).

Corollary 15.6 (Brouwer fixed point theorem). Let \( D^n \subset \mathbb{R}^n \) be the closed ball (Example 13.7), and suppose \( f : D^n \to D^n \) is a smooth map. Then there exists \( p \in D^n \) such that \( f(p) = p \).

The following lovely proof is due to Morris Hirsch.

![Figure 45. Proof of the Brouwer fixed point theorem](image)

Proof. If \( f : D^n \to D^n \) has no fixed point, then define a retraction \( r : D^n \to \partial D^n = S^{n-1} \) which sends \( x \in D^n \) to the intersection of the ray emanating from \( f(x) \) through \( x \) with \( \partial D^n \), as in Figure 45. But Theorem 15.2 rules out the existence of \( r \). To prove that \( r \) is smooth, consider

\[
F : D^n \times \mathbb{R}^>0 \to \mathbb{R}
\]

\[
x, t \mapsto \text{dist}(0, x + t(x - f(x))^2)
\]

The implicit function theorem implies that the function \( t(x) \) defined by \( F(x, t(x)) = 1 \) is smooth—check that \( \partial F/\partial t \neq 0 \) at points where \( F(x, t) = 1 \)—hence \( r(x) = x + t(x)(x - f(x)) \) is smooth. \( \square \)

Remark 15.8. The case \( n = 1 \) of Corollary 15.6 has an elementary proof: apply the intermediate value theorem to \( f(x) - x \). This case is illustrated in Figure 46, which indicates how the fixed point set of a map \( f : X \to X \) can be expressed as the intersection in \( X \times X \) of the graph of \( f \) and the diagonal \( \Delta = \{(x, x) : x \in X \} \).
Mod 2 degree: first attempt

(15.9) Setup. The degree of a map is defined in the following situation. Fix a positive integer \( n \). Let \( X \) be a compact \( n \)-manifold and \( Y \) a connected \( n \)-manifold. Notice that \( \dim X = \dim Y \). Suppose \( f: X \to Y \) is a smooth map. If \( q \in Y \) is a regular value, then \( f^{-1}(q) \) is a 0-dimensional submanifold of \( X \), hence a finite set of points, since \( X \) is compact. The degree counts the number of points in the inverse image.

(15.10) Dependence on the regular value. Figure 47 illustrates that \( \# f^{-1}(q) \) depends on the regular value \( q \). In this figure, \( f \) is orthogonal projection from the plane curve \( X \) onto the vertical line \( Y \), and \( q_1, q_2, q_3 \) are regular values. As we move \( q_1 \to q_2 \to q_3 \) the cardinality of the inverse image moves \( 0 \to 6 \to 2 \). So while the count is not independent of the regular value, its reduction modulo two is, and that is true in general. We can see intuitively what happens by examining the inverse images \( W_1 \) and \( W_2 \) of the closed intervals \([q_1, q_2]\) and \([q_2, q_3]\). By an easy adaptation of Theorem 13.27, \( W_1 \) and \( W_2 \) are 1-dimensional manifolds with boundary, and they are compact since \( X \) is compact. In fact, \( W_1 \) is a bordism from \( f^{-1}(q_1) \) to \( f^{-1}(q_2) \), and \( W_2 \) is a bordism from \( f^{-1}(q_2) \) to \( f^{-1}(q_3) \). The observed fact that the number of inverse image points modulo two is unchanged when passing through a critical value follows from Corollary 14.3: the number of boundary points of a compact 1-manifold with boundary is even. It is useful to think of a movie where time is \( t \in [q_1, q_2] \) and we watch \( f^{-1}(t) \) as \( t \) evolves. We can observe the birth of pairs of
inverse image points as we pass through critical values. Continuing the movie for \( t \in [q_2, q_3] \) we can see annihilations, or deaths, or pairs as we pass through critical values. These birth and death singularities are the essence of Morse/Cerf Theory.

**Remark 15.11.** We will introduce orientations later in the course, and then count points with sign to obtain an integer invariant (which in this example is zero).

(15.12) Main theorem; partial proof. Recall that a smooth homotopy of maps \( X \to Y \) between manifolds (no boundary) is a smooth map \( F: [0,1] \times X \to Y \). We write \( F_t : X \to Y \) for the restriction of \( F \) to \( \{t\} \times X \).

**Theorem 15.13.** Fix \( n \in \mathbb{Z}_{\geq 0} \) and let \( X \) be a compact \( n \)-manifold, \( Y \) a connected \( n \)-manifold, and \( f : X \to Y \) a smooth map.

1. The mod 2 cardinality \( \#f^{-1}(q) \) (mod 2) of the inverse image of a regular value \( q \in Y \) is independent of \( q \).
2. If \( F : [0,1] \times X \to Y \) is a smooth homotopy of maps, and \( q \in Y \) a simultaneous regular value of \( F, F_0, \) and \( F_1 \), then \( \#F_0^{-1}(q) = \#F_1^{-1}(q) \) (mod 2).

Statement (1) is the well-definedness of the mod 2 degree \( \deg_2 f \in \mathbb{Z}/2\mathbb{Z} \), and (2) implies that \( \deg_2 f \) is a smooth homotopy invariant. For the latter, we observe by Sard’s theorem (Corollary 8.3) that simultaneous regular values of \( F, F_0, F_1 \) exist.

![Figure 48. Homotopy invariance of \( \deg_2 f \)](image)

**Proof of (2).** Observe that \( \partial([0,1] \times X) = \{0\} \times X \sqcup \{1\} \times X \), and so \( \partial F = F_0 \sqcup F_1 \). Theorem 13.36 implies that \( W := F^{-1}(q) \) is a 1-dimensional submanifold of \( [0,1] \times X \)—see Figure 48—and that

\[
\partial W = W \cap (\{0\} \times X) \sqcup W \cap (\{1\} \times X)
= \{0\} \times F_0^{-1}(q) \sqcup \{1\} \times F_1^{-1}(q).
\]

Since \( \#\partial W \) is even, it follows that \( \#F_0^{-1}(q) = \#F_1^{-1}(q) \) (mod 2). \( \square \)

(15.15) Attempted proof of (1). Suppose \( q_0, q_1 \in Y \) are regular values. Since \( Y \) is connected, we can and do choose a motion \( t \mapsto q_t \) from \( q_0 \) to \( q_1 \). Its graph is a submanifold \( Z \subset [0,1] \times Y \); see Figure 49. Consider \( \text{id}_{[0,1]} \times f : [0,1] \times X \to [0,1] \times Y \). If \( (\text{id}_{[0,1]} \times f) \sqsupset Z \), then the inverse image of \( Z \) is a 1-dimensional submanifold (Theorem 13.38), and we can argue as above to prove (1). But
Families of maps and transversality

As a first step toward the approximation theorem, we consider a family of maps such that the entire family is transverse to a submanifold. The construction of such families is the subject of the next lecture. Here we assume we have such a family and prove the density of transverse maps in the family. For the first version of the theorem, we omit boundaries.

\[ \text{Figure 50. A transverse family of maps } X \to Y \text{ parametrized by } S \]

**Theorem 15.17.** Let \( X, Y, S \) be smooth manifolds, and \( Z \subset Y \) a submanifold. Suppose \( F: S \times X \to Y \) is a smooth map. If \( F \not\equiv Z \), then for a dense set \( s \in S \) we have \( F_s \not\equiv Z \).

Here, for \( s \in S \) we define

\[
F_s: X \to Y \\
p \mapsto F(s, p)
\]
**Proof.** By Theorem 11.38 the inverse image \( W := F^{-1}(Z) \subset S \times X \) is a submanifold. Let \( \pi : W \to S \) be the restriction of the projection \( \text{pr}_1 : S \times X \to S \):

\[
\begin{array}{ccc}
W & \xrightarrow{\pi} & S \\
\downarrow & & \downarrow \text{pr}_1 \\
S & & 
\end{array}
\]

(15.19)

Then we claim that \( s \in S \) is a regular value of \( \pi \) if and only if \( F_s \upharpoonright Z \). The theorem then follows from Sard, since regular values of \( \pi \) are dense in \( S \).

The claim is a linear algebra assertion about the differentials. Fix \( s \in S \) and \( p \in X \), and consider the diagram

\[
\begin{array}{cccccc}
0 & \to & T_pX & \xrightarrow{d(F_s)_p} & T_qY/T_qZ & \\
& & \downarrow & & \downarrow & \\
& & T_{(s,p)}W & \to & T_sS \oplus T_pX & \xrightarrow{dF(s,p)} T_qY/T_qZ & \to & 0 \\
& & \downarrow^{d\pi(s,p)} & & \downarrow^{\text{pr}_1} & \\
& & T_sS & \to & T_sS & \\
& & \downarrow & & \downarrow & \\
& & 0 & & 0 & 
\end{array}
\]

(15.20)

The claim is that \( \boxed{1} \) is surjective iff \( \boxed{2} \) is surjective. In the diagram, the two squares commute by (15.19) and (15.18). The middle row is exact since \( F \upharpoonright Z \). The middle column is exact by the definition of direct sum. These properties mean that (15.20) is symmetric about the diagonal from NW to SE, and that symmetry means that \( \boxed{2} \) surjective \( \implies \boxed{1} \) surjective follows from \( \boxed{1} \) surjective \( \implies \boxed{2} \) surjective. The latter is proved by a 4-step diagram chase:

(i) Fix \( \eta \in T_qY/T_qZ \).

(ii) By exactness of the middle row, choose \( \zeta \in T_sS \) and \( \xi \in T_pX \) such that

\[
(15.21) \quad dF(s,p) : \zeta + \xi \mapsto \eta.
\]

(iii) Since we assume \( \boxed{1} \) is surjective, choose \( \lambda \in T_{(s,p)}W \) such that

\[
(15.22) \quad d\pi(s,p) : \lambda \mapsto \zeta.
\]

(iv) Then \( \zeta + \xi - \lambda \in T_pX \) maps to \( \eta \) via \( d(F_s)_p \).
This proves the surjectivity of $\mathbb{2}$.

An almost identical argument proves the variation of Theorem 15.17 when $X$ has boundary.

**Theorem 15.23.** Let $X$ be a smooth manifold with boundary, $Y$, $S$ smooth manifolds, and $Z \subset Y$ a submanifold. Suppose $F: S \times X \to Y$ is a smooth map. If $F, \partial F \not\subset Z$, then for a dense set of $s \in S$ we have $F_s, \partial F_s \not\subset Z$.

---

**Lecture 16: Perturbing a map to achieve transversality**

(16.1) *Setup for this lecture.* Throughout this lecture $X$ is a smooth manifold with boundary, $Y$ is a smooth manifold, $Z \subset Y$ is a submanifold, and $f: X \to Y$ is a smooth map. (In the next lecture we assume that $Z \subset Y$ is closed, but that is not necessary in this lecture.) What we need to define topological invariants is that $f \not\subset Z$ and $\partial f \not\subset Z$. The goal is to prove that we can homotop $f$ to achieve this transversality. Our main tool is Theorem 15.23, which shows that if we can embed $f$ into a transverse family, then the generic map in that family is transverse. The work, then, is to construct transverse families.

*Remark 16.2.* In Theorem 12.17 we proved that transversality is stable under deformation. (For this result we do need that $Z \subset Y$ be closed.) This means that in a suitable topology on the space of smooth maps $X \to Y$, the subspace of maps transverse to $Z$ is open. The results in this lecture show that subspace is dense.

**Perturbations in affine targets**

![Figure 51. A map with affine codomain](image)

To begin, consider the special case in which $Y$ is an affine space $A$ with vector space of translations $V$. We use the transitive group of translations to perturb. Namely, define

$$F: V \times X \to A$$

$$\xi, \quad p \mapsto f(p) + \xi$$

(16.3)
This is a family of maps parametrized by $S = V$, and at the center of the family is the original map $F_0 = f$. Note that $\partial(V \times X) = V \times \partial X$.

**Proposition 16.4.** $F$ and $\partial F$ are submersions. In particular, $F, \partial F \sim Z$.

**Proof.** For any $\xi \in V$ and $p \in X$, the differential $dF(\xi, p) : V \oplus T_p X \to V$ restricts to $\text{id}_V$ on $V \oplus \{0\}$, so is surjective. The same holds for the differential of $\partial F$. □

**Statement of theorems**

![Figure 52. Perturbation by a uniform translation followed by projection](image)

(16.5) **Strategy.** Now suppose $Y$ is an arbitrary smooth manifold. By Theorem 11.25 we may assume that $Y$ is a submanifold of a finite dimensional affine space. The idea, then, is to perturb the composition $X \xrightarrow{f} Y \xrightarrow{\pi} A$ as in (16.3), control that the image of the perturbation lies in a small neighborhood $U \subset A$ of $Y$, and then compose with a submersion $\pi : U \to Y$. In Figure 52 we illustrate a small uniform perturbation. In general, though, the size of the perturbation may depend where we are on $Y$, since a noncompact manifold embedded in affine space may not have a neighborhood of uniform size. The task, then, is to first construct $U$ and $\pi$, and then to control the size of the translation to ensure that the perturbed maps have image in $U$.

Fix a norm on $V$.

**Definition 16.6.** For a smooth function $\epsilon : Y \to \mathbb{R}^\geq 0$, let

\[(16.7) \quad Y^\epsilon = \bigcup_{p \in Y} B_{\epsilon(p)}(p),\]

which is an open subset of $A$.

**Theorem 16.8.** Let $A$ be a finite dimensional real affine space and $Y \subset A$ a submanifold.

1. There exists an open neighborhood $U \subset A$ and a submersion $\pi : U \to Y$.
2. For any open neighborhood $U \subset A$ of $Y$, there exists a smooth function $\epsilon : Y \to \mathbb{R}^\geq 0$ such that $Y^\epsilon \subset U$. If $Y$ is compact, then we can choose $\epsilon$ to be a constant function.
The last statement follows since a continuous positive function on a compact space has a positive minimum.

The rest of the lecture is devoted to the proof of Theorem 16.8. For now we extract the statements about transversality we need.

**Corollary 16.9.** Let $X$ be a manifold with boundary, $A$ a finite dimensional real affine space with vector space $V$ of translations, $Y \subset A$ a submanifold, and $f : X \to Y$ a smooth map. Choose $U, \pi, \epsilon$ as in Theorem 16.8. Then the map

$$F : B_1(0) \times X \to Y$$

$$\xi , \ p \mapsto \pi(f(p) + \epsilon(f(p)) \xi)$$

(16.10)

is a submersion.

**Corollary 16.11.** Let $X$ be a manifold with boundary, $Y$ a smooth manifold, $Z \subset Y$ a submanifold, and $f : X \to Y$ a smooth map. Then there exists a smooth homotopy $H : [0, 1] \times X \to Y$ such that $H_0 = f$ and $H_1, \partial H_1 \to \sim Z$.

**Proof.** Embed $Y$ in a finite dimensional real affine space $A$ and construct the family of maps (16.10). Use Theorem 15.23 to choose $\xi \in B_1(0)$ so that $F_\xi, \partial F_\xi \to \sim Z$. Then define $i : [0, 1] \to B_1(0)$ by $i(t) = t\xi$ and set $H = F \circ (i \times \text{id}_X)$.

**Splittings of the normal bundle**

(16.12) **Splittings of a short exact sequence of vector spaces.** Let $V, V', V''$ be a vector spaces and

$$0 \to V' \xrightarrow{i} V \xrightarrow{j} V'' \to 0$$

(16.13)

a short exact sequence of linear maps.

**Definition 16.14.** A **splitting** of (16.13) is a right inverse $s : V'' \to V$ to $j$.

In other words,

$$j \circ s = \text{id}_{V''}.$$ 

(16.15)

A splitting is equivalent to a left inverse to $i$: either expresses $V$ as a direct sum $V = s(V'') \oplus i(V')$. Splittings exist since linear subspaces have linear complements. Observe that (16.15) is an affine equation in $s$: the left hand side is linear and the right hand side is constant. From this we deduce that splittings of (16.13) form an affine space over $\text{Hom}(V'', V')$.
(16.16) **Splittings of a short exact sequence of vector bundles.** We pass from vector spaces to vector bundles and use local triviality to deduce the existence of local splittings of short exact sequences. Then a partition of unity lets us pass from local to global.

**Proposition 16.17.** Let $Y$ be a smooth manifold, $\pi: E \to Y$, $\pi': E' \to Y$, $\pi'': E'' \to Y$ vector bundles over $Y$, and

$$
0 \to E' \xrightarrow{i} E \xrightarrow{j} E'' \to 0
$$

a short exact sequence of linear maps. Then there exists a splitting $s: E'' \to E$.

**Proof.** Construct an open cover $\{U_\alpha\}_{\alpha \in A}$ of $Y$ together with local trivializations (9.24) of each vector bundle over each $U_\alpha$. Then restricted to $U_\alpha$, (16.18) becomes a short exact sequence of linear maps $i(p), j(p), p \in U_\alpha$ between fixed vector spaces (16.13). Choose a complement to $i(p_0)(V') \subset V$ at some $p_0 \in U_\alpha$. It is a complement to $i(p)(V') \subset V$ for $p$ in an open neighborhood of $U_\alpha$, and by cutting down the $U_\alpha$ we can assume it is a complement for all $p \in U_\alpha$, and so find a splitting $s_\alpha$ of (16.18) restricted to $U_\alpha$. Let $\{\rho_\alpha\}_{\alpha \in A}$ be a partition of unity subordinate to $\{U_\alpha\}_{\alpha \in A}$, and define $s = \sum_{\alpha \in A} \rho_\alpha s_\alpha$. Then $s$ is the desired global splitting.

![Figure 53](image)

**Figure 53.** The normal space $\nu_p = T_pY/T_pM$ to $Y \subset M$ at $p \in M$

(16.19) **Recollection of the normal bundle.** You can recollect by rereading (9.71) and looking at Figure 53. Note in particular the short exact sequence (9.72), which we now know has a splitting.

**Proof of Theorem 16.8**

With these preliminaries we turn to our main task, split into the two parts of the theorem.

**Proof of (1).** Let $\rho: \nu \to Y$ be the normal bundle to $Y \subset A$. Use Proposition 16.17 to choose a splitting $s$ in

$$
0 \to TY \xrightarrow{s} A \times V \xleftarrow{\nu} \nu \to 0
$$
and set $\sigma = \text{pr}_2 \circ s : \nu \to V$. Define

\begin{align}
h : \nu \to A \\
\eta \mapsto \rho(\eta) + \sigma(\eta)
\end{align}

We will prove that $h$ restricts to a diffeomorphism of a neighborhood of the zero section in $\nu$ to a neighborhood of $Y$ in $A$.

First, for any $p \in Y$ we have $h(0_p) = p$, where $0_p \in \nu_p$ is the zero vector. Next, there is a natural isomorphism $T_pY \oplus \nu_p \cong T_0\nu_p$ which maps $\xi \in T_pY$ to the corresponding tangent vector to the zero section at $0_p$; see Figure 55. With this identification, we compute

\begin{align}
dh_0 : T_pY \oplus \nu_p &\to V \\
\xi + \eta &\mapsto \xi + \sigma(\eta)
\end{align}

By the splitting property, this is an isomorphism. Hence $h$ is a local diffeomorphism at $0_p$. Choose an open neighborhood $W_p \subset A$ of $p$ and a local inverse $g_p : W_p \to \nu$ to $h$. Define

\begin{align}
W = \bigcup_{p \in Y} W_p;
\end{align}

it is an open subset of $A$ and $\{W_p\}_{p \in Y}$ is an open cover of $W$. Since $W$ is a smooth manifold, it is paracompact (Theorem 10.6), so there is a countable locally finite refinement $\{W'_i\}_{i \in I}$ of $\{W_p\}_{p \in Y}$. Let $g_i : W'_i \to \nu$ be the local inverse to $h$ induced from the refinement function $I \to Y$ by restricting the appropriate $g_p$.

The local inverses need not agree on intersections, so define

\begin{align}
\tilde W = \{ q \in W : g_i(q) = g_j(q) \text{ if } q \in W'_i \cap W'_j \text{ for some } i, j \in I \}.
\end{align}
Then $Y \subset \tilde{W}$ since $g_i(p) = 0$, for all $p \in Y$. Also, for $p \in Y$ the set

$$(16.25) \quad I_p = \{i \in I : p \in W_i\}$$

is finite, from which $\bigcap_{i \in I_p} W'_i$ is an open neighborhood of $p$. Since $h$ is a local diffeomorphism at $p$, we can and do choose an open neighborhood $U_p \subset \bigcap_{i \in I_p} W'_i$ of $p$ on which $h$ is invertible, and by the uniqueness of inverses we have $U_p \subset \tilde{W}$. Set $U = \bigcup_{p \in Y} U_p$. The local inverses on $U_p$ patch to a function $g: U \to \nu$ which inverts $h$ restricted to $U$. Set $\pi = \rho \circ g: U \to Y$. Then $\pi$ is the composition of a submersion and a diffeomorphism, so is a submersion.

**Remark 16.26.** We have proved more than is stated in Theorem 16.8(1). Namely, we constructed a diffeomorphism $g$ of $U$ with a neighborhood $g(U)$ of the zero section in the normal bundle; under that identification $\pi$ is the restriction of the normal bundle $\rho$ to that neighborhood. This is a tubular neighborhood. The tubular neighborhood theorem asserts the existence of a tubular neighborhood for any submanifold $Y \subset M$ of any smooth manifold $M$.

The next proof once more illustrates the local-to-global technique using partitions of unity.

![Figure 56. Constructing the function $\epsilon: Y \to \mathbb{R}^{>0}$](image)

**Proof of (2).** For any $p \in Y$ there exists $\delta > 0$ such that $B_\delta(p) \subset U$, since $U$ is open. Then for all $p' \in B_{\delta/2}(p) \cap Y$, the triangle inequality implies $B_{\delta/2}(p') \subset U$. This solves the problem locally. Construct an open covering $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ together with $\epsilon_\alpha > 0$ such that $B_{\epsilon_\alpha}(p) \subset U$ for all $p \in U_\alpha$. Let $\{\rho_\alpha\}_{\alpha \in \mathcal{A}}$ be a partition of unity subordinate to the cover, and set $\epsilon = \sum_\alpha \epsilon_\alpha \rho_\alpha$. For $p \in Y$ define the finite set $\mathcal{A}_p = \{\alpha \in \mathcal{A} : \rho_\alpha(p) \neq 0\}$, and let $\alpha_p^-, \alpha_p^+ \in \mathcal{A}_p$ be elements at which $\alpha \mapsto \epsilon_\alpha$ achieves its minimum and maximum, respectively. Then $0 < \epsilon_{\alpha_p^-} \leq \epsilon(p)$, which proves that $\epsilon$ is a positive function. Also, $\epsilon(p) \leq \epsilon_{\alpha_p^+}$ from which $B_{\epsilon(p)}(p) \subset B_{\epsilon_{\alpha_p^+}}(p) \subset U$, since $p \in U_{\alpha_p^+}$. It follows that $Y^{\epsilon} \subset U$. 

**Lecture 17: Mod 2 Intersection Theory**

We still have one piece of unfinished business from Lecture 16: controlled perturbation to achieve transversality. After finishing that off we recall the discussion in Lecture 15 about the mod 2 degree,
which the now finished business finishes off. Then we introduce mod 2 intersection number and some applications.

**Controlled perturbations**

(17.1) **Motivation.** Suppose $X$ is a manifold with boundary, $Y$ a smooth manifold, $Z \subset Y$ a submanifold, and $f: X \to Y$ a smooth map. To construct topological invariants we need maps $f$ such that $f \not\pitchfork Z$. Corollary 16.11 tells that we can perturb (homotop) $f$ to make it transverse to $Z$. But now suppose $C \subset X$ is a subset on which our given $f$ is already transverse to $Z$. Then we would like to perturb $f$ only on the complement of $C$, as there is no reason to move it on $C$. To implement this kind of control we expect to use bump functions, constructed via a partition of unity, and those we glue on open subsets. So we would like the transversality to persist on an open neighborhood of $C$. Therefore, we require that $Z \subset Y$ be a closed submanifold, since transversality to a closed submanifold is an open condition. (This is a correction to Theorem 12.17.) Also, we require that $C$ be a closed subset (arbitrary, not necessarily a submanifold) of $X$ since asking for constancy of a homotopy on a subset is an equality that occurs on closed subsets.

**Theorem 17.2.** Let $X$ be a smooth manifold with boundary, $Y$ a smooth manifold, $Z \subset Y$ a closed submanifold, $C \subset X$ a closed subset, and $f: X \to Y$ a smooth map such that $f|_{C} \pitchfork f|_{X \setminus C} \not\pitchfork Z$. Then there exists a smooth homotopy $H: [0, 1] \times X \to Y$ such that $H_{0} = f$, $H_{1}, \partial H_{1} \not\pitchfork Z$, and $H_{t}|_{C} = f|_{C}$ for all $t \in [0, 1]$.

![Figure 57](image.png)

**Proof.** Since transversality to a closed submanifold is an open condition, choose an open set $U \subset X$ which contains $C$ such that $f|_{U}, \partial f|_{U} \not\pitchfork Z$. Separate the disjoint closed subsets $C$, $X \setminus U$ by open sets $W_{C}$, $W_{X \setminus U}$, and let $C' = X \setminus W_{X \setminus U}$. Then $C' \subset X$ is closed and satisfies $C \subset \text{Int} C' \subset C' \subset U$. Let $\{\rho_{U}, \rho_{X \setminus C'}\}$ be a partition of unity subordinate to the open cover $\{U, X \setminus C'\}$ of $X$. Set $\tau = \rho_{X \setminus C'}^{2}$. Then $\tau|_{C} = 0$; and if $p \in X$ satisfies $\tau(p) = 0$, then $d\tau|_{p} = 0$. Recall the perturbation $F$, defined in (16.10), which we use to achieve transversality without control. Define the controlled variation

(17.3) \[ G(\xi, p) = F(\tau(p)\xi, p), \quad \xi \in B_{1}(0), \quad p \in X, \]

where recall we have embedded $Y$ in a finite dimensional affine space and $B_{1}(0)$ is the unit ball in the normed vector space $V$ of translations. Notice that if $p \in C$, then $G(\xi, p) = f(p)$ for all $\xi$. We
claim that $G, \partial G \pitchfork Z$. Granting the claim, we argue as in the proof of Corollary 16.11 to complete the proof of Theorem 17.2.

To verify the claim, set

$$m: B_1(0) \times X \to B_1(0) \times X$$

$$(\xi, p) \mapsto (\tau(p)\xi, p)$$

(17.4)

Then $G = F \circ m$. The restriction of $m$ to $B_1(0) \times \tau^{-1}(\mathbb{R}^\geq 0) \subset B_1(0) \times X$ is a diffeomorphism---its inverse can be written explicitly---and so on that subset $G$ is the composition of a submersion and a diffeomorphism, hence is itself a submersion. In particular, it and its restriction to the boundary are transverse to $Z$. Now fix $p \in X$ such that $\tau(p) = 0$. Then $p \in U$ and for any $\xi \in B_1(0)$,

$$dm_{(\xi, p)}(\dot{\xi}, \dot{p}) = (d\tau_p(\dot{p})\xi + \tau(p)\dot{\xi}, \dot{p}) = (0, \dot{p}), \quad \dot{\xi} \in V, \quad \dot{p} \in T_p X,$$

from which

$$dG_{(\xi, p)}(\dot{\xi}, \dot{p}) = dF_{(0, p)}(0, \dot{p}) = df_p(\dot{p}).$$

(17.5)

Since $f, \partial f \pitchfork Z$ at $p$, we conclude $G, \partial G \pitchfork Z$ at $(\xi, p)$. \hfill \Box

**Mod 2 degree redux**

(17.7) *Completion of (15.15).* We resume the setup in (15.9). Given $f: X \to Y$ we can choose a regular value $q \in Y$ and count the number of inverse image points modulo two. Theorem 15.13(1) asserts that the count is independent of the choice of regular value $q$, so defines an invariant $\deg_2 f \in \mathbb{Z}/2\mathbb{Z}$. In (15.15) we sketched a proof of Theorem 15.13(1), but we fell short since we needed a controlled perturbation to achieve the desired transversality, as depicted in Figure 49. Theorem 17.2 applies to fill the gap in the proof there, as you should check carefully.

**Proposition 17.8.** Let $X$ be a compact connected manifold. Then $\text{id}_X$ is not smoothly homotopic to a constant map.

*Proof.* The mod 2 degree is defined for maps $X \to X$, and $\deg_2 \text{id}_X = 1$, since every point of $X$ is a regular value with a single inverse image point. On the other hand, the constant map $X \to X$ with value $p \in X$ has any $q \neq p$ as a regular value with empty inverse image, so the mod two degree of a constant map is zero. \hfill \Box

**Proposition 17.9.** Let $n$ be a positive integer, $W$ a compact $(n + 1)$-dimensional manifold with boundary, $Y$ a connected $n$-dimensional manifold, and $F: W \to Y$ a smooth map. Then the mod two degree of the restriction of $F$ to the boundary vanishes: $\deg_2 \partial F = 0$.

*Proof.* Let $q \in Y$ be a simultaneous regular value of $F, \partial F$. Then $F^{-1}(q) \subset W$ is a compact 1-dimensional submanifold with $\partial F^{-1}(q) = F^{-1}(q) \cap \partial W$. Now apply Corollary 14.3. \hfill \Box
Proposition 17.10. Let $X$ be a compact $n$-dimensional manifold. Then there exists $f : X \to S^n$ such that $\deg_2 f = 1$.

Given any point of $X$ we construct a map which wraps a ball centered at that point around the sphere and collapses the rest of $X$ to a point. We use a bump function to smooth out what might otherwise only be a continuous map.

Proof. Fix once and for all a smooth function $\rho : \mathbb{R}^{>0} \to \mathbb{R}$ such that $0 \leq \rho \leq \pi$ and

\begin{equation}
(17.11) \quad \rho(r) = \begin{cases} 
0, & r \leq 1/4; \\
\pi, & r \geq 3/4.
\end{cases}
\end{equation}

Let $D_1(\pi) \subset \mathbb{R}^n$ be the closed ball of radius $\pi$, and identify $\mathbb{R}^n$ with the tangent space to the unit sphere $S^n \subset \mathbb{A}^{n+1}$ at the north pole $n = (0, \ldots, 0, 1)$. Define $\phi : D_1(\pi) \to S^n$ to map $0 \in \mathbb{R}^n$ to the north pole $n$, and a nonzero vector $\xi \in D_1(\pi)$ to the endpoint of arc of length $\rho(\|\xi\|)$ along the half great circle emanating from $n$ with tangent $\xi$. Note that $\phi$ maps all vectors of norm $\geq 3/4$ to the south pole $s = (0, \ldots, 0, -1)$.

For any $p \in X$ choose a coordinate chart $x : U \to \mathbb{R}^n$ about $p$ such that $x(p) = 0$ and $x(U) \supset D_1(\pi)$. Transport $\phi$ to a map of $x^{-1}(D_1(\pi)) \to S^n$ and extend $\phi$ to all of $X$ by mapping the complement of $x^{-1}(D_1(\pi))$ to the south pole $s$. The resulting map $f : X \to S^n$ is smooth, and the regular value $(1, 0, \ldots, 0) \in S^n$ has a single inverse image point. \hfill \Box

Mod 2 intersection theory

(17.12) Motivation for setup. Let $Y$ be a smooth manifold and $X, Z \subset Y$ submanifolds of complementary dimension: $\dim X + \dim Z = \dim Y$. We would like to define the intersection number of $X$ and $Z$ in $Y$ by counting the elements of $X \cap Z \subset Y$. The first problem is that this intersection may contain infinitely many points. For example, let $Y = \mathbb{A}^2$ and $X = Z = \{(x, 0) : x \in \mathbb{R}\} \subset \mathbb{A}^2$. So we need to perturb one of the submanifolds, say $X$, to achieve a transverse intersection. Our techniques let us perturb maps rather than spaces, so we perturb the inclusion map $i_X : X \to Y$. Therefore, with no cost we generalize the setup to include an arbitrary smooth map $f : X \to Y$. Then Corollary 16.11 implies we can homotop $f$ to a map $g : X \to Y$ such that $g \cap Z$, and hence
$g^{-1}(Z) \subset X$ is a 0-dimensional submanifold, i.e., a discrete subset of $X$. But we want it to be a finite subset, and therefore we add the hypothesis that $X$ be compact. Finally, we want the mod 2 count of points in $g^{-1}(Z)$ to be independent of the perturbation, and that requires that $Z \subset Y$ be a closed submanifold. For example, consider $Y = \mathbb{A}^2$, $Z = \{(x,0) : x \in \mathbb{R} \neq 0\} \subset \mathbb{A}^2$, and $X = \{(x,y) : (x-1)^2 + y^2 = 1\}$. Then $(X \cap Z) = 1$, but any small nonzero translation of $X$ intersects $Z$ in 2 points; see Figure 58.

(17.13) **Setup for intersection theory.** Hence we arrive at the following collection of data:

<table>
<thead>
<tr>
<th>$X$</th>
<th>compact manifold</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y$</td>
<td>manifold</td>
</tr>
<tr>
<td>$Z \subset Y$</td>
<td>closed submanifold</td>
</tr>
<tr>
<td>$f : X \rightarrow Y$</td>
<td>smooth map</td>
</tr>
<tr>
<td>$\dim X + \dim Z = \dim Y$</td>
<td></td>
</tr>
</tbody>
</table>

**Lemma 17.15.** Let $g_0, g_1 : X \rightarrow Y$ be smoothly homotopic maps which satisfy $g_0, g_1 \upharpoonright Z$. Then

(17.16) $\#g_0^{-1}(Z) = \#g_1^{-1}(Z)$.

Each $g_i^{-1}(Z) \subset X$, $i = 1, 2$, is a compact 0-dimensional submanifold, hence a finite subset.

**Proof.** By perturbing a given smooth homotopy away from the boundary (Theorem 17.2) we may assume given a smooth homotopy $g : [0,1] \times X \rightarrow Y$ from $g_0$ to $g_1$ such that $g \upharpoonright Z$. Since $\partial g = g_0 \upharpoonright \partial Y$, we also have $\partial g \upharpoonright Z$. Hence $g^{-1}(Z) \subset [0,1] \times X$ is a compact 1-dimensional submanifold with $g^{-1}(Z) = g_0^{-1}(Z) \cup g_1^{-1}(Z)$. The result now follows from Corollary 14.3. \qed

**Definition 17.17.** Given the setup (17.14), define the mod 2 intersection number

(17.18) $\#_2(f, Z) = \#g^{-1}(Z)$

where $g \simeq f$ is any smoothly homotopic map such that $g \upharpoonright Z$.

Such maps exist by Corollary 16.11; the count is independent of the choice of $g$ by Lemma 17.15.

**Remark 17.19 (Intersections of submanifolds).** A special case of (17.14) is when $X \subset Y$ is a compact submanifold and $f = i_X$ is the inclusion map. Then we write $\#_2(f, Z) = \#_2(X, Z)$. The situation is not symmetric: whereas $X$ is compact and $Z$ is only assumed closed. If, however, we also assume that $Z$ is compact, then it is true that $\#_2(X, Z) = \#_2(Z, X)$. One can prove that by identifying each side as a mod 2 intersection number inside the Cartesian product $Y \times Y$. Namely, let $\Delta \subset Y \times Y$ be the diagonal submanifold, and then

(17.20) $\#_2^Y(X, Z) = \#_2^Y(Z, X) = \#_2^{Y \times Y}(i_X \times i_Z, \Delta),$

where for clarity we include the ambient manifold in the notation.
Properties of the mod 2 intersection number. The following properties are analogous to properties of the mod 2 degree; see Theorem 15.13(2) and Proposition 17.9.

**Proposition 17.22.** Suppose given the setup (17.14).

1. If \( f_0 \simeq f_1 \) are smoothly homotopic, then \( \#_2(f_0, Z) = \#_2(f_1, Z) \).
2. If \( W \) is a compact \((n + 1)\)-dimensional manifold with boundary \( \partial W = X \), and \( F : W \to Y \) a smooth map such that \( \partial F = f \), then \( \#_2(f, Z) = 0 \).

To prove (1), perturb a given homotopy to achieve transversality and apply the argument for Theorem 15.13(2). To prove (2), perturb \( F \) keeping it fixed on \( X \) to achieve transversality and then consider the inverse image of \( Z \).

**Examples, applications and variations**

**Example 17.23.** Let \( Y = S^1 \times S^1 \) be a 2-torus, and consider the submanifolds \( X = S^1 \times \{0\} \) and \( Z = \{0\} \times S^1 \). Then \( \#_2(X, Z) = 1 \). On the other hand, \( \#_2(X, X) = \#_2(Z, Z) = 0 \). These mod 2 intersection numbers organize into the 2 \( \times \) 2 intersection matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

**Example 17.24** (a nonzero self-intersection number). Let \( Y = \mathbb{RP}^2 \) be the real projective plane, and \( X = \mathbb{RP}^1 \subset \mathbb{RP}^2 \) a projective line. Then \( \#(X, X) = 1 \). To compute this, we perturb the inclusion \( i : \mathbb{RP}^1 \to \mathbb{RP}^2 \) to achieve transversality with the given line \( X \), something we can achieve by choosing a transverse line. In terms of \( \mathbb{RP}^2 = \mathbb{P}(\mathbb{R}^3) \), a projective line is a 2-dimensional subspace of \( \mathbb{R}^3 \), and two transverse 2-dimensional subspaces intersect in a 1-dimensional subspace. That is, two projective lines in the projective plane intersect in a point.

**Remark 17.25.** Note that two transverse affine lines in an affine plane can intersect in a point or be disjoint (parallel). The lack of compactness prevents mod 2 intersection theory from working for affine lines; Example 17.24 shows how compactification produces an arena in which nontrivial topology emerges.

**Remark 17.26.** There is an analog of Example 17.24 in the complex projective plane \( \mathbb{CP}^2 \): two distinct complex projective lines intersect in a single point.

Next, we apply the mod 2 intersection number to distinguish two manifolds.

**Theorem 17.27.** The 2-torus \( S^1 \times S^1 \) is not diffeomorphic to the 2-sphere \( S^2 \).

**Proof.** If there is a diffeomorphism, then by Example 17.23 we can find two 1-dimensional submanifolds of \( S^2 \) with nonzero mod 2 intersection number. However, any map \( f : X \to S^2 \) with \( \dim X = 1 \) is not surjective, and via stereographic projection from a point not in the image, followed by a 1-parameter family of homotheties, we can homotop \( f \) to a constant map to a point which does not lie on any given 1-dimensional submanifold \( Z \subset S^2 \), hence the mod 2 intersection number vanishes. \( \square \)
Cubics in \( \mathbb{R}P^2 \). Now we move from lines in the real projective plane to solutions to a cubic equation: cubic curves. (We skipped over quadrics—solutions to quadratic equations—and you may want to consider what happens there in parallel.) Figure 59 shows two families of cubic curves parametrized by \( c \in \mathbb{R} \). We consider the intersection with the \( x \)-axis, which is the set of roots of a real cubic equation. In the first family, for any value of \( c \) there are three real roots, though two of the roots can coincide. In the second family, for \( c > 0 \) there are three distinct real roots, whereas for \( c < 0 \) there is only one real root. As \( c \) passes from positive to negative, the real roots all come together at \( c = 0 \) and then two of them disappear. Of course, they do not disappear if we use complex coefficients; they become a pair of complex conjugate roots to the real cubic equation. In other words, if we consider the cubic in \( \mathbb{C}P^2 \) rather than \( \mathbb{R}P^2 \), then there are always three roots, so three intersection points with the line \( y = 0 \). When we come to oriented intersection theory, we will be able to count these three intersection points. But in \( \mathbb{R}P^2 \) we can still use the mod 2 theory to detect the intersection of a line and a cubic.

(17.29) Universal family of cubics. Rather than work with special one-parameter families, we can consider the universal family. Let \([x, y, z]\) denote a point of \( \mathbb{R}P^2 \), where \( x, y, z \in \mathbb{R} \) are not all zero. The equivalence relation is \([\lambda x, \lambda y, \lambda z] = [x, y, z]\) for all \( \lambda \in \mathbb{R}^\neq 0\). A cubic curve is the vanishing set of a homogeneous cubic polynomial

\[
a_1 x^3 + a_2 y^3 + a_3 z^3 + a_4 x^2 y + a_5 x^2 z + a_6 xy^2 + a_7 xz^2 + a_8 y^2 z + a_9 yz^2 + a_{10} xyz,
\]

where not all coefficients \( a_1, \ldots, a_{10} \) vanish. Furthermore, proportional cubics give the same vanishing set. Hence the cubics are parametrized by the projective space \( \mathbb{R}P^9 \) with homogeneous coordinates \( a_1, \ldots, a_{10} \). Note that some of these cubics are degenerate. For example, the zero set of \( xyz \) is a set of three lines transverse lines in the plane (which form a triangle); the zero set of \( x^3 \) is a single triple line. But bear in mind that these lines are projective: each is diffeomorphic to a circle. Of course, the zero set of (17.30) is a closed subset of \( \mathbb{R}P^2 \), and since \( \mathbb{R}P^2 \) is compact, it too is compact. By Theorem 14.1 if it is a manifold, then it is a union of circles. As illustrated in the second family of Proposition 7.22, the number of circles can change.

---

16If \( V \) is a 3-dimensional real vector space, and we replace \( \mathbb{R}P^2 \) by \( \mathbb{P}V \), then the projective space which parametrizes the lines in \( \mathbb{P}V \) is the dual projective space \( \mathbb{P}V^* \), and the vector space which parametrizes the cubics is \( \mathbb{P}\text{Sym}^3 V^* \). We have not yet defined symmetric powers of a vector space: \( \text{Sym}^3 V^* \) is (isomorphic to) the vector space of symmetric trilinear functions \( V \times V \times V \to \mathbb{R} \).
Intersection theory for the universal family. The mod 2 intersection of a line and a cubic curve is defined, as long as the cubic curve is a submanifold. We expect that mod 2 intersection number, which is $1 \in \mathbb{Z}/2\mathbb{Z}$ from the examples shown, to be independent of the smooth cubic curve. That is true, but it is not covered by the setup (17.14) since the topology is changing. The following theorem, whose proof is a homework problem, covers this situation.

**Theorem 17.32.** Let $X, Y$ be smooth manifolds, $S$ a connected smooth manifold, $Z \subset Y$ a closed submanifold, and suppose in the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{G} & S \times Y \\
\downarrow{F} & & \downarrow{\text{pr}_1} \\
S & \xleftarrow{} & Y
\end{array}
$$

the map $G$ is an embedding and $F$ is proper. Assume that $(\dim X - \dim S) + \dim Z = \dim Y$. Suppose that $s_0, s_1 \in S$ are regular values of $F$. Then

$$
\#_2^Y (F^{-1}(s_0), Z) = \#_2^Y (F^{-1}(s_1), Z),
$$

where we use $G$ to embed $F^{-1}(s_i)$ as submanifolds of $Y$.

To apply the theorem to the universal family of cubics, set $S = \mathbb{RP}^9$, $Y = \mathbb{RP}^2$, $Z \subset Y$ a fixed line $\mathbb{RP}^1 \subset \mathbb{RP}^2$, and define

$$
X = \left\{ ([a_1, \ldots, a_{10}], [x, y, z]) : (17.30) \text{ vanishes} \right\} \subset \mathbb{RP}^9 \times \mathbb{RP}^2.
$$

Then $X$ is a submanifold, since the vanishing of (17.30) is a transverse condition. The maps $F$ and $G$ are the restriction of projection onto a factor of $\mathbb{RP}^9 \times \mathbb{RP}^2$.

---

**Lecture 18: Mod 2 winding number, Jordan-Brouwer, and Borsuk-Ulam**

In the first part of the lecture we add the mod 2 winding number to our arsenal of mod 2 invariants, which up to now consists of the mod 2 degree and mod 2 intersection number. We then apply it to prove a generalization of the classical Jordan curve theorem. As bonus material we didn’t get to in lecture, I include an account of the Borsuk-Ulam theorem as well. The Jordan-Brouwer and Borsuk-Ulam theorems in topology are proved in a continuous setting; our methods grounded in calculus are suitable for the smooth setting. There are approximation theorems from which one can deduce the general continuous statements from the smooth ones.
(18.1) Setting for the lecture. We use the following data throughout:

- \( n \): positive integer
- \( A \): real affine space of dimension \( n + 1 \)
- \( V \): tangent space to \( A \), equipped with an inner product
- \( X \): compact \( n \)-dimensional manifold
- \( f : X \to A \): smooth map

In our discussion of the Jordan-Brouwer theorem the map \( f \) is an embedding, and we identify \( X \) with its image \( f(X) \subset A \), a codimension one submanifold (hypersurface) in the affine space \( A \). The topological invariants do not depend on the inner product on \( V \), which is a contractible choice.

![Diagram](image)

**Figure 60.** Definition of the mod 2 winding number

Mod 2 winding number

(18.3) Definition and homotopy invariance. Given the data (18.2), choose \( q \in A \setminus f(X) \). (By Sard’s theorem, \( f(X) \neq A \).) Let \( S = S(V) \subset V \) be the \( n \)-sphere of unit norm vectors. Define

\[
   w_q : X \to S
\]

\[
   p \mapsto \frac{f(p) - q}{\|f(p) - q\|}
\]

(18.4)

**Definition 18.5.** The mod 2 winding number of \( f \) about \( q \) is

\[
   W_2(f, q) = \deg_2 w_q.
\]

(18.6)

**Remark 18.7.** If \( n = 1 \) and \( X = S^1 \), then this is the classical case encountered in complex analysis, for example. There one writes an integral formula for the winding number. There are similar integral formulæ for the general case of Definition 18.5.
**Proposition 18.8.** Given the data (18.2), the mod 2 winding number \( W_2(f, q) \) depends only on the path component of \( q \) in \( A \setminus f(X) \). Also, \( W_2(f, q) \) is unchanged under smooth homotopies of \( f \) which do not contain \( q \) in the image.

**Proof.** Both statements follow from the homotopy invariance of the mod 2 degree (Theorem 15.13(2)). For the first choose a path \( t \mapsto q_t \) connecting two points in the complement of \( f(X) \). For the second let \( t \mapsto f_t \) be a homotopy such that \( f_t(p) \neq q \) for all \( t, p \). In each case we obtain a homotopy of (18.4), well-defined since the vector in the denominator is nonzero. \( \square \)

(18.9) **Computation of the mod 2 winding number via extension.** We give two methods to compute. In the first we write \( f \) as the boundary of a map out of a compact manifold with boundary. (This is possible if \( X \) is null bordant.) In the second we express the mod 2 winding number as the mod 2 intersection number with a ray.

**Theorem 18.10.** Given the data (18.2), suppose in addition that \( W \) is a compact \((n+1)\)-manifold with \( \partial W \), and \( F: W \to A \) a smooth map with \( \partial F = f \). Let \( q \in A \setminus f(X) \) be a regular value of \( F \). Then \( W_2(f, q) = \# F^{-1}(q) \) (mod 2).

Notice that \( q \) is trivially a regular value of \( \partial F = f \) since it is not in its image. Regular values of \( q \) in each path component of \( A \setminus f(X) \) exist by Sard’s theorem.

![Figure 61. Computation of \( W_2 \) via extension and counting](image)

**Proof.** If \( F^{-1}(q) = \emptyset \), then the map \( w_q \) extends to \( W \) using the formula in (18.4) with \( f \) replaced by \( F \). The vanishing of the mod 2 winding number follows from Proposition 17.9.

Hence we may assume \( F^{-1}(q) = \{p_1, \ldots, p_N\} \) is a nonempty finite set. Since \( q \) is a regular value, \( F \) is a local diffeomorphism at each \( p_i \). Choose balls\(^{17} \) \( B_i \subset W \) containing \( p_i \) and \( B \subset A \) containing \( q \) such that \( F|_{B_i}: B_i \to B \) is a diffeomorphism. Apply the first paragraph of the proof to \( W \setminus \bigcup_{i=1}^N B_i \) to conclude

\[
(18.11) \quad W_2(f, q) = \sum_{i=1}^N W_2(f, q_i),
\]

where \( f_i = F|_{\partial D_i} \) and \( D_i = \overline{B_i} \). Finally, since \( f_i \) is a diffeomorphism, and the mod 2 degree of a composition is the product of the mod 2 degrees, the mod 2 degree of (18.4) for \( f = f_i \) equals the

\(^{17}\) A ‘ball’ in a manifold means the inverse image of an open ball under a coordinate chart. We assume the closure of the ball is a manifold with boundary, the pullback of a closed disk in the affine space of the coordinate chart.
mod 2 degree of the map with domain $\partial D$, $D = \overline{B} \subset A$, and the latter clearly has mod 2 degree equal to one. □

(18.12) Computation of the mod 2 winding number via ray crossing. For $q \in A$ and $\xi \in V$ a nonzero vector, define the ray

$$Z_q(\xi) = \{q + t\xi : t > 0\}. \tag{18.13}$$

Observe that $Z_q(\xi) \subset A \setminus \{q\}$ is a closed submanifold of the deleted affine space.

Theorem 18.14. Given the data (18.2), fix $q \in A \setminus f(X)$ and let $Z = Z_q(\xi)$ be a ray in $A$ emanating from $q$ in the direction $\xi \neq 0 \in V$. Then if $f \not\parallel Z$,

$$W_2(f, q) = \#_2(f, Z), \tag{18.15}$$

where the intersection number is computed in $Y = A \setminus \{q\}$.

![Figure 62. Computation of $W_2$ via intersection with a ray](image)

**Proof.** Suppose $p \in X$ satisfies $f(p) \in Z$. Then the tangent space to the unit sphere of $V$ at the unit vector $\frac{f(p) - q}{\|f(p) - q\|} \in S$ is the orthogonal complement to $\xi$. Now

$$d\left(\frac{f - q}{\|f - q\|}\right)_p = \frac{df_p}{\|f(p) - q\|} - \frac{\langle df_p, f(p) - q \rangle}{\|f(p) - q\|^3} [f(p) - q] \tag{18.16}$$

has image a subspace of that orthogonal complement. The image is the entire orthogonal complement—$p$ is a regular point of $w_q$—if and only if the image of $df_p$ is a complement to the span of $\xi$, i.e., $f \not\parallel p Z$. Then $\xi$ is a regular value of $w_q$ iff $f \not\parallel Z$, in which case $f^{-1}(Z) = w_q^{-1}(\xi)$. □

**The Jordan-Brouwer Separation Theorem**

The setup for this section is (18.2) with $f = i_X : X \to A$ the inclusion of a submanifold.
Theorem 18.17. Given the data (18.2), assume $X \subset A$ is a compact connected hypersurface. Then $A \setminus X$ has two components $D_0, D_1$; exactly one component, say $D_1$, is bounded. The closure $\overline{D_1}$ is a compact manifold with boundary $X$. For $q \in A \setminus X$, we have $q \in D_j$ iff $W_2(i_X, q) = j$, $j \in \mathbb{Z}/2\mathbb{Z}$.

Proof. Fix $q \in A \setminus X$. Let $X' \subset X$ be the subset of $p \in X$ such that for any open neighborhood $U \subset A$ of $p$ there exists a smooth motion in $A \setminus X$ which begins at $q$ and terminates at a point of $U$. We claim $X'$ is nonempty, for if $p_0 \in X$ minimizes the positive distance function $p \mapsto \|p - q\|$ on the compact manifold $X$, then the constant velocity motion from $q$ to $p$ enters any neighborhood $U$ of $p$ before intersecting $X$. Also, $X'$ is closed: if $p_n \to p$ is a convergent sequence in $X$ with $p_n \in X'$, and $U$ is an open neighborhood of $p$, then $U$ is an open neighborhood of $p_n$ for $n$ sufficiently large. Finally, $X'$ is open. To prove this, let $p \in X'$ and choose a submanifold chart on a connected open neighborhood $U_1$ of $p$ such that $U_1 \cap X$ is also connected, as in Figure 63. Fix a motion $\gamma$ in $A \setminus X$ from $q$ to $U_1$. If $p' \in U' \cap X$, and $U \subset A$ is any open neighborhood of $p'$, then $U \cap U'$ intersects both components of $U' \setminus X \cap U'$. Hence we can extend the motion $\gamma$ to terminate in $U \cap U'$. Since $X$ is connected, the nonempty open and closed subset $X'$ is equal to $X$. It follows that $A \setminus X$ has at most two (path) components.

Figure 63. Joining $q \in A \setminus X$ to a neighborhood of $p \in X$

Figure 64. Mod 2 winding number as an invariant of $\pi_0(A \setminus X)$

Fix $q_0 \in A \setminus X$ and a ray $Z_{q_0}(\xi)$ such that $Z_{q_0}(\xi) \cap X \neq \emptyset$ and $Z_{q_0}(\xi) \cap X$. Let $q_1 \in Z_{q_0}(\xi) \setminus X$ be a point on the ray past the first intersection point with $X$, as in Figure 64. Apply Theorem 18.14 to $Z_{q_0}(\xi)$ and $Z_{q_1}(\xi)$ to conclude that $W_2(i_X, q_0) \neq W_2(i_X, q_1)$. Then the first statement in Proposition 18.8 implies that $q_0$ and $q_1$ lie in different path components of $A \setminus X$. Hence $\#\pi_0(A \setminus X) = 2$ and the path components are

$$d_j = \{q \in A \setminus X : W_2(i_X, q) = j\}, \quad j \in \mathbb{Z}/2\mathbb{Z}.$$
Since $X$ is compact, there is a closed disk $C \subset A$ such that $X \subset C$, as in Figure 64. Let $q \in A \setminus C$, choose $s \in \partial C$ which minimizes the distance from $q$ to $\partial C$, and let $A' = s + T_s \partial C$ be the affine hyperplane tangent to $\partial C$ at $s$. Then $X$ lies in the half space with boundary $A'$ which does not contain $q$, and so $Z_q(\xi) \cap X = \emptyset$, where $\xi = q - s$. Therefore, $q \in D_0$. Hence $D_1 \subset C$ is bounded and $D_0$ is unbounded.

Now $\overline{D_1} = D_1 \cup X$ is compact, and $D_1 \subset A$ is an open submanifold. To prove that $\overline{D_1}$ is a manifold with boundary, for $p \in X$ we must produce a boundary chart. We can do so by restricting a submanifold chart for $X$ at $p$; see Figure 63. □

(18.19) An application to a nonembedding theorem. Recall our discussion in Example 11.8 of the minimal embedding dimension of real projective spaces. Here we apply Theorem 18.17 to prove that $\mathbb{R}P^2$ does not embed in $\mathbb{A}^3$. However, we do not quite have all the tools to carry out the proof, so beware that the last step requires more technical foundations.

**Proposition 18.20.** Given the data (18.2), assume $X \subset A$ is a compact connected hypersurface. Then the normal bundle $\nu \to X$ is orientable.

In fact, the normal bundle carries a canonical orientation, which we choose in the proof.

**Proof.** Let $p \in X$; then the normal space is $\nu_p = V/T_pX$. A vector $\xi \in V \setminus T_pX$ has a nonzero image in the quotient space $\nu$, and for all sufficiently small $t > 0$ we have either $p + t\xi \in D_0$ or $p + t\xi \in D_1$. (Argue as in the proof of Theorem 18.17 based on Figure 63.) We say $[\xi] \in \nu_p \setminus \{0\}$ is positively oriented if $p + t\xi \in D_0$ for all sufficiently small $t > 0$. □

Now suppose there exists an embedding $\mathbb{R}P^2 \hookrightarrow \mathbb{A}^3$. Let $Z = \mathbb{R}P^1 \subset \mathbb{R}P^2$. Then at $p \in Z$ we have the full flag

\begin{equation}
0 \subset T_pZ \subset T_p\mathbb{R}P^2 \subset \mathbb{R}^3
\end{equation}

in the vector space $\mathbb{R}^3$. Since $\mathbb{R}P^1 \approx S^1$ we can consistently orient $T_pZ$ for all $p \in Z$. By Proposition 18.20 the quotient $\mathbb{R}^3/T_p\mathbb{R}P^2$ is oriented. Choose $e_1 \in T_pZ$ be positively oriented and choose $e_3 \in \mathbb{R}^3 \setminus T_p\mathbb{R}P^2$ so that the image of $e_3$ in the normal line is positively oriented. Relying on your knowledge of an orientation of $\mathbb{R}^3$ (the “right hand rule”), choose $e_2 \in T_p\mathbb{R}P^2 \setminus T_pZ$ so that $e_1, e_2, e_3$
is a positively oriented basis of \( \mathbb{R}^3 \). Rigidify these constructions: choose \( e_1 \) to have unit length, \( e_3 \) to be a unit vector orthogonal to \( T_p \mathbb{R}P^2 \), and \( e_2 \) to complete to a normal basis. Then \( e_2 \) is a nonzero normal vector field to \( Z \subset \mathbb{R}P^2 \). Use it to “push” \( Z \) off of itself to a parallel circle in \( \mathbb{R}P^2 \). Then \( \#_2(Z, Z) = 0 \) in \( \mathbb{R}P^2 \) since by a homotopy we have deformed \( i_Z \) to have image disjoint from \( Z \). This contradicts Example 17.24.

The Borsuk-Ulam Theorem

For the standard \( n \)-sphere (3.4), the antipodal map \( \alpha: S^n \to S^n \) is the map \( x \mapsto -x \). A map \( g: S^n \to V \) into a vector space \( V \) is odd if \( g(\alpha(p)) = -g(p) \) for all \( p \in S^n \).

**Theorem 18.22.** Fix a positive integer \( n \). Let \( V \) be a real vector space of dimension \( n \) and \( W \) a real vector space of dimension \( n + 1 \).

1. If \( f: S^n \to V \) is a smooth map, then there exists \( p \in S^n \) such that \( f(\alpha(p)) = f(p) \).
2. If \( g: S^n \to V \) is an odd map, then there exists \( p \in S^n \) such that \( g(p) = 0 \).
3. If \( h: S^n \to W \) is an odd map, and \( 0 \in h(S^n) \), then \( W_2(h, 0) = 1 \).

**Proof.** If (3) is true, and \( g: S^n \to V \) is odd, set \( W = V \oplus \mathbb{R} \) and

\[
(18.23) h: S^n \longrightarrow W \quad p \longrightarrow g(p) \oplus 0
\]

If \( g \) never vanishes, then \( h(S^n) \cap (V \oplus \{0\}) = \emptyset \), which implies \( W_2(h, 0) = 0 \) in contradiction to (3).

If (2) is true, and \( f: S^n \to V \) is given, set \( g(p) = f(\alpha(p)) - f(p) \) and then deduce (1) from (2).

We prove (3). Fix an inner product on \( W \). Set

\[
(18.24) \varphi = \frac{h}{\|h\|}: S^n \longrightarrow S(W) \cong S^n;
\]

then

\[
(18.25) \varphi(\alpha(p)) = \alpha(\varphi(p)), \quad p \in S^n,
\]

and by (18.6) we have \( W_2(h, 0) = \deg_2 \varphi \). We proceed by induction on \( n \).

For \( n = 1 \) write \( \varphi: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) and use covering space theory to lift to a smooth function \( \tilde{\varphi}: \mathbb{R} \to \mathbb{R} \) which satisfies \( \tilde{\varphi}(x + 1) = \tilde{\varphi}(x) + d \) for some \( d \in \mathbb{Z} \); then \( \deg_2 \varphi = d \) (mod 2). But by (18.25) we also have \( \tilde{\varphi}(x + 1/2) = \tilde{\varphi}(x) + 1/2 + e \) for some \( e \in \mathbb{Z} \). Iterating we deduce \( d = 2e + 1 \).

It remains to prove the inductive step. Let \( \varphi: S^{n+1} \to S = S(W) \) and assume the theorem holds in lower dimensions. Choose \( \xi \in S \) such that \( \xi \) is a regular value of \( \varphi \) and \( \xi \notin \varphi(S^n) \). Then \( \deg_2 \varphi = \# \varphi^{-1}(\xi) \) (mod 2). Compose \( \psi = \varphi \big|_{S^n} \) with stereographic projection from \( -\xi \) onto \( T_\xi S \); then \( 0 \notin \psi(S^n) \). Use the restriction of \( \varphi \) to the upper hemisphere \( D^{n+1} \) as an extension of \( \psi \) and apply Theorem 18.10 to compute

\[
(18.26) W_2(\psi, 0) = \# \left( \varphi \big|_{D^{n+1}} \right)^{-1}(\xi) = \deg_2 \varphi.
\]

By the inductive hypothesis, \( W_2(\psi, 0) = 1 \). \(\square\)
Lectures 19–23: Exterior algebra, differential forms, orientations

These are the bulk of the lectures in the class which cover calculus on manifolds. The first topic is algebraic: the exterior algebra of a vector space. Then we turn to differential calculus, going through the progression

(19.1) vector spaces \(\rightarrow\) affine spaces \(\rightarrow\) smooth manifolds

Then we develop integral calculus with the same progression, a development which continues in the next lecture. The material on exterior algebra and for calculus in affine spaces is written up in the notes on Multivariable Analysis. Here I will only discuss the local-to-global process which takes us from open sets in affine space to smooth manifolds. As you might guess, partitions of unity are a key tool.

Differential forms on manifolds

(19.2) 0-forms and 1-forms. Let \(X\) be a smooth manifold. The real vector spaces

\[
\Omega^0(X) = \{\text{functions } X \to \mathbb{R}\}
\]

\[
\Omega^1(X) = \{\text{sections of } T^*X \to X\}
\]

of functions and 1-forms have already been defined, and the differential is a linear map

(19.4) \(\Omega^0(X) \xrightarrow{d} \Omega^1(X)\)

which satisfies the Leibniz rule

(19.5) \(d(f_1 f_2) = d(f_1) f_2 + f_1 d(f_2).\)

Our first goal is to extend to higher degree forms.
(19.6) Bundles of exterior algebras. Let $E \to X$ be a real vector bundle. Recall from Lecture 9 that we can construct new vector bundles from a given vector bundle from a functorial construction on vector spaces. The dual bundle (9.51) is an example; the general procedure is discussed in (9.44). Here for each $k \in \mathbb{Z}_{\geq 0}$ we construct a vector bundle $\wedge^k T^*X \to X$ from the cotangent bundle $T^*X \to X$ using Theorem 9.45. The same construction gives a bundle $\bigwedge^k T^*X \to X$ whose fibers are exterior algebras. Define the vector space of differential $k$-forms

$$\Omega^k(X) = \{ \text{sections of } \wedge^k T^*X \to X \}.$$ 

The algebra of differential forms is the direct sum

$$\Omega^\bullet(X) = \bigoplus_{k=0}^{\infty} \Omega^k(X)$$

with the wedge product defined pointwise; it may be identified as the algebra of sections of $\wedge^\bullet T^*X \to X$.

(19.9) Vector fields. A vector field$^{18}$ on a manifold is a smooth choice of tangent vector at each point. The vector space of vector fields on $X$ is

$$\mathfrak{X}(X) = \{ \text{sections of } TX \to X \}.$$ 

(19.11) Differential forms as functionals of vector fields. Recall that if $V$ is a real vector space, then there is a duality pairing

$$\bigwedge^k V^* \times \bigwedge^k V \to \mathbb{R}.$$ 

Combining with the multiplication map $V \times \cdots \times V \to \bigwedge^k V$ we construct an isomorphism between $\bigwedge^k V^*$ and the vector space of $k$-linear alternating maps $V \times \cdots \times V \to \mathbb{R}$.

If $X$ is a smooth manifold and $\alpha \in \Omega^k(X)$ is a differential $k$-form, then pointwise evaluation determines an alternating $k$-linear map

$$\tilde{\alpha} : \mathfrak{X}(X) \times \cdots \times \mathfrak{X}(X) \to \Omega^0(X)$$

where for $\xi_1, \ldots, \xi_k \in \mathfrak{X}(X)$ we have

$$\tilde{\alpha}(\xi_1, \ldots, \xi_k)(p) = \alpha_p(\xi_1(p), \ldots, \xi_k(p)), \quad p \in X.$$ 

$^{18}$In general an $x$-field on a manifold is a smooth choice of $x$ at each point, where $x$ can be ‘vector’, ‘covector’, ‘tensor’, ‘scalar’, ‘spinor’, ...
There is an additional important property: \( \hat{\alpha} \) is linear over functions. Namely, if \( f_1, \ldots, f_k \in \Omega^0(X) \) are functions, and \( \xi_1, \ldots, \xi_k \in \mathfrak{X}(X) \) are vector fields, then

\[
\hat{\alpha}(f_1\xi_1, \ldots, f_k\xi_k) = f_1 \cdots f_k \hat{\alpha}(\xi_1, \ldots, \xi_k).
\]

The following proposition constructs a differential form from a map (19.13).

**Proposition 19.16.** Let

\[
\hat{\alpha}: \mathfrak{X}(X) \times \cdots \times \mathfrak{X}(X) \to \Omega^0(X)
\]

be a \( k \)-linear alternating map which is linear over functions. Then there is a unique differential \( k \)-form \( \alpha \in \Omega^k(X) \) such that (19.14) holds.

A function (19.17) which is linear over functions is said to be tensorial. Proposition 19.16 holds for other kinds of tensor fields on a smooth manifold. For example, a Riemannian metric is (determined by) a tensorial positive definite symmetric bilinear form on vector fields.

**Example 19.18.** Let \( U \subset A \) be an open subset of an affine space \( A \) with tangent space \( V \), and fix a smooth function \( f \in \Omega^0(U) \) which is not locally constant. Define

\[
\hat{\alpha}_f(\xi_1, \xi_2) = \xi_2 f - \xi_1 f, \quad \xi_1, \xi_2 \in \mathfrak{X}(U).
\]

Each term in (19.19) is an iterated directional derivative. Then \( \hat{\alpha}_f \) is bilinear and alternating, but it is not linear over functions so does not define a 2-form on \( U \).

**Proof of Proposition 19.16.** For ease of notion we take \( k = 1 \). For \( p \in X \) and \( \xi_p \in T_pX \), we would like to define

\[
\alpha_p(\xi_p) = \hat{\alpha}(\xi)(p),
\]

where \( \xi \in \mathfrak{X}(X) \) is any vector field such that \( \xi(p) = \xi_p \), i.e., \( \xi \) is an extension of the vector \( \xi_p \in T_pX \) to a vector field. We must prove that (19.20) is independent of the extension \( \xi \). Equivalently, if \( \eta \in \mathfrak{X}(X) \) satisfies \( \eta(p) = 0 \), then we claim \( \hat{\alpha}(\eta)(p) = 0 \).

To prove the claim, let \( (U; x^1, \ldots, x^n) \) be a standard coordinate chart about \( p \) and let \( \rho: X \to \mathbb{R} \) be a function with \( \text{supp}(\rho) \subset U \) and \( \rho(p) = 1 \). Write \( \eta|_U = f^i \partial_i / \partial x^i \) for functions \( f^i \in \Omega^0(U) \), so \( f^i(p) = 0 \) and

\[
\eta = (\rho f^i) \left( \frac{\partial}{\partial x^i} \right) + (1 - \rho^2) \eta.
\]

Notice that each factor in each term is a global function or a global vector field on \( X \). By the hypothesis that \( \hat{\alpha} \) is linear over functions, we deduce

\[
\hat{\alpha}(\eta)(p) = \rho(p) f^i(p) \hat{\alpha} \left( \frac{\partial}{\partial x^i} \right)(p) + (1 - \rho^2(p)) \hat{\alpha}(\eta)(p)
\]

\[= 0 \quad \square\]
The Cartan $d$ operator

(19.23) The de Rham complex. The goal is to construct the de Rham complex

\[
0 \rightarrow \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(X) \rightarrow 0,
\]

assuming $\dim X = n$. In the multivariable notes we construct the de Rham complex on an open subset of affine space, as summarized in the following theorem.

**Theorem 19.25.** Let $A$ be an affine space and $U \subset A$ an open subset. Then there exists a unique map $d: \Omega^*(U) \rightarrow \Omega^*(U)$ of degree $+1$ such that

(i) $d$ is linear,
(ii) $d(\alpha_1 \wedge \alpha_2) = d\alpha_1 \wedge \alpha_2 + (-1)^k \alpha_1 \wedge d\alpha_2$, \hspace{1cm} $\alpha_1 \in \Omega^k(U)$, $\alpha_2 \in \Omega^*(U)$,
(iii) $d^2 = 0$,
(iv) $d$ agrees with the usual differential on $\Omega^0(U)$.

Furthermore, for all $\alpha \in \Omega^*(U)$ we have $\text{supp}(d\alpha) \subset \text{supp}(\alpha)$.

The fact that $d$ does not increase supports means that $d$ is local.

(19.26) $d$ commutes with pullbacks. In the multivariable notes we also prove that $d$ commutes with pullbacks under smooth maps. So if $A'$ is an affine space, $U' \subset A'$ an open subset, and $\varphi: U' \rightarrow A'$ a smooth map with $\varphi(U') \subset U$, then for all $\alpha \in \Omega^*(U)$ we have

\[
\varphi^* d\alpha = d\varphi^* \alpha.
\]

The locality of $d$ and the fact that $d$ commutes with diffeomorphisms are key ingredients in the proof of the following.

**Theorem 19.28.** Let $X$ be a smooth manifold. Then there exists a unique map $d: \Omega^*(X) \rightarrow \Omega^*(X)$ of degree $+1$ such that

(i) $d$ is linear,
(ii) $d(\alpha_1 \wedge \alpha_2) = d\alpha_1 \wedge \alpha_2 + (-1)^k \alpha_1 \wedge d\alpha_2$, \hspace{1cm} $\alpha_1 \in \Omega^k(X)$, $\alpha_2 \in \Omega^*(X)$,
(iii) $d^2 = 0$,
(iv) $d$ agrees with the usual differential on $\Omega^0(X)$.

Furthermore, for all $\alpha \in \Omega^\bullet(X)$ we have $\text{supp}(d\alpha) \subset \text{supp}(\alpha)$. Also, if $\varphi : X' \to X$ is a smooth map of manifolds, and $\alpha \in \Omega^\bullet(X)$, then

$$
\varphi^* d\alpha = d\varphi^* \alpha.
$$

**Proof.** Let $\{(U_i, x_i)\}_{i \in I}$ be a cover of $X$ by coordinate charts, and choose a partition of unity $\{\rho_i\}_{i \in I}$ subordinate to the cover. For any $\alpha \in \Omega^\bullet(X)$ we have $\text{supp}(\rho_i \alpha) \subset U_i$ for all $i$. Define

$$
d\alpha = \sum_{i \in I} d((x_i^{-1})^*(\rho_i \alpha)),
$$

where the differential of $(x_i^{-1})^*(\rho_i \alpha) \in \Omega^\bullet(x_i(U_i))$ is the one determined uniquely on affine space in Theorem 19.25. Since $\alpha = \sum_{i \in I} \rho_i \alpha$ and since $d$ is assumed linear in (i), the definition (19.30) is uniquely determined. But it remains to prove that it is independent of the choice of cover and partition of unity.

Suppose $\{(V_a, y_a)\}_{a \in A}$ is another cover by coordinate charts, and $\{\sigma_a\}_{a \in A}$ is a subordinate partition of unity. Then

$$
\sum_{i \in I} d((x_i^{-1})^*(\rho_i \alpha)) = \sum_{i \in I} \sum_{a \in A} d((x_i^{-1})^*(\rho_i \sigma_a \alpha))
\]

$$
\]

$$
(19.31)
$$

Notice that the form $\rho_i \sigma_a \alpha = \sigma_a \rho_i \alpha$ has support in $U_i \cap V_a$. The passage from the first to the second line is (19.29) applied to $\varphi = y_a \circ x_i^{-1} : x_i(U_i \cap V_a) \to y_a(U_i \cap V_a)$. \qed
Orientations on manifolds

(19.32) Vector spaces and affine spaces. In lecture we discussed orientation and volume forms in finite dimensional real vector spaces and in affine spaces over them. Recall that if \( V \) is a finite dimensional real vector space, then an orientation of \( V \) is \( \sigma \in \pi_0(\text{Det} V \setminus \{0\}) \), a choice of path component in the determinant line of \( V \) minus the zero vector. An orientation of \( V \) determines an orientation of \( V^* \) and vice versa; use the duality between \( \text{Det} V^* \) and \( \text{Det} V \). A volume form on \( V \) is a nonzero element \( \omega \in \text{Det} V^* \).

![Figure 70. The two orientations of a finite dimensional real vector space \( V \)](image-url)

(19.33) Orientation of a smooth manifold. Let \( X \) be a smooth manifold. Carry out the construction indicated in Figure 70 on each tangent space of \( X \), and use local trivializations of the tangent bundle \( TX \to X \) to construct a fiber bundle

\[
\hat{X} \to X
\]

whose fiber at \( p \in X \) is \( \pi_0(\text{Det} T_pX \setminus \{0\}) \). Then (19.34) is a double cover, called the orientation double cover of \( X \).

Definition 19.35. Let \( X \) be a smooth manifold of dimension \( n \).

1. An orientation \( \sigma \) of \( X \) is a section of the orientation double cover.
2. \( X \) is orientable if a section of the orientation double cover exists.
3. A volume form on \( X \) is a nowhere vanishing \( n \)-form \( \omega \in \Omega^n(X) \).

An orientation is data; orientability is a condition. A volume form induces an orientation.

Example 19.36 (\( \mathbb{R}P^2 \) is not orientable). We sketch the argument in Figure 71. Represent \( \mathbb{R}P^2 \) as the unit disk \( D^2 \subset \mathbb{A}^2 \) with antipodal points of the boundary \( \partial D^2 = S^1 \) identified. The blue line segment pictured is an embedded \( \mathbb{R}P^1 \subset \mathbb{R}P^2 \), which is geometrically a circle. Suppose the pictured basis \( e_1, e_2 \) at the left endpoint of \( \mathbb{R}P^1 \) is positively oriented. Then transporting it along \( \mathbb{R}P^1 \) we must get positively oriented bases. By the time we end up at the right endpoint we deduce that the pictured basis \( \tilde{e}_1, \tilde{e}_2 \) is also positively oriented. However, under the differential of the antipodal map we have \( \tilde{e}_1 = e_1 \) and \( \tilde{e}_2 = -e_2 \), from which \( \tilde{e}_1 \wedge \tilde{e}_2 = -e_1 \wedge e_2 \), and hence the two bases are oppositely oriented. This contradiction shows that \( \mathbb{R}P^2 \) is not orientable.
Definition 19.38. Let \( X \) be an oriented manifold. A standard chart \((U; x^1, \ldots, x^n)\) with values in \( \mathbb{A}^n \) is \textit{oriented} if for each \( p \in U \) the basis \( \partial/\partial x^1 \big|_p, \ldots, \partial/\partial x^n \big|_p \) is a positively oriented basis of \( T_p X \).

If \( U \) is connected and \( n \geq 1 \), then if the standard chart \((U; x^1, \ldots, x^n)\) is not oriented, then the standard chart \((U; -x^1, x^2, \ldots, x^n)\) is oriented.

Proposition 19.39. Let \( X \) be an oriented manifold and suppose \((U; x^1, \ldots, x^n), (V; y^1, \ldots, y^n)\) are oriented charts. Then

\[
\text{det} \left( \frac{\partial x^i}{\partial y^a} \right)_{1 \leq i, a \leq n} > 0
\]

on \( U \cap V \).

Here we write \( x^i = x^i(y^1, \ldots, y^n) \); the matrix in (19.40) is the differential of the coordinate change.

Proof. Let \( J: U \cap V \to \mathbb{R} \) be the function on the left hand side of (19.40). Then

\[
dx^1 \wedge \cdots \wedge dx^n = J dy^1 \wedge \cdots \wedge dy^n,
\]

and since both \( dx^1(p), \ldots, dx^n(p) \) and \( dy^1(p), \ldots, dy^n(p) \) are positively oriented bases of \( T_p^* X \), it follows that \( J(p) > 0 \). \( \square \)

Lecture 24: Integration on manifolds

In the previous lecture we passed from differential calculus on affine space to differential calculus on smooth manifolds—that is, differential calculus of differential forms. In this lecture we do the
same for integral calculus. The natural objects to integrate on manifolds are densities, which are twisted differential forms. In the presence of an orientation on a manifold, densities are canonically equivalent to differential forms, and it is the integration of differential forms over oriented manifolds that we treat here. (It is a small variation to treat densities.)

The key is the change of variables formula for the integral in affine space, which we review first. The globalization of the integral is parallel to the globalization of the differential (Theorem 19.28). We prove Stokes’ theorem, which is a generalization of the fundamental theorem of calculus that relates integration and differentiation.

I recommend Lecture 19 of the multivariable analysis notes for motivational material on why it is differential forms that are natural geometric objects to integrate. That lecture also contains a rigorous treatment of the integral of a 1-form over a 1-dimensional manifold.

Change of variables formula in flat space

**Theorem 24.1** (change of variables). Let $U, U' \subset \mathbb{A}^n$ be open sets and $f: U \to \mathbb{R}$ a bounded function of compact support which is integrable. Suppose $\varphi: U' \to U$ is a $C^1$ diffeomorphism. Then $\varphi^* f: U' \to \mathbb{R}$ is integrable and

\begin{equation}
\int_U f = \int_{U'} \varphi^* f |\det d\varphi|.
\end{equation}

The determinant factor is the continuous function which is the composition

\begin{equation}
U' \xrightarrow{d\varphi} \text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \xrightarrow{\det} \mathbb{R} \xrightarrow{|\cdot|} \mathbb{R}
\end{equation}

It tells the instantaneous stretching factor on volumes of the diffeomorphism $\varphi$.

**Example 24.4.** The special case $n = 1$ of (24.2) is not the usual change of variables formula you first learned in calculus, which is for definite integrals. Thus if into an integral you substitute

\begin{equation}
x = -2y \\
dx = -2dy
\end{equation}

you obtain formulas such as

\begin{equation}
\int_2^4 x^2 \, dx = \int_{-1}^{-2} (4y^2)(-2dy).
\end{equation}

On the other hand, Theorem 24.1 addresses the integral over a subset, which here we apply to a closed subset, to obtain instead

\begin{equation}
\int_{[2,4]} x^2 \, |dx| = \int_{[-2,-1]} 4y^2 \, 2|dy|.
\end{equation}

In (24.2) we did not write the standard measure on $\mathbb{A}^n$, which here we render in standard affine coordinates $x^1, \ldots, x^n$ as $|dx^1 \cdots dx^n|$. The absolute value is consonant with the change of variables formula.
Remark 24.8. The change of variables formula (24.6) treats $x^2\,dx$ as a differential 1-form on $[2, 4]$, whereas (24.7) is for the integral of the density $x^2|dx|$. The integration theory discussed up to now is for densities, not differential forms. We tell a bit about integration of differential forms below, but for completeness we first define densities.

**Interlude: densities.** Let $V$ be an $n$-dimensional real vector space and $\mathcal{B}(V)$ the set of bases, i.e., isomorphisms $\mathbb{R}^n \to V$.

**Definition 24.10.** The line of densities of $V$ is

$$|\text{Det } V^*| = \{\mu : \mathcal{B}(V) \to \mathbb{R} : \mu(b \cdot g) = |\det g|\mu(b) \text{ for all } b \in \mathcal{B}(V), \ g \in \text{GL}_n\mathbb{R}\}.$$  

Let $|\text{Det } V^*|_+ \subset |\text{Det } V^*|$ be the ray of positive functions. A (positive) density is an element of $|\text{Det } V^*|_+$.  

Recall that $\mathcal{B}(V)$ is a right $\text{GL}_n\mathbb{R}$-torsor, that is, the group of invertible $n \times n$ matrices acts simply transitively on $\mathcal{B}(V)$ by right composition. A density $\mu$ is a volume function on parallelepipeds in $V$. If $A$ is an affine space over $V$, then $\mu$ defines a translation-invariant density on $A$, in particular a translation-invariant volume function on parallelepipeds in $A$.

More generally, we can consider variable densities

$$\mu : U \to |\text{Det } V^*|$$

defined on an open set $U \subset A$. The product of a function and a density is a density, so for example in (24.7), the integrand $x^2|dx|$ is a variable density on $[2, 4]$.

(24.13) **The integral on functions.** Let $U \subset \mathbb{A}^n$ be an open subset, and let $\Omega^0_c(U)$ denote the vector space of compactly supported smooth functions. The integral in (24.2) is a map

$$\int_U : \Omega^0_c(U) \to \mathbb{R}$$

$$f \mapsto \int_U f$$

which (i) is linear, and (ii) satisfies the change of variables formula (24.2).

(24.15) **The integral on $n$-forms.** On standard flat space we use the isomorphism

$$\Omega^0_c(U) \xrightarrow{\cong} \Omega^n_c(U)$$

$$f \mapsto \omega_f = f \,dx^1 \wedge \cdots \wedge dx^n$$
to port the integral (24.14) of functions to an integral

\[
\int_U : \Omega^n_c(U) \rightarrow \mathbb{R}
\]

(24.17)

\[\omega_f \mapsto \int_U f\]

of n-forms. Note that under the diffeomorphism \( \varphi : U' \rightarrow U \) in Theorem 24.1 we have

(24.18)

\[\varphi^* \omega_f = \det d\varphi \cdot \omega_{\varphi^*f}.\]

Therefore, the integral (24.17) (i) is linear, and (ii) satisfies the change of variables formula

(24.19)

\[\int_U \omega = \int_{U'} \varphi^* \omega\]

if \( \varphi \) is orientation-preserving so that \( \det d\varphi_{p'} > 0 \) for all \( p' \in U' \).

Remark 24.20. The change of variables formula (24.19) can be used to define the integral on open sets of any oriented finite dimensional affine space (not equipped with a choice of affine coordinates).

Integration of differential forms on oriented manifolds

We globalize the integral (24.17) from affine space to smooth manifolds equipped with an orientation.

**Theorem 24.21.** Let \( X \) be an oriented manifold. Then there exists a unique linear map

(24.22)

\[
\int_X : \Omega^n_c(X) \rightarrow \mathbb{R}
\]

such that if \((U, x)\) is an oriented chart and \( \omega \in \Omega^n_c(U) \), then

(24.23)

\[\int_X \omega = \int_{x(U)} (x^{-1})^* \omega.\]

We illustrate (24.23) in Figure 72.

**Proof.** Let \( \{(U_i, x_i)\}_{i \in I} \) be a cover of \( X \) by oriented charts, and let \( \{\rho_i\} \) be a subordinate partition of unity. Then \( \omega = \sum_{i \in I} (\rho_i \omega) \), and \( \text{supp}(\rho_i \omega) \subset \text{supp}(\rho_i) \cap \text{supp}(\omega) \) is a compact subset of \( U_i \). The integral of \( \rho_i \omega \) is determined by (24.23), so we must have

(24.24)

\[\int_X \omega = \sum_i \int_{x_i(U_i)} (x_i^{-1})^* (\rho_i \omega).\]
This proves uniqueness, and for existence we define (24.22) and check it is independent of choices. Thus let \( \{ (V_a, y_a) \}_{a \in A} \) be another cover of \( X \) by oriented charts and \( \{ \sigma_a \}_{a \in A} \) a subordinate partition of unity. Then

\[
\int_X \omega = \sum_i \int_{x_i(U_i)} (x_i^{-1})^*(\rho_i \omega) \\
= \sum_i \sum_a \int_{x_i(U_i \cap V_a)} (x_i^{-1})^*(\rho_i \sigma_a \omega) \\
= \sum_a \sum_i \int_{y_a(U_i \cap V_a)} (x_i \circ y_a^{-1})^*(x_i^{-1})^*(\rho_i \sigma_a \omega) \\
= \sum_a \sum_i \int_{y_a(U_i \cap V_a)} (y_a^{-1})^*(\rho_i \sigma_a \omega) \\
= \sum_a \int_{y_a(V_a)} (y_a^{-1})^*(\sigma_a \omega)
\]

as desired. The passage to the third line is the change of variables formula (24.19), which applies since all charts are positively oriented.

\[\quad\]

**Proposition 24.26.** The integral (24.22) satisfies the following properties.

1. Let \(-X\) denote the oppositely oriented manifold to \( X \). Then

\[
\int_{-X} \omega = -\int_X \omega, \quad \omega \in \Omega_c^n(X).
\]

2. Let \( X' \) be an oriented manifold and \( \varphi: X' \to X \) an orientation-preserving diffeomorphism. Then

\[
\int_{X'} \varphi^* \omega = \int_X \omega, \quad \omega \in \Omega_c^n(X).
\]

**Example 24.29.** Let \( S^2 \subset \mathbb{R}^3_{x,y,z} \) be the standard unit sphere. For now we assume an orientation; below we discuss a canonical orientation of the boundary of an oriented manifold with boundary, and so \( S^2 \) inherits an orientation by virtue of being the boundary of the closed unit ball \( D^3 \). Define

\[
\omega = x \, dy \wedge dz + y \, dx \wedge dz + z \, dx \wedge dy.
\]
Then $\omega \in \Omega^3(\mathbb{R}^3)$, and it restricts to an element of $\Omega^2(S^2)$, which necessarily has compact support since $S^2$ is compact. To integrate it we may omit a set of measure zero from $S^2$ and parametrize the complement. That we do via the parametrization

\begin{equation}
\varphi: (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3
\end{equation}

\begin{equation}
\phi, \theta \to (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)
\end{equation}

This embeds an open rectangle in $\mathbb{R}^2_{\phi,\theta}$ into the unit sphere in $\mathbb{R}^3$. To compute we write

\begin{align}
x &= \sin \phi \cos \theta \\
y &= \sin \phi \sin \theta \\
z &= \cos \phi
\end{align}

and then apply $d$:

\begin{align}
dx &= \cos \phi \cos \theta \, d\phi - \sin \phi \sin \theta \, d\theta \\
dy &= \cos \phi \sin \theta \, d\phi + \sin \phi \cos \theta \, d\theta \\
dz &= -\sin \phi \, d\phi.
\end{align}

These are identities among functions and 1-forms on $S^2$. Substitute (24.32) and (24.33) into (24.30) and use the rules of exterior algebra to deduce

\begin{equation}
\varphi^* \alpha = \sin \phi \, d\phi \wedge d\theta.
\end{equation}

(In computations it is customary to omit ‘$\varphi^*$’ in (24.34).) Now apply (24.23) and (24.17) to compute

\begin{equation}
\int_{S^2} \omega = \int_{(0, \pi) \times (0, 2\pi)} \sin \phi \, d\phi \wedge d\theta = \int_0^\pi \, d\phi \int_0^{2\pi} \sin \phi \, d\theta = 4\pi,
\end{equation}

where in the second line we use Fubini’s theorem.

The boundary orientation

(24.36) **Quotient Before Sub.** Suppose

\begin{equation}
0 \to V' \xrightarrow{i} V \xrightarrow{j} V'' \to 0
\end{equation}
is a short exact sequence of finite dimensional vector spaces. It is elementary to prove

\[(24.38)\quad \dim V'' + \dim V' = \dim V.\]

The corresponding statement for determinant lines is a canonical isomorphism\(^{19}\)

\[(24.39)\quad \text{Det } V'' \otimes \text{Det } V' \longrightarrow \text{Det } V.\]

We define (24.39) in terms of three choices:

\[(24.40)\quad e'_1, \ldots, e'_k \quad \text{basis of } V'\]

\[(24.40)\quad e''_1, \ldots, e''_\ell \quad \text{basis of } V''\]

\[(24.40)\quad \tilde{e}'_1, \ldots, \tilde{e}'_\ell \quad \text{lifts of the } e_j \text{ to } V\]

Then \(e''_1, \ldots, \tilde{e}'_\ell, e'_1, \ldots, e'_k\) is a basis of \(V\), and we define (24.39) on basis elements:

\[(24.41)\quad (e''_1 \wedge \cdots \wedge e''_\ell) \otimes (e'_1 \wedge \cdots \wedge e'_k) \longmapsto \tilde{e}'_1 \wedge \cdots \wedge \tilde{e}'_\ell \wedge i(e'_1) \wedge \cdots \wedge i(e'_k)\]

The mnemonic “Quotient Before Sub” comes from the ordering in (24.41), which by experience is a convention which gives nice formulas in many situations.

\[(24.42)\quad 2 \text{ out of 3. Recall that an orientation of a finite dimensional real vector space is an orientation of its determinant line. Suppose two out of the three vector spaces } V', V'', V \text{ are equipped with an orientation. Then the isomorphism (24.39) can be used to orient the remaining vector space.}\]

\[\text{Figure 73. Boundary orientation in a standard boundary chart}\]

\[\text{19One can combine (24.38) and (24.39) into an isomorphism of } \mathbb{Z}\text{-graded lines. Also, there is a generalization for not-short but finite length exact sequences: the alternating product of the (}\mathbb{Z}\text{-graded) determinant lines has a canonical nonzero element, which is the determinant of the exact sequence.}\]
The induced boundary orientation. Suppose $X$ is an oriented manifold with boundary. Then at $p \in \partial X$ we have the short exact sequence

$$0 \rightarrow T_p(\partial X) \rightarrow T_pX \rightarrow \nu_p \rightarrow 0$$

where $\nu_p$ is the normal line to the boundary. In this exact sequence $\nu_p$ is oriented by the outward normal; see (13.15). Since $T_pX$ is oriented, by the 2-out-of-3 rule there is an induced orientation of $T_p(\partial X)$. This construction proves that the boundary of an orientable manifold is orientable, and furthermore gives a canonical orientation of the boundary of an oriented manifold. Figure 73 illustrates the boundary orientation in a standard boundary chart, which is arranged so that the Quotient Before Sub convention is compatible with the standard orientation of $\mathbb{A}^n$. See (13.19) as well as Remarks 13.20 and 13.21.

Lecture 25: Stokes’ theorem; oriented degree and applications

Stokes’ Theorem

The integral in Theorem 24.21 extends to manifolds with boundary in a straightforward manner. In the following Stokes’ theorem we use the boundary orientation induced on an oriented manifold with boundary; see (24.43).

**Theorem 25.1.** Let $X$ be an oriented $n$-dimensional manifold with boundary, and suppose $\omega \in \Omega^n_{\text{c}}(X)$. Denote the inclusion of the boundary as $i : \partial X \to X$. Then

$$\int_X d\omega = \int_{\partial X} i^* \omega.$$  

**Proof.** Since both sides of (25.2) are linear in $\omega$, in view of the definition (24.24) of the integral we reduce to the case in which supp($\omega$) is contained in the domain $U$ of a single standard chart $(U, x)$. As illustrated in Figure 74 there are two types of charts: Type I contained in Int($X$) and Type II if the intersection with $\partial X$ is nonempty; see Lecture 13. We assume $\omega$ has been transported to $x(U) \subset \mathbb{A}^n$, and write

$$\omega = f_i \, dx^1 \wedge \cdots \wedge dx^{i-1} \wedge df_i^{i+1} \wedge \cdots \wedge dx^n,$$

where $f_i$, $i = 1, \ldots, n$, is a smooth function with compact support in $x(U)$ and the expression is summed over $i$. Then

$$d\omega = \sum_{i=1}^n (-1)^{i-1} \frac{\partial f_i}{\partial x^i} \, dx^1 \wedge \cdots \wedge dx^n.$$
and, substituting $x^1 = 0$ into (25.3),

$$i^* \omega = f_1(0, x^2, \ldots, x^n) \, dx^2 \wedge \cdots \wedge dx^n.$$  \hfill (25.5)

If $(U, x)$ is of Type I, then the left hand side of (25.2) reduces to

$$\sum_{i=1}^{n} \int_{\mathbb{A}^n} (-1)^{i-1} \frac{\partial f_i}{\partial x^i} |dx^1 \cdots dx^n|.$$  \hfill (25.6)

By Fubini we can first integrate the $i^{th}$ term over $x^i$, and that definite integral vanishes by the fundamental theorem of calculus, since $f_i$ has compact support. The right hand side of (25.2) vanishes since $\text{supp}(\omega) \cap \partial X = \emptyset$.

If $(U, x)$ is of Type II, then the left hand side reduces to (25.6), and the argument for $i = 2, \ldots, n$ is as before: those terms vanish. Only the term with $i = 1$ contributes to

$$\int_{x(U)} d\omega = \int_{\mathbb{A}^{n-1}} |dx^2 \cdots dx^n| \int_{-\infty}^{0} dx^1 \frac{\partial f_1}{\partial x^1}$$

$$= \int_{\mathbb{A}^{n-1}} |dx^2 \cdots dx^n| f_1(0, x^2, \ldots, x^n)$$

$$= \int_{\partial x(U)} i^* \omega.$$  \hfill (25.7)

At the last stage we use the fact that the oriented chart $(U; x^1, \ldots, x^n)$ of $X$ restricts to an oriented chart $(U \cap \partial X; x^2, \ldots, x^n)$ of $\partial X$; see Figure 73.  \hfill \Box

**Example 25.8.** We continue Example 24.29. From (24.30) we compute $d\omega = 3 \, dx \wedge dy \wedge dz$. Stokes’ theorem implies

$$\int_{S^2} \omega = \int_{D^3} d\omega = 3 \text{vol}(D^3) = 3\left(\frac{4}{3}\pi\right) = 4\pi.$$  \hfill (25.9)
which agrees with (24.35), as it must. We leave the reader to check that with the standard ori-
entation of \( D^3 \), which we use to identify \( dx \wedge dy \wedge dz \) with the standard density on \( \mathbb{A}^3 \), the parametrization (24.31) is positively oriented.

**Example 25.10.** Theorem 25.1 generalizes the fundamental theorem of calculus, though one cannot derive the latter from the former since the proof reduces the former to the latter. Nonetheless, it is instructive to observe that if \( X = [a, b] \subset \mathbb{A}^1 = \mathbb{R} \) with the standard orientation, and \( f : [a, b] \to \mathbb{R} \) is a smooth function, then

\[
\int_{[a,b]} df = \int_a^b f'(x) \, dx
\]

and

\[
\int_{[a,b]} f = f(b) - f(a);
\]

the equality of (25.11) and (25.12) is the fundamental theorem of calculus. Equation (25.12)
requires additional discussion, which we provide in the next section.

The general Stokes’ Theorem 25.1 generalizes other forms you may have seen: Green theorem in
the plane, Stokes’ theorem for surfaces, Gauss’ theorem in 3-space, etc.

**Orientations and integrals in zero and one dimensions**

(25.13) *The zero-dimensional vector space.* For any finite dimensional real vector space \( V \), the
determinant line is the highest exterior power \( \text{Det} V = \bigwedge^n V, \, n = \text{dim} V \). So, if \( V = 0 \) we have
\( \text{Det} V = \bigwedge^0 V = \mathbb{R} \). The line \( \mathbb{R} \) has a canonical orientation given by the positive real numbers
\( \mathbb{R}^{>0} \subset \mathbb{R}^{\neq 0} \). Hence there is a canonical orientation of \( V = 0 \), as pictured in Figure 75. We call this
canonical orientation ‘+’ and the opposite orientation ‘−’.

![Figure 75. The canonical orientation of \( \mathbb{R} = \text{Det} 0 \), and the boundary orientation of \([0, 1]\)](image)

(25.14) *Orientations on a zero-dimensional manifold.* A 0-manifold \( S \) is a finite or countable set
of points, and by (25.13) an orientation of \( S \) is a function

\[
\phi : S \to \{+, -\}.
\]
Definition 25.16. Let $S$ be a compact 0-manifold with orientation $o$. Then the signed count of $S = \{p_1, \ldots, p_N\}$ is

\[(25.17) \quad \#_s S = \sum_{i=1}^{N} o(p_i).\]

Of course, in the sum we interpret $o(p_i)$ as $+1$ or $-1$.

(25.18) Boundaries of oriented 1-manifolds. Let $X = [0,1]$ be equipped with the standard orientation in which the vector field $\partial/\partial x$ is positively oriented. We compute the induced orientation (24.43) on the boundary $\partial X = \{0,1\}$. At the point $p \in \partial X$ the short exact sequence (24.44) reduces to

\[(25.19) \quad 0 \rightarrow T_p(\partial X) \rightarrow T_p X \rightarrow \nu_p \rightarrow 0.
\]

The induced orientation of the zero vector space subspace is $+$ or $-$ according as the isomorphism $T_p X \rightarrow \nu_p$ preserves or reverses orientation. The vector $\partial/\partial x \in T_p X$ is positively oriented for $p \in \{0,1\}$. At $p = 1$ the outward normal is $\partial/\partial x$, and so the induced orientation on the boundary is $+$, whereas at $p = 0$ the outward normal is $-\partial/\partial x$, and so the induced orientation on the boundary is $-$; see the second drawing in Figure 75.

Theorem 25.20. Let $X$ be a compact oriented 1-manifold with boundary. Then $\#_s \partial X = 0$.

Proof. The Classification Theorem 14.1 of 1-manifolds implies that $X$ is a finite union of circles and closed intervals. The result now follows since $\#_s \partial [0,1] = 0$ by (25.18). \[\square\]

(25.21) Stokes’ theorem on a closed interval. Let $a < b$ be real numbers and let $f : [a,b] \rightarrow \mathbb{R}$ be a smooth function. Stokes’ Theorem 25.1 implies

\[(25.22) \quad \int_{[a,b]} df = \int_{\partial[a,b]} f.
\]

The right hand side is the integral of a function over an oriented 0-manifold, and for that we need a special definition (or give a slightly modified exposition of Theorem 24.21; the existing account does not do well in dimension zero). Namely, we sum the values of the function over the points of the manifold and weight using the sign given by the orientation (25.15). With that understood, and using $df = f'(x) \, dx$, (25.22) reduces to the fundamental theorem of calculus

\[(25.23) \quad \int_{a}^{b} f'(x) \, dx = f(b) - f(a).
\]
Orientation of a Cartesian product

25.24 Direct sum of vector spaces. Let $V', V''$ be finite dimensional oriented real vector spaces. There is an induced direct sum orientation\(^\text{20}\) on $V = V' \oplus V''$. Namely, suppose given

\begin{align*}
&\quad e'_1, \ldots, e'_k \quad \text{basis of } V' \\
&\quad e''_1, \ldots, e''_\ell \quad \text{basis of } V''
\end{align*}

Then define an isomorphism

\begin{equation}
\det V' \otimes \det V'' \to \det V
\end{equation}

by

\begin{equation}
(e'_1 \wedge \cdots \wedge e'_k) \otimes (e''_1 \wedge \cdots \wedge e''_\ell) \mapsto e'_1 \wedge \cdots \wedge e'_k \wedge e''_1 \wedge \cdots \wedge e''_\ell
\end{equation}

We use this isomorphism to induce an orientation of $V$ from orientations of $V'$ and $V''$.

25.28 Cartesian products. If $X', X''$ are smooth manifolds, then the Cartesian product is naturally a smooth manifold (Example 3.8) and at $x' \in X', \ x'' \in X''$ we have

\begin{equation}
T_{(x', x'')} (X' \times X'') = T_{x'} X' \oplus T_{x''} X''.
\end{equation}

Hence if $X', X''$ are oriented, there is an induced orientation on $X' \times X''$.

In particular, if $X$ is any oriented manifold the Cartesian product $[0, 1] \times X$ is an oriented manifold with boundary, and by the discussion in (25.18) we deduce a canonical isomorphism

\begin{equation}
\partial([0, 1] \times X) = \{1\} \times X - \{0\} \times X
\end{equation}

of oriented manifolds, where on the right hand side we implicitly use the + orientation of $\{1\}$ and $\{0\}$.

Oriented degree

25.31 Setup. We resume (15.9) with the addition of orientations. Thus let $X$ be an oriented compact manifold, $Y$ an oriented connected manifold, $f : X \to Y$ a smooth map, and assume $\dim X = \dim Y = n$.

\(^{20}\)We might be tempted to apply (24.36) by writing the direct sum as a short exact sequence. But there are two ways to do so—we can put either $V'$ or $V''$ as the sub—and if $\dim V', \dim V''$ are both odd we get opposite orientations of $V$. So we use the order of the direct sum to define the orientation.
(25.32) The (oriented) degree. At a regular point \( p \in X \) of \( f \), the differential \( df_p : T_pX \to T_{f(p)}Y \) is an isomorphism.

**Definition 25.33.** The local degree of \( f \) at \( p \) is +1 if \( df_p \) is orientation-preserving and is −1 if \( df_p \) is orientation-reversing.

![Figure 76. The local degree \( \deg_p f \) of \( f \) at \( p \)](image)

If \( q \in Y \) is a regular value, define

\[
(25.34) \quad \deg f = \sum_{p \in f^{-1}(q)} \deg_p f.
\]

The right hand side depends on \( q \), whereas the left hand side purports not to. The following summarizes the basic properties of the degree.

**Proposition 25.35.**

1. The right hand side of (25.34) is independent of the regular value \( q \).
2. If \( f : [0, 1] \times X \to Y \) is a smooth homotopy, then \( \deg f_0 = \deg f_1 \).
3. If \( W \) is a compact oriented manifold with boundary and \( F : W \to Y \) is a smooth map, then \( \deg \partial F = 0 \).
4. If \( X \to Y \to Z \) is a sequence of smooth maps, where \( X, Y \) are compact, \( Y, Z \) are connected, and \( \dim X = \dim Y = \dim Z \), then \( \deg (g \circ f) = (\deg g)(\deg f) \).

The proof depends on yet more constructions with orientations, which we defer to the next lecture. Namely, if \( W, Y \) are oriented manifold, \( Z \subset Y \) an oriented submanifold, and \( F : W \to Y \) a smooth map transverse to \( Z \), then the submanifold \( F^{-1}(Z) \subset W \) is orientable and in fact has a canonical orientation. Furthermore, if \( \dim W = \dim Y \) and \( Z = \{q\} \subset Y \) is a single point, then the induced orientation at \( p \in F^{-1}(q) \) agrees with the local degree \( \deg_p F \). We also implicitly use a comparison of the boundary orientation of a transverse inverse image and the orientation of the transverse inverse image of a boundary, another fact we treat in the next lecture. We suggest the reader to refer back to (15.12) and (17.7).

**Proof.** Let \( t \mapsto q_t \) be a smooth path in \( Y \) between regular values \( q_0 \) and \( q_1 \). As in Figure 49 form the map \( \text{id}_{[0, 1]} \times f : [0, 1] \times X \to [0, 1] \times Y \) and let \( Z \subset [0, 1] \times Y \) be the graph of \( t \mapsto q_t \). Use Theorem 17.2 to perturb \( \text{id}_{[0, 1]} \times f \) to a map \( F \) which is transverse to \( Z \) and which agrees with \( f \) on the boundary. Then \( S := F^{-1}(Z) \) is a 1-dimensional submanifold of \([0, 1] \times X \) and
\( \partial S = \{0\} \times f^{-1}(q_0) \cup \{1\} \times f^{-1}(q_1) \). There is a canonical orientation on \( S \), and the induced boundary orientation agrees with the local degree of \( f \), up to sign. Now

\[
(25.36) \quad \sum_{p \in f^{-1}(q_1)} \deg_p f = \sum_{p \in f^{-1}(q_1)} \deg_p f
\]

follows from Theorem 25.20.

The proof of (2) is the same as that of Theorem 15.13(2), except that we include orientations and use Theorem 25.20 in place of Corollary 14.3.

The argument for (3) follows that for Proposition 17.9. Let \( q \in Y \) be a simultaneous regular value of \( F, \partial F \). Then \( F^{-1}(q) \subset W \) is a compact 1-dimensional submanifold with \( \partial F^{-1}(q) = F^{-1}(q) \cap \partial W \). It inherits an orientation. Now apply Theorem 25.20.

For (4), let \( r \in Z \) be a simultaneous regular value of \( g, g \circ f \). Then all points of \( g^{-1}(r) \) are regular values of \( f \). Now for \( p \in (g \circ f)^{-1}(q) \) we have \( d(g \circ f)_p = dg_{f(p)} \circ df_p \), and so \( \deg_p (g \circ f) = \deg_{f(p)}(g) \cdot \deg_p(f) \). Hence

\[
(25.37) \quad \deg (g \circ f) = \sum_{p \in (g \circ f)^{-1}(r)} \deg_p (g \circ f) \\
= \sum_{p \in (g \circ f)^{-1}(r)} \deg_{f(p)}(g) \cdot \deg_p(f) \\
= \sum_{q \in g^{-1}(r)} \deg_q(g) \sum_{p \in f^{-1}(q)} \deg_p(f) \\
= (\deg g)(\deg f).
\]

Applications

\( (25.38) \) The degree of the antipodal map. Let \( S^n \subset \mathbb{R}^{n+1} \) be the unit sphere and \( \alpha : S^n \to S^n \) the antipodal map. Note that \( \alpha \) is the restriction of the map \(-1 : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \).

Theorem 25.39. \( \alpha \) is not homotopic to the identity map if \( n \) is even.

Proof. We have \( \deg(\text{id}_{S^n}) = 1 \), as it is for any identity map. We claim \( \deg(\alpha) = (-1)^{n+1} \); then the theorem follows immediately from Proposition 25.35(2). For the claim, take \( q = (-1,0,\ldots,0) \), so that \( f^{-1}(q) \) consists of the single point \( p = (+1,0,\ldots,0) \). Orient \( S^n \) as the boundary of the closed unit ball \( D^{n+1} \). Then both \( T_p S^n \) and \( T_q S^n \) are identified with the subspace \( \mathbb{R}^n \subset \mathbb{R}^{n+1} \) of vectors with first coordinate zero. However, the natural isomorphism \( \mathbb{R}^n \to T_p S^n \) is orientation-preserving, whereas the natural isomorphism \( \mathbb{R}^n \to T_q S^n \) is orientation-reversing. The differential \( d\alpha_p : T_p S^n \to T_q S^n \) is \(-1 : \mathbb{R}^n \to \mathbb{R}^n \) under these isomorphisms, which is orientation-preserving if \( n \) is even. The degree computation follows.\[ \Box \]
(25.40) The hairy ball theorem. If \( p \in S^n \), then
\[
T_p S^n = \{ \xi \in \mathbb{R}^{n+1} : \langle p, \xi \rangle = 0 \},
\]
which is obtained by differentiating the defining equation \( \langle p, p \rangle = 1 \) of \( S^n \subset \mathbb{R}^{n+1} \). If \( n \) is odd, then
\[
\xi_p = (x^2, -x^1, x^4, -x^3, \ldots), \quad p = (x^1, x^2, \ldots, x^{n+1}) \in S^n,
\]
is a nowhere vanishing vector field on \( S^n \).

**Corollary 25.43.** If \( n \) is even, then \( S^n \) does not admit a nowhere vanishing vector field.

**Proof.** If \( \xi \) is a nowhere vanishing vector field on \( S^n \), define
\[
f_t(p) = \cos(t)p + \sin(t)\xi_p, \quad t \in [0, 1].
\]
This is a smooth homotopy from \( f_0 = \text{id}_{S^n} \) to \( f_1 = \alpha \), which contradicts Theorem 25.39. \( \square \)

(25.45) Real projective space. If \( n \) is odd, then the antipodal map \( \alpha : S^n \to S^n \) is orientation-preserving. It is the deck transformation of the double cover \( \pi : S^n \to \mathbb{RP}^n \), and so an orientation on \( S^n \) induces an orientation of the quotient \( \mathbb{RP}^n \).

**Corollary 25.46.** If \( n \) is even, then \( \mathbb{RP}^n \) is not orientable.

**Proof.** Assume \( \mathbb{RP}^n \) is oriented. Then from the commutative diagram
\[
\begin{array}{ccc}
S^n & \xrightarrow{\alpha} & S^n \\
\pi \downarrow & & \downarrow \pi \\
\mathbb{RP}^n & \xrightarrow{\text{id}_{\mathbb{RP}^n}} & \mathbb{RP}^n
\end{array}
\]
and Proposition 25.35(4) we deduce
\[
\deg(\pi) \deg(\alpha) = \deg(\text{id}) \deg(\pi),
\]
which contradicts \( \deg(\alpha) = -1 \). \( \square \)
Lecture 26: Preimage orientation; oriented degree and differential forms

Orientation of a transverse preimage

(26.1) Setup. Let $X, Y$ be oriented manifolds, $Z \subset Y$ an oriented submanifold, and $f: X \to Y$ a map such that $f \pitchfork Z$. Set $S := f^{-1}(Z) \subset X$.

(26.2) Induced orientation. In this situation the manifold $S$ is orientable, and in fact it carries a canonical orientation. We use the “2 out of 3” rule (24.42), following “Quotient Before Sub” (24.36), applied twice to the short exact sequence (7.9) of tangent spaces. Namely, for $p \in S$ we have the commutative diagram

$$
\begin{array}{cccccc}
0 & \to & T_p S & \to & T_p X & \to & \nu_p(S \subset X) & \to & 0 \\
\downarrow d_p & & \downarrow d_f & & \cong & & \downarrow d_f \\
0 & \to & T_{f(p)} Z & \to & T_{f(p)} Y & \to & \nu_{f(p)}(Z \subset Y) & \to & 0
\end{array}
$$

(26.3)

in which the rows are short exact sequences. The differential $df_p: T_p X \to T_{f(p)} Y$ induces the left vertical map on subspaces and therefore the right vertical map on the quotients. The latter is an isomorphism since $f \pitchfork_p Z$.

The induced orientation on $T_p S$ is obtained by a 3-step procedure:

1. Use the orientations of $Z$ and $Y$ to induce an orientation of $\nu_{f(p)}(Z \subset Y)$.
2. Use the right vertical isomorphism to transport that orientation to $\nu_p(S \subset X)$.
3. Combine with the orientation of $X$ to induce an orientation of $T_p S$.

In steps (1) and (3) we deploy the isomorphism (24.39) to induce an orientation.

Remark 26.4. The same procedure works if $X$ is a manifold with boundary and $\partial f \pitchfork Z$ (as well as $f \pitchfork Z$). In this case there are two natural orientations of $\partial S$, which we compare below.

![Figure 78. Orientation of a transverse preimage](image_url)

Example 26.5. Let $X \subset \mathbb{R}^2_{x,y}$ be the closed unit disk, $Y = \mathbb{R}$, $Z = \{0\} \subset \mathbb{R}$, and $f: X \to \mathbb{R}$ the function $f(x, y) = y$. Then $S := f^{-1}(0)$ is a diameter of the disk. Use the standard orientation
of $\mathbb{A}^2$ in which $\partial/\partial x, \partial/\partial y$ is an oriented basis. The real line $\mathbb{R}$ is oriented as usual, and we take the + orientation of the point $Z = \{0\} \subset \mathbb{R}$. In this case the bottom row of (26.3) degenerates: the sub is the zero vector space and the quotient map is an isomorphism. Hence the quotient is canonically isomorphic to $T_{f(p)}\mathbb{R}$, and since we use the + orientation on the sub the induced orientation (1) of the quotient agrees with that of the ambient space: it is the usual orientation of $\mathbb{R}$. In step (2) the normal to $S$ in $X$ has positive orientation pointing up in the figure. In step (3) a vector pointing up is the first vector of an oriented basis of $X$, and so we see that $-\partial/\partial x$ is positively oriented on $S$.

The induced boundary orientation (24.43) of $\partial S$ is $+$ at the left endpoint $(-1,0)$ and $-$ at the right endpoint $(+1,0)$. We can also orient $\partial S$ as the transverse inverse image of $Z \subset Y$ via the map $\partial f: \partial X \to Y$. Thus consider the 3-step procedure at $p = (-1,0)$. Each sub in (26.3) is the zero vector space. Steps (1) and (2) are the same as in the previous paragraph. Then at $p$ the vector $\partial/\partial y$ is positively oriented in the normal space to $p$ in $\partial X$ but is negatively oriented in the tangent space to $\partial X$. Hence the induced orientation on $\partial S$ at $p$ is $-$. We leave the reader to check that the induced orientation of $\partial S$ at $(+1,0)$ is $+$. Observe that the two natural induced orientations of $\partial S$ are opposite in this case.

(26.6) Comparison of boundary orientations. In some applications it is only important to know that the two orientations of $\partial S$ are equal or opposite; the precise sign does not matter. Nonetheless, we compute it.

Proposition 26.7. Let $X$ be an oriented manifold with boundary, $Y$ an oriented manifold, $Z \subset Y$ an oriented submanifold, and $f: X \to Y$ a map such that $f, \partial f \cap_{Z} Y$. Then as oriented manifolds,

\begin{equation}
\partial[f^{-1}(Z)] = (-1)^{\text{codim}(Z \subset Y)}(\partial f)^{-1}(Z).
\end{equation}

The left hand side uses the boundary orientation whereas the right hand side uses the inverse image orientation.

Proof. Set $S = f^{-1}(Z)$. We compute at $p \in \partial S$, but for convenience leave off the point in the notation. We have the exact sequences

\begin{equation}
\begin{align*}
0 \to T(\partial S) & \to TS & \to \nu(\partial S \subset S) & \to 0 \\
0 \to T(\partial X) & \to TX & \to \nu(\partial X \subset X) & \to 0 \\
0 \to TS & \to TX & \to \nu(S \subset X) & \to 0 \\
0 \to T(\partial S) & \to T(\partial X) & \to \nu(\partial S \subset \partial X) & \to 0
\end{align*}
\end{equation}

Let $b$ denote the boundary orientation and $ii$ the inverse image orientation. The dual space to a line $L$ is denoted $L^{-1}$. Repeatedly apply (24.39) with care about the order, which encodes
orientations, and use the convention that \(-L\) is the oppositely oriented line to an oriented line:

\[
\text{Det } T(\partial S)^{(b)} \cong \nu(\partial S \subset S)^{-1} \otimes \text{Det } TS \\
\cong \nu(\partial X \subset X)^{-1} \otimes \text{Det } TS \\
\cong \nu(\partial X \subset X)^{-1} \otimes \text{Det } \nu(S \subset X)^{-1} \otimes \text{Det } TX \\
\cong \nu(\partial X \subset X)^{-1} \otimes \text{Det } \nu(Z \subset Y)^{-1} \otimes \text{Det } TX \\
\cong \nu(\partial X \subset X)^{-1} \otimes \text{Det } \nu(Z \subset Y)^{-1} \otimes \text{Det } T(\partial X) \\
\cong (-1)^{\text{codim}(Z \subset Y)} \text{Det } \nu(Z \subset Y)^{-1} \otimes \text{Det } T(\partial X) \\
\cong (-1)^{\text{codim}(Z \subset Y)} \text{Det } T(\partial S)^{(ii)}.
\]

Each isomorphism preserves orientation. The transversality \(f \widehat{\Delta} Z\) is used to pass from line 3 to line 4, and the transversality \(\partial f \widehat{\Delta} Z\) is used to pass from line 6 to line 7.

\[\square\]

Transitivity of diffeomorphisms on a connected manifold

As preparation for proving a relationship between degrees and integration of differential forms, we prove that on a connected manifold we can move any point to any other via a diffeomorphism. In fact, the statement is stronger: the diffeomorphism can be chosen to be smoothly homotopic to the identity map and to be the identity outside a compact set.

**Definition 26.11.** Let \(Y\) be a smooth manifold.

1. The *support* of a diffeomorphism \(\psi: Y \to Y\) is

\[
\text{supp } \psi = \{ y \in Y : \psi(y) \neq y \}.
\]

2. A smooth homotopy \(\varphi: [0, 1] \times Y \to Y\) is an *isotopy* if each \(\varphi_t: Y \to Y\) is a diffeomorphism.
3. An isotopy \(\varphi: [0, 1] \times Y \to Y\) has *compact support* if there exists a compact subset \(K \subset Y\) such that \(\text{supp } \varphi_t \subset K\) for all \(t \in [0, 1]\).

**Theorem 26.13.** Let \(Y\) be a smooth connected manifold and \(q_0, q_1 \in Y\). Then there exists a compactly supported isotopy \(\varphi: [0, 1] \times Y \to Y\) such that \(\varphi_0 = \text{id}_Y\) and \(\varphi_1(q_0) = q_1\).

**Proof.** Define a relation \(\sim\) on \(Y\) by letting \(q_0 \sim q_1\) if there exists a compactly supported isotopy from the identity to a diffeomorphism which maps \(q_0\) to \(q_1\). Then \(\sim\) is an equivalence relation.\(^\text{21}\)

We claim that each equivalence class is open. If so, then its complement is a union of open sets, so each equivalence class is also closed. It follows that there is a single equivalence class, since \(Y\) is connected, and that proves the theorem.

---

\(^\text{21}\)Reflexivity and symmetry are immediate. For transitivity one needs to glue isotopies \(\varphi: [0, 1] \times Y \to Y\) and \(\psi: [1, 2] \times Y \to Y\) along \([1] \times Y\). While continuity of the glued function is assured, smoothness is not. To execute such gluings one can employ smooth cutoff functions on the time intervals to reparametrize \(\varphi\) and \(\psi\) so that they are constant on \((0, 0.9]\) and \([1, 1.1)\), respectively. Then the glued isotopy is smooth. The same device is used to glue smooth homotopies in general, and so prove that smooth homotopy is an equivalence relation.
For the claim it suffices to work locally in a coordinate chart, so in standard affine space $\mathbb{A}^n$. We must prove that there exists $\varepsilon > 0$ so that for all $q \in B_r(0)$ there exists a compactly supported isotopy from $\text{id}_{\mathbb{A}^n}$ to a diffeomorphism which maps the origin to $q$. Fix smooth cutoff functions $\rho: \mathbb{A}^1 \to \mathbb{R}_{\geq 0}$ and $\sigma: \mathbb{A}^{n-1} \to \mathbb{R}_{\geq 0}$ such that $\rho(0) = 1$, supp $\rho \subset (-1, 1)$, $\sigma(0) = 1$, and supp $\sigma \subset B_1(0)$. It suffices to take $q = (r; 0, \ldots, 0)$ for some $r > 0$, since we can always compose with a rotation. Define the map $\varphi: [0, 1] \times \mathbb{A}^n \to \mathbb{A}^n$ by

$$
\varphi_t(y; \bar{y}) = (y + t \rho(y) \sigma(\bar{y})r; \bar{y}), \quad y \in \mathbb{A}^1, \quad \bar{y} \in \mathbb{A}^{n-1}.
$$

Then $\varphi_0 = \text{id}_{\mathbb{A}^n}$, $\varphi_1(0; 0) = q$, the map $\varphi$ has compact support, and $\varphi_t(-; \bar{y})$ is a monotonic nondecreasing function $\mathbb{R} \to \mathbb{R}$, so $\varphi_t$ is bijective for all $t \in [0, 1]$. It remains to prove that $\phi_t^{-1}$ is smooth. That follows from the inverse function theorem if we can show the differential is bijective. Letting $I_n$ denote the $(n-1) \times (n-1)$ identity matrix, we have

$$
d(\varphi_t(y; \bar{y})) = \begin{pmatrix}
1 + t \rho'(y) \sigma(\bar{y})r & * \\
0 & I_{n-1}
\end{pmatrix}
$$

as a block $(1 + (n-1)) \times (1 + (n-1))$ matrix. Choose $\epsilon = 1/(2 \max |\rho'(y)|)$. Then the upper left entry is positive for all $t, y, \bar{y}$, from which $d(\varphi_t)$ is invertible. Apply the inverse function theorem to complete the proof. □

Oriented degree and integration

(26.16) Main result. In this section we generalize the change of variables formula Proposition 24.26(2), which applies to orientation-preserving diffeomorphisms, to arbitrary maps.

Theorem 26.17. Let $X$ be a compact oriented manifold, $Y$ a connected oriented manifold, $f: X \to Y$ a smooth map, and $\omega \in \Omega^n_c(Y)$, where we assume $n = \dim X = \dim Y$. Then

$$
\int_X f^* \omega = \deg(f) \int_Y \omega.
$$

Note the support of $f^* \omega$ is compact, since $X$ is assumed compact. If we normalize $\omega$ so that $\int_Y \omega = 1$, then (26.18) is an integral formula for the degree. We treat some preliminaries before proving Theorem 26.17.

Remark 26.19. This formula for degree expresses a global topological invariant—the degree—in terms of a local quantity—the differential form $\omega$. Formulæ which relate local and global permeate differential topology and differential geometry.
(26.20) Oriented bordism. I have used the term ‘bordism’ a few times, but have neglected to give a formal definition. The one here is not perfect, since to glue bordisms we need to impose product structures near the boundary (see footnote 21), which I do not do explicitly here. I only define oriented bordism, whereas the notion is much more general.

Definition 26.21. Let $X_0, X_1$ be closed oriented manifolds of the same dimension.

(1) An oriented bordism from $X_0$ to $X_1$, denoted $W: X_0 \to X_1$, is a compact oriented manifold $W$ equipped with a partition $\partial W = \partial W_0 \sqcup \partial W_1$ of its boundary and orientation-preserving diffeomorphisms $X_i \to \partial W_i, i = 0, 1$.

(2) We say $X_0$ is oriented bordant to $X_1$ if there exists an oriented bordism $W: X_0 \to X_1$.

(3) Let $Y$ be a smooth manifold and let $f_i: X_i \to Y$ be smooth maps. An oriented bordism from $f_0$ to $f_1$ is the data in (1) and a smooth map $f: W \to Y$ such that $\partial f = f_0 \sqcup f_1$.

Oriented bordism is an equivalence relation on diffeomorphism classes of closed oriented manifolds of a fixed dimension. The equivalence classes form a finitely generated abelian group. These abelian groups were introduced in a sense by Poincaré, further studied by Pontrjagin and especially Thom, and arise in many interesting geometric contexts, as well as in theoretical physics.

Remark 26.22. A product bordism is one diffeomorphic to $[0, 1] \times X_0$. A smooth homotopy is an example of a bordism between smooth maps.

(26.23) An oriented bordism invariant. The following simple application of Stokes’ theorem shows how an integral can produce a topological invariant. It is crucial for the proof of Theorem 26.17.

Proposition 26.24. Let $W: X_0 \to X_1$ be an oriented bordism of $n$-manifolds $X_0$ and $X_1$. Suppose $Y$ is a smooth manifold, $f: W \to Y$ a smooth map, and $\omega \in \Omega^n(Y)$. Denote $\partial f = f_0 \sqcup f_1$. Then

\[
\int_{X_0} f_0^* \omega = \int_{X_1} f_1^* \omega.
\]

We need not assume $\omega$ has compact support since $X_0, X_1$ are compact, so the integrals in (26.25) make sense. Proposition 26.24 applies in particular to a smooth homotopy.

\[\text{Recall that a closed manifold is a compact manifold without boundary.}\]
Proof. Apply Stokes theorem to the form $f^*\omega$ on $W$ and observe $df^*\omega = f^*d\omega = 0$ since every $(n+1)$-form on an $n$-manifold is identically zero. \qed


\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure80}
\caption{An evenly covered open subset of $Y$}
\end{figure}

Proof of Theorem 26.17. Let $q \in Y$ be a regular value of $f$, set $f^{-1}(q) = \{p_1, \ldots, p_N\}$, and define $\epsilon_i = \pm 1$, $i = 1, \ldots, N$, where the sign tells if $df_{p_i}$ is orientation-preserving or orientation-reversing. Apply the inverse function theorem to find open neighborhoods $U_i \subset X$ of $p_i$ and $V \subset Y$ of $q$ such that $f^{-1}(V) = U_1 \cap \cdots \cap U_N$ and $f|_{U_i}: U_i \to V$ is a diffeomorphism. Then Proposition 24.26 implies that (26.18) holds if $\text{supp}(\omega) \subset Y$:
\[
\int_X f^*\omega = \sum_{i=1}^{N} \int_{U_i} f^*\omega = \sum_{i=1}^{N} \epsilon_i \int_Y \omega = (\deg f) \int_V \omega.
\]

(26.27)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure81}
\caption{A cover of $Y$ by diffeomorphic images of $V$}
\end{figure}
Now Theorem 26.13 applies to construct a cover \( \{ \varphi^{(i)}(V) \}_{i \in I} \) of \( Y \) by images of \( V \) under diffeomorphisms isotopic to the identity; in particular, these diffeomorphisms are orientation-preserving. Choose a partition of unity subordinate to this cover and use it to write any \( \omega \in \Omega^n(Y) \) as a sum of forms with support in an open set of the cover. It suffices to prove (26.18) for one of these forms, since each side is linear in \( \omega \). If \( \text{supp}(\omega) \subset \varphi^{(i)}(V) \), then

\[
\int_X f^* \omega = \int_X (\varphi^{(i)} \circ f)^* \omega \\
= \int_X f^* (\varphi^{(i)})^* \omega \\
= \int_Y (\varphi^{(i)})^* \omega \\
= \int_Y \omega
\]

(26.28)

In the first and last line we use the homotopy invariance of the integral (Proposition 26.24), and in the third line we use the special case proved in (26.27).

□

Example: the winding number as an integral

In this section we illustrate Theorem 26.17 via an extended example. Take \( Y = \mathbb{A}^2 \setminus \{(0,0)\} \) and \( X = S^1 \), both equipped with the standard orientation.23

(26.29) The differential form. In standard real coordinates \( x, y \) on \( \mathbb{A}^2 \), set

\[
\omega = \frac{1}{2\pi} \frac{xdy - ydx}{x^2 + y^2}.
\]

(26.30)

It is useful to write (26.30) in polar coordinates \( r, \theta \). Note that whereas \( r : Y \to \mathbb{R} \) is a global function, a maximal domain for \( \theta \) is \( Y \) with an (open) ray emanating from the origin deleted. The following computation is valid no matter which ray is chosen. Differentiate24

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{y}{x}
\]

(26.31)

to deduce

\[
\omega = \frac{1}{2\pi} d\theta.
\]

(26.32)

From this we conclude

\[
d\omega = 0.
\]

23Well, which orientation that is on \( S^1 \) depends on which \( S^1 \) you take. We might in general view \( S^n \subset \mathbb{A}^{n+1} \) as the boundary of the unit ball \( D^{n+1} \), give \( \mathbb{A}^{n+1} \) its standard orientation, and use the boundary orientation on \( S^n \).

24Equation (26.31) is only valid where \( x \neq 0 \), but the result holds everywhere \( \theta \) is defined.
**Theorem 26.34** *Some complex differential forms.* Let $\mathbb{C}^\times$ denote the space of nonzero complex numbers; it is identified with $Y$ by writing $\lambda \in \mathbb{C}^\times$ as $\lambda = x + yi$ for $i = \sqrt{-1}$. In terms of polar coordinates we have\(^{25}\) $\lambda = re^{i\theta}$. Thus

\begin{equation}
(26.35) \quad d\lambda = dr \, e^{i\theta} + re^{i\theta} \, d\theta,
\end{equation}

from which

\begin{equation}
(26.36) \quad \frac{1}{2\pi i} \frac{d\lambda}{\lambda} = \frac{dr}{ir} + d\theta = d(-i \log r) + d\theta
\end{equation}

and so setting $h = -i \log(r) : Y \rightarrow \mathbb{R}$ we have

\begin{equation}
(26.37) \quad \frac{1}{2\pi i} \frac{d\lambda}{\lambda} = \omega + dh \quad \text{on } \mathbb{C}^\times.
\end{equation}

Let $\mathbb{T} \subset \mathbb{C}^\times$ be the unit circle, defined by $r = |\lambda| = 1$. Then $h$ vanishes on $\mathbb{T}$, and so

\begin{equation}
(26.38) \quad \frac{1}{2\pi i} \frac{d\lambda}{\lambda} = \omega \quad \text{on } \mathbb{T}.
\end{equation}

Remove the subset of measure zero $\{1\} \subset \mathbb{T}$ and parametrize the remainder by $\lambda = e^{i\theta}$, $\theta \in (0, 2\pi)$, to carry out the computation\(^{26}\)

\begin{equation}
(26.39) \quad \int_{\mathbb{T}} \omega = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{d\lambda}{\lambda} = \frac{1}{2\pi} \int_{0}^{2\pi} d\theta = 1.
\end{equation}

---

\(^{25}\)This and the equation $\lambda = x + yi$ are equalities between complex-valued functions on $Y = \mathbb{C}^\times$.

\(^{26}\)We use the orientation of $\mathbb{T}$ discussed in footnote\(^{23}\).
(26.40) **Application to winding number.** Let $w: S^1 \to \mathbb{T}$ be a smooth function. Then by Theorem 26.17 we have

$$
(26.41) \quad \text{deg}(w) = \int_{S^1} w^* \omega. \n$$

Suppose $f: S^1 \to Y$ is a smooth function. Normalizing as in (18.4) (with $q = (0, 0)$) we obtain a function $w_f: S^1 \to \mathbb{T}$. The integer-valued **winding number** about the origin is defined as

$$
(26.42) \quad W(f) = \text{deg}(w_f). \n$$

We claim the following integral formula for the winding number

$$
(26.43) \quad W(f) = \int_{S^1} f^* \omega. \n$$

Observe that the normalized function $w$ is smoothly homotopic to $f$ as functions $S^1 \to Y$. Let $[0, 1] \times S^1 \to Y$ be the homotopy. Then by Stokes’ theorem

$$
(26.44) \quad \int_{S^1} f^* \omega - \int_{S^1} w^* \omega = \int_{[0,1] \times S^1} d\omega = 0. \n$$

(26.45) **Counting zeros of complex functions.** Let $W \subset \mathbb{C}$ be a compact manifold with boundary, and assume for simplicity that $\partial W \approx S^1$. Suppose $f: W \to \mathbb{C}$ is a smooth function with isolated zeros on a finite subset $\text{Zero}_W(f) = \{p_1, \ldots, p_N\} \subset \text{Int}(W)$. We would like to count the number of zeros of $f$ in $W$. From our experience with oriented degree and oriented intersection number, and also the experience before in the mod 2 case, we know the answer which is stable under perturbations is not simply ‘$N’ in general. We must worry about transversality, signs, etc. In the case of counting zeros, this is encoded by a ‘multiplicity’ of the zero. In other words, we would like to attach an integer $n_i$ to each zero $p_i$ which represents its multiplicity. This is familiar in the real case too: the function $x \mapsto x^2$ on the real line has a zero of multiplicity two at the origin. You know how to define the multiplicity for real polynomials, and so too for complex functions $f$ which are polynomials. The question is how to define the multiplicity more generally.

An inspired observation is the following. Let us suppose $f$ extends to a complex polynomial

$$
(26.46) \quad P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0, \quad a_0, \ldots, a_{n-1} \in \mathbb{C} \n$$

on the entire complex line, and let $p \in \mathbb{C}$ be an isolated zero of multiplicity $m$. Then

$$
(26.47) \quad P(z) = (z - p)^m g(z) \n$$
for some polynomial \( g \) with \( g(p) \neq 0 \). Choose \( \delta > 0 \) so that \( g \) does not vanish on the closed disk \( D_\delta \) of radius \( \delta \) about \( p \). Compute

\[
P^* \omega = \frac{1}{2\pi i} \frac{dP}{P} = \frac{1}{2\pi i} \left[ \frac{m \, dz}{z - p} + \frac{dg}{g} \right].
\]

Integrate (26.48) over \( \partial D_\delta \). Since \( g \) does not vanish on \( D_\delta \), the closed 1-form \( dg/g \) extends over \( D_\delta \), and by Stokes' theorem its integral over \( \partial D_\delta \) vanishes. To integrate the first term, parametrize \( \partial D_\delta \) minus a point by \( z = p + \delta e^{i\theta}, \theta \in (0,2\pi) \). Then \( dz = \delta ie^{i\theta} d\theta \), and

\[
\int_{\partial D_\delta} P^* \omega = \frac{1}{2\pi} \int_0^{2\pi} m \, d\theta = m.
\]

This is an integral formula for the local multiplicity of the zero of a polynomial.

Apply Stokes’ theorem to the closed 1-form \( \omega \) on \( W' = W \setminus U_t \cup \dot{B}_\delta(p_i) \). Orient \( W \) with the standard orientation on \( \mathbb{A}^2 \); the induced boundary orientations are indicated in Figure 83. Then

\[
\int_{\partial W} \omega = \sum_{i=1}^N \int_{\partial D_\delta(p_i)} \omega,
\]

and the right hand side is the sum of the multiplicities of the zeros of the polynomial \( P \). Therefore,

\[
\#_m \text{Zero}_W(P) = \frac{1}{2\pi i} \int_{\partial W} \frac{dP}{P},
\]

where ‘\( \#_m \)’ indicates the count with multiplicity.

**Remark 26.52.** We can use (26.51) to prove the fundamental theorem of algebra. Choose \( W \) to be a disk of radius \( R \) with center \( 0 \in \mathbb{C} \), where \( R \) is chosen so that

\[
|a_{n-1}|R^{n-1} + \cdots + |a_0| < R^n.
\]
Then $P$ does not have any zeros on $\partial W$. Define the homotopy

$$P_t(z) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_0), \quad t \in [0, 1],$$

and apply Proposition 26.24 and (26.51) to conclude that the polynomial $P_0(z) = z^n$ has the same number of zeros in $W$ as does $P_1(z) = P(z)$. The latter has $n$ zeros. We give a different, but closely related, proof in the next lecture.

(26.55) **Local multiplicity of holomorphic functions.** Inspired by (26.49) we might define the multiplicity of an isolated zero of a complex function $f : W \to \mathbb{C}$ to be

$$\int_{\partial D_\delta} f^* \omega = \frac{1}{2\pi i} \int_{\partial D_\delta} \frac{df}{f}.$$  

For a holomorphic function $f$ this is justified by the theorem which states that we can write $f$ as in (26.47). The manipulations in this section with Stokes’ theorem and differential forms, and the relationship to topological quantities such as the winding number, degree, and number of zeros are all ideas you saw in a first course on complex variables.

**Lecture 27: Oriented intersection number; linking number**

We begin by reproving the fundamental theorem of algebra. (We sketched a proof in Remark 26.52.) Then we turn to the oriented intersection number, developing some basic theory and examples. We apply this to define the linking number in the first nontrivial case, and use it to detect that the Hopf map $S^3 \to S^2$ is not homotopically trivial.

**Fundamental theorem of algebra**

**Theorem 27.1.** Fix $n \in \mathbb{Z}_{>0}$ and suppose $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ is a nonconstant polynomial with complex coefficients $a_0, \ldots, a_{n-1} \in \mathbb{C}$. Then there exists $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.

In fact, $P$ has $n$ zeros counted with multiplicity; see (26.45).

**Proof.** Let $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ be the complex projective line. Represent points in $\mathbb{CP}^1$ as equivalence classes of ordered pairs $[z, w]$ of complex numbers, not both zero. Identify $z \in \mathbb{C}$ with $[z, 1] \in \mathbb{CP}^1$. Then we claim $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ defined by

$$f([z, 1]) = [P(z), 1]$$

$$f([1, 0]) = [1, 0]$$

(27.2)
is smooth. Namely,

\[(27.3)\quad f([1, w]) = \left[ 1, \frac{1}{P(\frac{1}{w})} \right],\]

and

\[(27.4)\quad \frac{1}{P(\frac{1}{w})} = \frac{w^n}{1 + a_{n-1}w + \cdots + a_0w^n}\]

is smooth near \(w = 0\) and converges to 0 as \(w \to 0\).

The homotopy (26.54) induces a homotopy \(f_t: \mathbb{CP}^1 \to \mathbb{CP}^1\), and \(\text{deg}(f) = \text{deg}(f_1) = \text{deg}(f_0)\) by the homotopy invariance of degree. Now \(f_0(z) = z^n\) has 1 \(\in \mathbb{C}\) as a regular value, and \(f_0^{-1}(1) = \{1, \omega, \ldots, \omega^{n-1}\}\), where \(\omega = e^{2\pi i/n}\) is an \(n\)th root of unity. The differential \(df_{\omega_i}\) at \(\omega_i\) is the complex linear map \(\mathbb{C} \to \mathbb{C}\) which is multiplication by \(n\omega^{i(n-1)}\). This is the composition of a homothety and a rotation, each of which is orientation-preserving. Hence the local degree is +1 at each inverse image point, and so \(\text{deg}(f_0) = n\).

If \(P\) has no zeros, then \([0, 1] \in \mathbb{CP}^1\) is not in the image of \(f\), from which \(\text{deg}(f) = 0\). This contradiction proves the theorem. \(\Box\)

**Oriented intersection number**

(27.5) **Setup.** The setup is (17.13) with the addition of orientations. Namely, we have

\[
\begin{align*}
X & \quad \text{compact oriented manifold} \\
Y & \quad \text{oriented manifold} \\
Z \subset Y & \quad \text{closed oriented submanifold} \\
f: X \longrightarrow Y & \quad \text{smooth map} \\
\dim X + \dim Z = \dim Y
\end{align*}
\]
**Definition and basic properties.** If \( f \not\sim Z \), then \( S := f^{-1}(Z) \subset X \) is a 0-dimensional submanifold, and it inherits an orientation by the 3-step procedure in (26.2). We work out the induced orientation of \( S \) in (27.13) below.

**Definition 27.8.** The oriented intersection number is

\[
\#^Y (f, Z) = \#(f, Z) = \#_s S.
\]

The oriented intersection number satisfies the following.

**Proposition 27.10.** Assume the setup of (27.6).

1. If \( f : [0, 1] \times X \to Y \) is a smooth homotopy, then \( \#(f, Z) = \#(f_1, Z) \).
2. If \( W \) is a compact oriented manifold with boundary and \( F : W \to Y \), then \( \#(\partial F, Z) = 0 \).

Observe that (1) is a special case of (2).

**Remark 27.11.** Recall the logic of the theory. We first define the oriented intersection number by (27.9), assuming \( f \not\sim Z \). Then we prove Proposition 27.10(1) assuming that \( f_0, f_1 \not\sim Z \). Corollary 16.11 is then used to define the intersection number in general, and the properties in Proposition 27.10 hold.

**Figure 85.** Homotopy invariance of the oriented intersection number

**A bit of proof.** We give the proof of Proposition 27.10(1) assuming that \( f_0, f_1 \not\sim Z \). The first step is to apply a controlled perturbation of \( f \) which leaves it as is on \( \{0\} \times X \sqcup \{1\} \times X \) and gives a homotopic map \( \tilde{f} : [0, 1] \times X \to Y \) such that \( \tilde{f} \not\sim Z \). We already have \( \partial \tilde{f} \not\sim Z \), and so \( T := \tilde{f}^{-1}(Z) \subset [0, 1] \times X \) is a 1-dimensional submanifold, oriented by (26.2). Now apply Theorem 25.20 to deduce that the signed count \( \#_s \partial T = 0 \), where \( \partial T \) has the boundary orientation. Define the 0-manifolds \( S_0 = f_0^{-1}(Z) \) and \( S_1 = f_1^{-1}(Z) \) with the induced orientation defined in (26.2). Then Proposition 26.7 implies \( \partial T = S_0 \sqcup -S_1 \) as oriented 0-manifolds. (This uses (25.30).) Therefore, \( 0 = \#_s \partial T = \#_s S_0 - \#_s S_1 \), which proves the desired smooth homotopy invariance.
(27.13) The inverse image orientation. In the situation of (27.6) suppose that \( f \not\sim Z \) so that \( S := f^{-1}(Z) \subset X \) is a 0-dimensional submanifold. Then for \( p \in S \) the diagram (26.3) simplifies:

\[
\begin{array}{cccc}
0 & T_p S & T_p X & \nu_p(S \subset X) \\
\| & \| & \| & \| \\
0 & T_{f(p)} Z & T_{f(p)} Y & \nu_{f(p)}(Z \subset Y) \\
\end{array}
\]

Hence the orientation \( \pm 1 \) of \( T_p S \) tells if the isomorphism in the first line preserves or reverses the orientation of \( T_p X \). From the last line this, in turn, tells whether the natural isomorphism

\[
\text{Det} \, df_p(T_p X) \otimes \text{Det} \, T_{f(p)} Z \overset{\cong}{\longrightarrow} \text{Det} \, T_{f(p)} Y
\]

preserves or reverses orientation. It is this latter that we use in examples. The order \((X \text{ before } Z)\) follows the order in ‘\(#(f, Z)\)’.

**Example 27.16.** Consider Figure 86 in which \( Y \) is an affine 2-plane in affine 3-space with a handle attached, \( Z \subset Y \) is a circle wrapping around the handle, \( X \) is a circle, and the map \( f \) wraps \( X \) around the handle transversely to \( Z \). Orient \( Y \) as the boundary of the 3-dimensional region which lies below it in the picture, and orient \( Z \) and \( X \) as indicated by the little arrow. We use the standard “right hand rule” orientation of affine 3-space. The sign at the intersection point is then +, since the ordered basis of 3-space \{outward normal, oriented tangent to \( f(X) \), oriented tangent to \( Z \)\} is positively oriented.

(27.17) Oriented degree redux. Suppose that in (27.6) we have \( \dim Y = \dim X \) and \( Z = \{q\} \subset Y \) is a single point. Assume that \( Y \) is connected. Then comparing the definitions we see that the oriented intersection number reduces to the degree \( #(f, Z) = \deg f \).
Example 27.18 (transverse intersection of complex manifolds). Let $V$ be a 3-dimensional complex vector space and $W \subset V$ a 2-dimensional subspace. Then $Y = \mathbb{P}V$ is a compact 4-manifold (a complex projective plane) and $X = Z = \mathbb{P}W$ is a compact 2-dimensional submanifold (a complex projective line). Let $i_X: X \rightarrow Y$ denote the inclusion map. We compute $\#^Y(i_X, Z)$. To do so we need to homotop $i_X$ to a map which is transverse to $Z$; the inclusion of a distinct complex projective line will do. The problem, then, is to compute the intersection number of two lines in a plane, here the lines and planes are complex projective. We have not yet said how to orient $X, Y,$ and $Z$. Each is a complex manifold, an object we have not formally defined, and so each tangent space is a complex vector space. In fact, if $L \subset V$ is a line, so $L \in \mathbb{P}V$ a point, then we know $T_L \mathbb{P}V \cong \text{Hom}(L, V/L)$ which is a complex vector space.

In general, if $U$ is a finite dimensional complex vector space and $U_\mathbb{R}$ the underlying real vector space, let $I: U_\mathbb{R} \rightarrow U_\mathbb{R}$ denote the real linear map which is multiplication by $\sqrt{-1}$ on $U$. Then if $e_1, \ldots, e_m$ is a (complex) basis of $U$, the $2m$ vectors $e_1, Ie_1, e_2, Ie_2, \ldots, e_m, Ie_m$ is a (real) basis of $U_\mathbb{R}$. We convene that this is a positively oriented basis of $U_\mathbb{R}$. It is easy to check that this orientation of $U_\mathbb{R}$ is independent of the basis.

At a transverse intersection point the sign (27.15) is determined by comparing orientations of $df_p(T_pX) \oplus T_{f(p)}Z$ and $T_{f(p)}Y$. Each vector space has a complex structure, and the direct sum is compatible with the complex structures. It follows that the natural orientations agree, so the local oriented intersection number is $+1$.

Remark 27.19. The argument in Example 27.18 proves that transverse intersection points of complex submanifolds of a complex manifold contribute positively to the oriented intersection number. It is nonetheless possible to have negative self-intersection numbers in complex geometry.
A symmetric version. We can treat $X$ and $Z$ symmetrically if we assume $Z$ is a compact oriented manifold rather than a closed submanifold. Hence modify (27.6) to:

\[
X, Z \quad \text{compact oriented manifolds}
\]
\[
Y \quad \text{oriented manifold}
\]

(27.21)
\[
f: X \to Y \quad \text{smooth map}
\]
\[
g: Z \to Y \quad \text{smooth map}
\]

\[
\dim X + \dim Z = \dim Y
\]

If $f \not\sim g$, and $x \in X, z \in Z$ satisfy $f(x) = g(z)$ for some $y \in Y$, then define the local intersection number $\pm 1$ according as the isomorphism

(27.22)
\[
\text{Det } df_x(T_X X) \otimes \text{Det } dg_z(T_z Z) \cong \text{Det } T_y Y
\]

is orientation-preserving or orientation-reversing. Define $\#^Y(f,g) \in \mathbb{Z}$ by summing the local intersection numbers. In case $g$ is the inclusion of a submanifold, this agrees with Definition 27.8; see (27.15).

Observe that

(27.23)
\[
\{(x, z) \in X \times Z : f(x) = g(z)\} = (f \times g)^{-1}(\Delta),
\]

where

(27.24)
\[
f \times g: X \times Z \to Y \times Y
\]

is the Cartesian product of $f$ and $g$, and $\Delta \subset Y \times Y$ is the diagonal, a closed submanifold. This gives the opportunity to reduce the intersection number $\#^Y(f,g)$ to $\#^{Y \times Y}(f \times g, \Delta)$ as in Definition 27.8, at least up to sign. The following enables this comparison.

**Lemma 27.25.** If $(x, z) \in X \times Z$ satisfies $f(x) = g(z) = y \in Y$, then $f \times g \not\sim (x, z) \Delta$ if and only if $df_x(T_x X) \oplus dg_z(T_z Z) = T_y Y$. If so, the local intersection numbers are related by

(27.26)
\[
\#^Y_{(x,z)}(f,g) = (-1)^{\dim Z} \#^{Y \times Y}_{(x,z)}(f \times g, \Delta).
\]

**Proof.** Set $A = df_x(T_x X)$, $B = dg_z(T_z Z)$, and $C = T_y Y$; then $A$ and $B$ are subspaces of $C$. Fix oriented bases $a_1, \ldots, a_k$ of $A$ and $b_1, \ldots, b_\ell$ of $B$. Set $a = a_1 \wedge \cdots \wedge a_k \in \text{Det } A$ and $b = b_1 \wedge \cdots \wedge b_\ell \in \text{Det } B$. Recall the natural isomorphism $\wedge^\bullet (C \oplus C) \cong \wedge^\bullet C \otimes \wedge^\bullet C$. Then inclusion $A \oplus 0 \to C \oplus C$ induces a linear map $\text{Det } A \mapsto \wedge^\bullet C \otimes \wedge^\bullet C$, and we denote the image of $a$ as $a^{(1)}$. Including in the other summand we obtain $a^{(2)}$, and similar elements $b^{(1)}$ and $b^{(2)}$. Compute

(27.27)
\[
a^{(1)} \wedge b^{(2)} \wedge (a_1 \otimes 1 + 1 \otimes a_1) \wedge \cdots \wedge (a_k \otimes 1 + 1 \otimes a_k)
\]
\[
\wedge (b_1 \otimes 1 + 1 \otimes b_1) \wedge \cdots \wedge (b_\ell \otimes 1 + 1 \otimes b_\ell)
\]
\[
= a^{(1)} \wedge b^{(2)} \wedge a^{(2)} \wedge b^{(1)}
\]
\[
= (-1)^{\dim A \dim B + \dim B^2 + \dim A \dim A} a^{(1)} \wedge b^{(1)} \wedge a^{(2)} \wedge b^{(2)}
\]
\[
= (-1)^{\dim B} a^{(1)} \wedge b^{(1)} \wedge a^{(2)} \wedge b^{(2)}
\]
The first expression is nonzero iff \( f \times g \cong_{(x,z)} \Delta \) and the last is nonzero iff \( A \oplus B = C \); the first assertion of the lemma follows. Assuming this, \( a \land b \in \text{Det } C \) is nonzero and is positively or negatively oriented according to \( \#^Y_{(x,z)}(f, g) \). In either case \( a^{(1)} \land b^{(1)} \land a^{(2)} \land b^{(2)} \in \text{Det}(C \oplus C) \) is positively oriented. Let \( D \subset C \oplus C \) be the diagonal. Then

\[
(a_1 \otimes 1 + 1 \otimes a_1) \land \cdots \land (a_k \otimes 1 + 1 \otimes a_k) \land (b_1 \otimes 1 + 1 \otimes b_1) \land \cdots \land (b_\ell \otimes 1 + 1 \otimes b_\ell)
\]

in \( \text{Det } D \) is positively or negatively oriented according to \( \#^Y_{(x,z)}(f, g) \). Putting everything together, (27.27) implies (27.26).

\[
\text{Corollary 27.29. In the setup (27.21),}
\]

\[
\#^Y(f, g) = (-1)^{\dim Z} \#^Y \times_Y(f \times g, \Delta).
\]

Finally, we have the symmetry property

\[
\#^Y(g, f) = (-1)^{\dim X \dim Z} \#^Y(f, g).
\]

\[
(\text{27.32) Intersection of submanifolds. A special case of (27.20) is when } f = i_X \text{ and } g = i_Z \text{ are inclusions of submanifolds. Then we denote the intersection number as } \#^Y(X, Z). \text{ Writing it as } \#^Y(f, g) \text{ enables us to use our theory to wiggle the submanifolds to achieve transversality. The symmetry (27.31) now reads}
\]

\[
\#^Y(Z, X) = (-1)^{\dim X \dim Z} \#^Y(X, Z).
\]

This immediately implies the following.

\[
(\text{Proposition 27.34. If } \dim Y = 4k + 2 \text{ for some } k \in \mathbb{Z}_{\geq 0}, \text{ and } X \subset Y \text{ is a compact orientable submanifold of dimension } 2k + 1 \text{ such that } \#^Y_{X}(X, X) \neq 0, \text{ then } Y \text{ is not orientable.}
\]

For example, \( \mathbb{R}P^{4k+2} \) is not orientable. (We already proved a stronger statement in Corollary 25.46.)

\[
(\text{Linking number})
\]

Beware that some of the arguments in this section are sketchy as presented here; they can be made rigorous.
(27.35) The Gauss map. Let $A$ be an affine space over a 3-dimensional oriented real inner product space $V$, and let $S^2$ be the unit sphere in $V$. Suppose $K_1, K_2 \subset A$ are disjoint oriented compact 1-dimensional submanifolds. Define the Gauss map (Figure 88)

$$f : K_1 \times K_2 \rightarrow S^2$$

(27.36)

$$p_1, p_2 \mapsto \frac{p_1 - p_2}{\|p_1 - p_2\|}$$

Definition 27.37. The linking number of $K_1, K_2$ is $L(K_1, K_2) = \deg f$.

We use the Cartesian product orientation on $K_1 \times K_2$ and orient $S^2 = S(V)$ as the boundary of the closed unit ball in $V$. It follows from (25.38) that $L(K_2, K_1) = L(K_1, K_2)$. Also, if we reverse orientation, then $L(-K_1, K_2) = L(K_1, -K_2) = -L(K_1, K_2)$.

Remark 27.38. There is no self-linking number $L(K, K)$ without additional data, for example a nonzero normal vector field to $K \subset A$.

Example 27.39 (Hopf link). Choose $A = \mathbb{R}^3_{x,y,z}$ with standard orientation and inner product on $\mathbb{R}^3$, let $K_1$ be the unit circle in the $x$-$y$ plane with center $(0,0,0)$, and let $K_2$ be the unit circle.
in the $x$-$z$ plane with center $(-1, 0, 0)$. Orient $K_1, K_2$ as in Figure 89. Then $(0, 0, 1) \in S^2 \subset \mathbb{R}^3$ is a regular value of the Gauss map $f$ whose inverse image is a single point:

\[(27.40) \quad (p_1, p_2) = ((-1, 0, 0), (-1, 0, -1)) \in K_1 \times K_2.\]

The differential of $f$ maps $-\partial/\partial y \in T_{p_1} K_1$ to $-\partial/\partial y$ and $\partial/\partial x \in T_{p_2} K_2$ to $-\partial/\partial x$, so is orientation-reversing. Hence \(^{27}\) $L(K_1, K_2) = -1.$

\[(27.41) \quad \text{Computation by a bounding surface.} \quad \text{The linking number is an intersection number, but} \]

of a curve with a surface with boundary, and so we must take care to keep the curve away from the boundary lest an intersection point disappear off the edge of the surface.

**Proposition 27.42.** Let $W_1 \subset A$ be a compact oriented 2-manifold with boundary $\partial W_1 = K_1$, and suppose $K_2 \subset A$ is a compact oriented 1-manifold. Assume $\partial W_1 \cap K_2 = \emptyset$ and $W_1 \cap K_2$. Then

$L(K_1, K_2) = \#(W_1, K_2)$ where the latter is the signed count $\#_s(W_1 \cap K_2)$.

We need not assume the transversality $W_1 \cap K_2$: we can perturb $W_1$ and/or $K_2$ to achieve it. We sketch (literally) a proof, cognizant that we have not filled in all details.

![Figure 90. Reduction of $K_1$ to a union of “small” loops](image)

**Proof.** Write $W_1 \cap K_2 = \{p_1, \ldots, p_N\}$. For each $i \in \{1, \ldots, N\}$ fix an open ball\(^{28}\) $B_i \subset W_1$ which contains $p_i$ and such that $B_i \cap B_j = \emptyset$ for all $i, j$. Set $D_i = \overline{B_i}$, $S_i = \partial D_i$, and $\tilde{K}_1 = \bigcup_i S_i$. Then $W_1 \setminus \bigcup_i B_i$ is an oriented bordism from $\tilde{K}_1$ to $K_1$, and the bordism invariance of degree implies

\[(27.43) \quad L(K_1, K_2) = L(\tilde{K}_1, K_2) = \sum_i L(S_i, K_2).\]

\(^{27}\)I must be making a sign mistake here, since the linking number should be $+1$, but for the moment I cannot find the error.

\(^{28}\)This means an open subset contained in the domain of a chart whose image is a ball in affine space.
To compute $L(S_i, K_2)$ we execute the surgery indicated in Figure 91. Namely, cut out $S^0 \times D^1 \subset K_2$, which is the union of two closed intervals, and glue in $D^1 \times S^0$ along $\partial(K_2 \setminus S^0 \times D^1) \approx S^0 \times S^0$. The result is the disjoint union of a circle $M_i$ and a 1-manifold $N_i$. There is an oriented bordism $K_2 \to M_i \amalg N_i$ which is a product away from the surgery and is $D^1 \times D^1$ at the surgery point. (We must smooth corners.) The bordism invariance of degree implies

\begin{equation}
(27.44)
L(S_i, K_2) = L(S_i, M_i) + L(S_i, N_i).
\end{equation}

Since $D_i \cap N_i = \emptyset$, we have $L(S_i, N_i) = 0$, again by the bordism invariance of degree. Finally, the circles $S_i, M_i$ form a Hopf link (Example 27.39), and so $L(S_i, M_i) = \pm 1$ where the sign depends on the orientations. Tracing through, $S_i$ is oriented as the boundary of $D_i$, and $D_i$ inherits its orientation from that of $W$. $K_2$ is assumed oriented, and $M_i$ inherits an orientation. One can check\footnote{Perhaps one can, but I can't since I don't have the sign right in the Hopf link.} that the sign is equal to the local intersection number $\#^A(D_i, M_i)$.

\begin{equation}
(27.45)
\text{Links in } S^3. \text{ If now } S^3 \text{ is an oriented 3-sphere, and } K_1, K_2 \subset S^3 \text{ are disjoint oriented compact 1-dimensional submanifolds, then we can}^{30} \text{ bound } K_1 \text{ by an oriented surface } W_1 \subset S^3, \text{ use transversality to arrange that } K_2 \text{ intersects } W_1 \text{ transversely and only in the interior, and so define the linking number as } \#_*(W_1 \cap K_2). \text{ (Alternatively, we may delete a point of } S^3, \text{ compute the linking number in } S^3 \setminus \text{pt } \approx \mathbb{A}^3, \text{ and prove that the result is independent of the point.)}
\end{equation}

\begin{equation}
(27.46) \text{The Hopf invariant. Suppose } g: S^3 \to S^2 \text{ is a smooth map. The Hopf invariant is }
H(g) = L(g^{-1}(q_1), g^{-1}(q_2))
\end{equation}

for any distinct regular values $q_1, q_2 \in S^2$ of $g$. We fix orientations on the spheres and use the inverse image orientations of the closed 1-manifolds $g^{-1}(q_1)$ and $g^{-1}(q_2)$. The Hopf invariant does not change if $g$ undergoes a smooth homotopy.

\footnote{This requires proof, which we do not give here.}
Example 27.48 (The Hopf fibration). Let $\rho : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$ be the map which assigns to a nonzero vector its span. Then $\rho$ is a fiber bundle with fiber $\mathbb{C}^\times$ the nonzero complex numbers. (Better: $\mathbb{C}^\times$ acts on $\mathbb{C}^2 \setminus \{0\}$ by scalar multiplication, and $\rho$ is a quotient map for this group action.)

Restrict $\rho$ to the unit sphere $S^3 \subset \mathbb{C}^2 \setminus \{0\}$ to obtain a fiber bundle $\pi : S^3 \to \mathbb{C}P^1 \simeq S^2$. We claim\(^{31}\) that $H(\pi) = +1$. If so, then $\pi$ is not smoothly homotopic to a constant map, which clearly has zero Hopf invariant.

To compute $H(\pi)$ omit $(0, 1)$ from $S^3$ and by stereographic projection identify $S^3 \setminus \{(0, 1)\} \approx \mathbb{A}^3$. Every $q \in \mathbb{C}P^1$ is a regular value since $\pi$ is a submersion. Then

\[
K_1 := \pi^{-1}(\{1, 0\}) = \{(\lambda, 0) \in S^3 : |\lambda| = 1\}
\]

\[
K_2 := \pi^{-1}(\{0, 1\}) = \{(0, \mu) \in S^3 : |\lambda| = 1\}
\]

Under stereographic projection to $\mathbb{A}^3_{x,y,z}$ we can identify $K_1$ as the unit circle in the $x$-$y$ plane and $K_2 \setminus \{(0, 1)\}$ as the $z$-axis. This is the Hopf link (minus a point at $\infty$) so has linking number $+1$, which we can compute by writing $K_1$ as the boundary of the unit disk $W_1$ in the $x$-$y$ plane, and then $W_1 \cap K_2$ is a single point, so it only remains to get the orientations correct...

Remark 27.50 (Homotopy groups of $S^2$). Leaving off basepoints, the $q$th homotopy group, $q \in \mathbb{Z}_{\geq 0}$, is the set $[S^q, S^2]$ of homotopy classes of maps $S^q \to S^2$. We can take smooth maps and smooth homotopies. For $q = 0, 1$ we know by Sard’s theorem that every map is homotopic to a constant map. For $q = 2$ we have an invariant—the degree—which is a map

\[
(27.51) \quad \deg : [S^2, S^2] \longrightarrow \mathbb{Z}.
\]

In fact, it is an isomorphism, as follows from the Hopf degree theorem. In other words, the degree is a complete invariant of $[S^2, S^2]$. Now for $q = 3$ we have sketched a map

\[
(27.52) \quad H : [S^3, S^2] \longrightarrow \mathbb{Z},
\]

the Hopf invariant, and this too turns out to be an isomorphism. The degree and Hopf invariant illustrate how differential topology—the application of calculus to global geometry—can be used to construct interesting and effective invariants.

It turns out that $[S^q, S^2]$ has cardinality greater than one for infinitely many $q$.

---

Lecture 28: Euler numbers and Lefschetz numbers

In this lecture we take up the Euler number, a basic invariant of a smooth manifold. We define and study it not just for manifolds, but also for real vector bundles whose rank equals the dimension

\[^{31}\text{modulo our sign troubles}\]
of the base manifold. A generalization is the Lefschetz number of a self map on a manifold, and we use it to illustrate some computations.

Throughout the base manifold $X$ is compact, and we also assume it is oriented, though the orientation can be eliminated for the Euler number of a manifold and the Lefschetz number of a map; see Remark 28.4.

Euler numbers

(28.1) **Euler number of a compact manifold.**

**Definition 28.2.** Let $X$ be a compact oriented manifold. The *Euler number* of $X$ is

$$\chi(X) = \#^X \times X(\Delta, \Delta),$$

where $\Delta \subset X \times X$ is the diagonal.

The manifold $X \times X$ has the Cartesian product orientation, and we transport the orientation of $X$ to $\Delta$ via the diffeomorphism $X \to \Delta$ given by $p \mapsto (p, p)$. The symmetry property (27.33) implies $\chi(X) = 0$ if $\dim X$ is odd.

**Remark 28.4.** Imagine the first copy of $\Delta$ in (28.3) is perturbed to be transverse to $\Delta$ at some point $(p, p) \in X \times X$. Then infinitesimally we have the isomorphism

$$W \oplus D \xrightarrow{\cong} V \oplus V,$$

where $V = T_p X$ and $W \subset V \oplus V$ is transverse to the diagonal $D \subset V \oplus V$. We can take $W$ to be the graph of a linear map $V \to V$, and it is oriented by its projection to $V \oplus 0$. The local intersection number at $(p, p)$ is $\pm 1$ according as (28.5) is orientation-preserving or orientation-reversing. The answer to that question is independent of the orientation of $V$, and this explains why orientations are not necessary for Euler numbers of manifolds and, later, Lefschetz numbers of self maps.

(28.6) **Euler number of a real vector bundle.** Let $\pi_E : E \to X$ be a real vector bundle. It has a canonical section $s_0 : X \to E$, the zero section, whose image $Z_E = s_0(X) \subset E$ is a closed submanifold also termed ‘the zero section’. The section $s_0$ and projection $\pi_E$ give inverse diffeomorphisms of $X$ and $Z_E$. For $p \in X$, let $0_p = s_0(p) \subset E_p$ denote the zero vector of the vector space $E_p$. There are two natural submanifolds which pass through $0_p \in E$: the zero section $Z_E$ and the fiber $E_p$. Infinitesimally, that leads to a direct sum decomposition of the tangent space to the total space:

$$T_0 X \oplus E_p \cong T_0 Z_E \oplus T_0 E_p \cong T_0 E,$$

(28.8)

$\text{At any point } e \in E \text{ there is a short exact sequence}

$$0 \longrightarrow E_p \longrightarrow T_e E \xrightarrow{d(\pi_E)_e} T_p X \longrightarrow 0,$$

where $p = \pi_E(e)$. For $e \in Z_E$ there is a canonical splitting, written in (28.8), and we observe the ‘Quotient Before Sub’ rule in writing (28.8).
Here we use the canonical isomorphism of the tangent space of the vector space $E_p$ with $E_p$. The set of two orientations of each fiber $E_p$ form a double cover $\sigma(E) \to X$, the orientation double cover; an orientation of $E$ is a section of $\sigma(E) \to X$. Note from (28.8) that an orientation of $X$ and of $\pi_E$ induce an orientation of the total space $E$. Finally, recall that the rank of $\pi_E$ is a locally constant function $\text{rank} E: X \to \mathbb{Z}$.

![Figure 92. A vector bundle $\pi_E$ and its zero section $s_0$ with image $Z_E$](image)

**Definition 28.9.** Let $X$ be a compact oriented manifold and $\pi_E: E \to X$ an oriented real vector bundle. Assume $\text{rank} \pi_E = \dim X = n$ for some positive integer $n$. Then the Euler number of $\pi_E$ is

(28.10) \[ \chi(\pi_E) = \#^E(Z_E, Z_E). \]

The symmetry property (27.33) implies $\chi(\pi_E) = 0$ if $n$ is odd.

**Lemma 28.12.** There exists a section $s_1: X \to E$ of $\pi_E$ which is transverse to $Z_E$.

**Proof.** By Corollary 16.11 there exists a map $f: X \to E$ such that $f \not\sim Z_E$ and $f$ is homotopic to $s_0$. Recall the construction embeds $s_0$ in a family of maps $f_b: X \to E$ parametrized by a ball $B$, and Sard’s theorem implies that transversality is achieved for $b$ in a dense subset of $B$. The Stability Theorem 12.17(vii) implies that $\pi_E \circ f_b: X \to X$ is a diffeomorphism for all $b \in B$ in a neighborhood of $0 \in B$ (if we take $f_0 = s_0$). Choose $b$ in this neighborhood so that $f_b \not\sim Z_E$, and set $f = f_b$. Then $s := f \circ (\pi_E \circ f)^{-1}$ is a section of $\pi_E$ that is transverse to $X$. \( \square \)

**Properties of the Euler number.** We record two elementary facts which follow immediately from the definition.
Proposition 28.14. Let $\pi_E : E \to X$ be an oriented real vector bundle over a compact oriented manifold, and assume $\text{rank} \pi_E = \dim X$.

1. If $\dim X$ is odd, then $\chi(\pi_E) = 0$.
2. If $\pi_E$ admits a nowhere vanishing section, then $\chi(\pi_E) = 0$.

The converse of (2) is also true, though we do not prove it.

(28.15) Computation of local intersection numbers.

Lemma 28.16. Let $s : X \to E$ be a section of $\pi_E$, suppose $s(p) = 0$ for some $p \in X$, and assume $s \not\equiv 0$ $Z_E$. Assume $n = \dim X = \text{rank} E$ is even. Then the local intersection number $\#_p(s, Z_E)$ equals the degree of the isomorphism

\begin{equation}
I_p : T_p X \xrightarrow{ds_p} T_0 E \xrightarrow{\text{proj}} E_p.
\end{equation}

The projection $T_0 E \to E_p$ is defined by the splitting (28.8). If $n$ is odd, then the Euler number vanishes so we have no need for the local intersection number.

Proof. Let $\epsilon_p = \#_p(s, Z_E) = \pm 1$; it is computed by requiring that the isomorphisms

\begin{equation}
\text{Det} T_0 E \cong \epsilon_p \text{Det} ds_p(T_p X) \otimes \text{Det} T_p X
\end{equation}

be orientation-preserving. The degree $\delta_p = \pm 1$ of $I_p$ makes the isomorphisms

\begin{equation}
\text{Det} T_0 E \cong \text{Det} T_p X \otimes \text{Det} E_p
\end{equation}

orientation-preserving. The lemma follows by comparing (28.18) and (28.19), bearing in mind that $n$ is even so we can swap the factors in the tensor product without incurring a sign penalty. (To do the comparison, one must bear in mind what the isomorphism are.) \qed

(28.20) The tubular neighborhood theorem. A neighborhood of each point in a submanifold has a normal form; indeed, that is the very definition of a submanifold (Definition 6.20). The following theorem is a global version which gives control of an open neighborhood of the entire submanifold. Theorem 16.8 is the tubular neighborhood theorem for submanifolds of affine space; see Remark 16.26.

Theorem 28.21. Let $Y$ be a smooth manifold, $Z \subset Y$ a submanifold, and $\nu = \nu(Z \subset Y) \to Z$ the normal bundle. Then there exists an embedding $\varphi : \nu \to Y$ such that $\varphi|_{Z} = \text{id}_{Z}$ and $\varphi(\nu) \subset Y$ is an open subset.
The theorem is illustrated in Figure 93. We identify $Z$ with the image $Z_\nu \subset \nu$ of the zero section. Theorem 28.21 can be proved from Theorem 16.8, but we do not do so in these notes.

Remark 28.22. A variation of the tubular neighborhood theorem holds for a neighborhood of the boundary $\partial X$ of a manifold $X$ with boundary. Namely, there exists a collar: an embedding $[0,1) \times \partial X \hookrightarrow X$ which is the identity on $\{0\} \times \partial X$.

(28.23) The normal bundle to the diagonal. Let $X$ be a smooth manifold. The diagonal $\Delta \subset X \times X$ is a submanifold, so there is a short exact sequence

\[(28.24)\quad 0 \rightarrow T\Delta \rightarrow TX \oplus TX \rightarrow \nu(\Delta \subset X \times X) \rightarrow 0\]

of vector bundles over $\Delta$. For $p \in X$ the coset of $(\xi_1, \xi_2) \in T_pX \oplus T_pX$ under the diagonal action of $T_pX$ by translation contains a unique vector of the form $(\xi, 0)$. Hence there is a splitting $\nu(\Delta \subset X \times X) \rightarrow TX \oplus TX$ whose image is $TX \oplus 0$. This proves the following.

Lemma 28.25. The normal bundle $\nu(\Delta \subset X \times X) \rightarrow \Delta$ is canonically isomorphic to the tangent bundle $TX \rightarrow X$. 

Figure 94. A tubular neighborhood of $\Delta \subset X \times X$
Theorem 28.27. Let \( X \) be a compact oriented manifold and \( \pi_{TX} : TX \to X \) its tangent bundle. Then we have the equality of Euler numbers

\[
\chi(X) = \chi(\pi_{TX}).
\]

\[\tag{28.28}\]

Proof. Let \( \xi : X \to TX \) be a vector field which is transverse to the zero section (Lemma 28.12), and fix a tubular neighborhood \( \varphi : TX \hookrightarrow X \times X \) of the diagonal \( \Delta \).

\[
\chi(\pi_{TX}) = \#^{TX}(\xi, Z_{TX}) = \#^{X \times X}(\varphi \circ \xi, \Delta) = \chi(X).
\]

\[\tag{28.29}\]

Corollary 28.30. Let \( X \) be a compact oriented manifold.

1. If \( \dim X \) is odd, then \( \chi(X) = 0 \).
2. If \( X \) admits a nowhere vanishing vector field, then \( \chi(X) = 0 \).

(28.31) A special case of Poincaré-Hopf. If \( \xi \) is a vector field which vanishes at \( p \in X \), then (28.17) is the composition

\[
T_pX \xrightarrow{d\xi_p} T_0TX \xrightarrow{\text{proj}} T_pX
\]

which is usually identified as the differential of the vector field. By Lemma 28.16 it is invertible iff \( \xi \) is transverse to the zero section at \( p \). In that case its degree is called the index of \( \xi \) at \( p \), denoted \( \text{ind}_p \xi = \pm 1 \).

Theorem 28.33. Let \( X \) be a compact oriented manifold and \( \xi \) a vector field which is transverse to the zero section. Then

\[
\chi(X) = \sum_{p \in \text{Zero}(\xi)} \text{ind}_p \xi.
\]

(28.34)

There is a generalization of the index to an isolated zero of a vector field for which (28.34) still holds. We discuss that, or at least its analog for isolated fixed points of a map \( X \to X \), in the next lecture.

Example 28.35 (Euler number of \( S^n \)). Construct \( S^n \) by the surjection \( \mathbb{A}^n \cup \mathbb{A}^n \to S^n \) in which two copies of affine space are glued on the complement of a point by inversion. Namely, identify \( \mathbb{A}^n \setminus \{0\} \approx \mathbb{R}^{>0} \times S^{n-1} \) (“polar coordinates”). Then the overlap map is

\[
\mathbb{R}^{>0} \times S^{n-1} \longrightarrow \mathbb{R}^{>0} \times S^{n-1}
\]

\((r, \Theta) \mapsto (r^{-1}, \Theta)\)
More simply the map is \( s = r^{-1} \), and so \( ds = -r^{-2} dr \) from which \( r \partial / \partial r \rightarrow -s \partial / \partial s \) under (28.36). The latter radial vector field glues then to a global vector field \( \xi \) on \( S^n \) which vanishes transversely at the two poles of \( S^n \). The differential of \( r \partial / \partial r \) at \( r = 0 \) is the identity map, so for \( \xi \) the differential at one pole is \( \text{id} \) and at the other is \(- \text{id}\). Now Theorem 28.33 gives

\[
\chi(S^n) = 1 + (-1)^n = \begin{cases} 
0, & n \text{ odd;}
2, & n \text{ even.}
\end{cases}
\]

**Lefschetz numbers**

The basic definition is a variant of Definition 28.2.

**Definition 28.38.** Let \( X \) be a compact oriented manifold and \( f: X \to X \) a smooth map. The **Lefschetz number** of \( f \) is

\[
L(f) = \#^{X \times X} (\Gamma(f), \Delta),
\]

where \( \Gamma(f) \subset X \times X \) is the graph of \( f \).

Observe that \( \Gamma(f) \cap \Delta = \text{Fix}(f) \) is the fixed point set of \( f \).

The following properties are immediate from Proposition 27.10(1) and Definition 28.2.

**Proposition 28.40.**

1. If \( f_0 \simeq f_1 \) are smoothly homotopic maps, then \( L(f_0) = L(f_1) \).
2. If \( f \simeq \text{id}_X \), then \( L(f) = \chi(X) \).
3. If \( L(f) \neq 0 \), then \( \text{Fix}(f) \neq \emptyset \).

Assertion (2) leads to effective computations of the Euler number of a manifold, as we illustrate below. Assertion (3) is a fixed point theorem, effective if we have a method of computing the Lefschetz number. We discuss this more in the next lecture.

**Example 28.41.** Let \( G \) be a positive dimensional Lie group and \( g \in G \) a non-identity element which is connected to the identity by a smooth path. Let \( L_g: G \to G \) be left multiplication by \( g \), i.e., the diffeomorphism \( x \mapsto gx \). Then \( L_g \simeq \text{id}_G \) and \( \text{Fix}(L_g) = \emptyset \). Therefore, \( \chi(G) = 0 \).
**Lefschetz fixed points.** The following is in keeping with our theme that transverse intersections are special.

**Definition 28.43.** Let $X$ be a compact oriented manifold and $f: X \to X$ a smooth map. Then $p \in \text{Fix}(f)$ is a **Lefschetz fixed point** of $f$ if $\Gamma(f) \cap_{(p,p)} \Delta$.

We compute the local intersection number at a Lefschetz fixed point.

**Proposition 28.44.** Let $X$ be a compact oriented manifold and $f: X \to X$ a smooth map.

1. $p \in \text{Fix}(f)$ is Lefschetz iff $1 - df_p: T_pX \to T_pX$ is invertible.
2. If $p$ is a Lefschetz fixed point, then
   \[ \#_{(p,p)}^{X \times X}(\Gamma(f), \Delta) = \deg(1 - df_p) = \pm 1. \]

Here and hereafter we use ‘1’ in place of ‘id$_{T_pX}$’ for ease of reading. If $p$ is a Lefschetz fixed point, then we define the **local Lefschetz number**

\[ L_p(f) = \deg(1 - df_p) = \pm 1. \]

Note that this equals $\text{sign det}(1 - df_p)$. In the next lecture we generalize the local Lefschetz number from Lefschetz fixed points to general isolated fixed points.

**Remark 28.46.** The map $1 - df_p$ is invertible iff the map $df_p$ has no nonzero fixed vector. In other words, $p$ is a Lefschetz fixed point of $f$ iff the linearization of $f$ at $p$ has a single fixed point: the zero vector. In still other words, the condition is that $1$ is not an eigenvalue of the differential $df_p$.

**Proof.** Set $V = T_pX$ and $T = df_p: V \to V$. The nonlinear maps $X \to X \times X$ with image $\Gamma(f)$ and $\Delta$, respectively, have differentials the linear maps $V \to V \oplus V$ given by

\[ \xi \mapsto (\xi, T\xi) \]

\[ \xi \mapsto (\xi, \xi) \]

Compose with the orientation-preserving automorphism

\[ V \oplus V \to V \oplus V \]

\[ (\xi_1, \xi_2) \mapsto (\xi_1 - \xi_2, \xi_2) \]
to obtain the maps

(28.49) \[
    \xi \mapsto ((1 - T)\xi, T\xi) \\
    \xi \mapsto (0, \xi)
\]

The images are transverse iff \(1 - T\) is invertible, and if so the map

(28.50) \[
    V \oplus V \to V \oplus V \\
    (\xi_1, \xi_2) \mapsto ((1 - T)\xi_1, T\xi_1 + \xi_2)
\]

preserves or reverses orientation according as \(1 - T: V \to V\) preserves or reverses orientation. □

(28.51) \textit{Lefschetz maps.} More variations on our theme that transversality is generic follow.

**Definition 28.52.** Let \(X\) be a compact oriented manifold and \(f: X \to X\) a smooth map. Then \(f\) is \textit{Lefschetz} if \(\Gamma (f) \not\subseteq \Delta\).

If \(f\) is Lefschetz, then its global Lefschetz number is the sum of local Lefschetz numbers:

(28.53) \[
    L(f) = \sum_{p \in \text{Fix}(f)} L_p(f).
\]

**Theorem 28.54.** Let \(X\) be a compact manifold and \(f: X \to X\) a smooth map. Then there exists a smooth homotopy \(f_t: X \to X, t \in [0, 1]\) such that \(f_0 = f\) and \(f_1\) is Lefschetz.

**Proof.** By Theorem 11.11 we can and do embed \(X\) in an affine space \(A\) over a normed linear space \(V\). Let \(S = B_1(0) \subset V\) be the unit ball. In Corollary 16.9 we constructed a submersion \(F: S \times X \to X\) such that \(F(0, p) = f(p)\) for all \(p \in X\) and the partial differential \(dF_{(s,p)}^1: V \to T_{F(s,p)}X\) is surjective for all \(s \in S\) and \(p \in X\). Define

(28.55) \[
    G: S \times X \to X \times X \\
    (s, p) \mapsto (p, F(s, p))
\]

Then the differential of \(G\) has the form \(dG = \begin{pmatrix} 0 & \text{id} \\ dF_{1} & dF_{2} \end{pmatrix}\), so \(G\) is a submersion and hence \(G \not\subseteq \Delta\). It follows from Theorem 15.17 that for a dense set of \(s \in S\) the map \(p \mapsto (p, F(s, p))\) is transverse to \(\Delta\). For any such \(s\) set \(f_1(p) = F(s, p)\). □

(28.56) \textit{Computations of Euler number via Lefschetz maps.} We illustrate on real projective space and the first real Grassmannian which is not a projective space. You can generalize to general Grassmannians and to the complex case as well.
Example 28.57 (Euler number of real projective space). Let

\[
(28.58) \quad T = \begin{pmatrix} 1 & & & \\ 2 & \ddots & & \\ & \ddots & \ddots & \\ & & n+1 & \end{pmatrix}.
\]

This linear transformation of \(\mathbb{R}^{n+1}\) induces the eigenspace decomposition

\[
(28.59) \quad \mathbb{R}^{n+1} \cong L_1 \oplus \cdots \oplus L_{n+1}
\]

where \(L_i\) is the \(i\)th coordinate line and \(T\) acts as multiplication by \(i\) on \(L_i\). The linear map \(T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}\) induces a projective linear map \(f = f_T : \mathbb{RP}^n \to \mathbb{RP}^n\) which is Lefschetz (as we will see) with \(\text{Fix}(f) = \{L_1, \ldots, L_{n+1}\}\). Recall that for any vector space \(V\) and line \(L \subset V\) we have \(T_L^*V \cong \text{Hom}(L, V/L)\). In this case we identify

\[
(28.60) \quad T_L^*\mathbb{RP}^n \cong \text{Hom}(L_i, L_1) \oplus \cdots \oplus \text{Hom}(L_i, L_{i-1}) \oplus \cdots \oplus \text{Hom}(L_i, L_{n+1}).
\]

The differential of \(f\) is the map induced on the Hom spaces by the linear map \(T\), which acts by conjugation on an element of \(\text{Hom}(L_i, L_j)\), so acts as multiplication by \(j/i\). In other words, \((28.60)\) is the eigenspace decomposition of \(df_{L_i}\), so too of \(1 - df_{L_i}\):

\[
(28.61) \quad 1 - df_{L_i} = \left(1 - \frac{1}{i}\right) \oplus \left(1 - \frac{2}{i}\right) \oplus \cdots \oplus \left(1 - \frac{i}{i}\right) \oplus \cdots \oplus \left(1 - \frac{n+1}{i}\right).
\]

Thus \(1 - df_{L_i}\) is invertible, and the local Lefschetz number is the parity of the number of negative eigenvalues, which is \((-1)^{n+1-i}\). Since \(T\) is homotopic to the identity matrix, it follows that \(f \cong \text{id}_{\mathbb{RP}^n}\) and so

\[
(28.62) \quad \chi(\mathbb{RP}^n) = L(f) = \sum_{i=1}^{n+1} (-1)^{n+1-i} = \begin{cases} 0, & n \text{ odd;} \\ 1, & n \text{ even.} \end{cases}
\]

Remark 28.63. Note that \(\mathbb{RP}^n\) is not orientable if \(n\) is even, but nonetheless the computation is valid; see Remark 28.4. Also, the Euler number is multiplicative for a finite covering space, and that is borne out in this example by comparing \((28.62)\) with \((28.37)\).

Example 28.64 (The Grassmannian \(\text{Gr}_2(\mathbb{R}^4)\)). Set \(X = \text{Gr}_2(\mathbb{R}^4)\), and let \(f : X \to X\) be the map on 2-planes induced from the linear transformation

\[
(28.65) \quad T = \begin{pmatrix} 1 & & \\ 2 & 3 & \\ & & 4 \end{pmatrix}
\]
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of \( \mathbb{R}^4 \). Then the fixed point set consists of the 6 coordinate 2-planes

\[(28.66) \quad \text{Fix}(f) = \{ L_1 \oplus L_2, L_1 \oplus L_3, L_1 \oplus L_4, L_2 \oplus L_3, L_2 \oplus L_4, L_3 \oplus L_4 \}\]

The tangent space is hom from the sub to the quotient, so for example

\[(28.67) \quad T_{L_1 \oplus L_2}X \cong \text{Hom}(L_1 \oplus L_2, L_3 \oplus L_4) \cong \text{Hom}(L_1, L_3) \oplus \text{Hom}(L_1, L_4) \oplus \text{Hom}(L_2, L_3) \oplus \text{Hom}(L_2, L_4).\]

The differential \( df_{L_1 \oplus L_2} \) is computed from \( (28.65) \) as in the previous example, so \( (28.67) \) is the eigenspace decomposition and \( df_{L_1 \oplus L_2} \) acts as \( \frac{3}{4} \oplus \frac{3}{4} \oplus \frac{3}{4} \oplus \frac{3}{4} \). We see that \( f \) is Lefschetz at this fixed point (and at the other 5); the Lefschetz number counts the parity of the number of eigenvalues greater than 1. Hence \( L_{L_1 \oplus L_2}(f) = +1 \). The result at all fixed points:

\[
\begin{align*}
L_{L_1 \oplus L_2}(f) &= +1 \\
L_{L_1 \oplus L_3}(f) &= -1 \\
L_{L_1 \oplus L_4}(f) &= +1 \\
L_{L_2 \oplus L_3}(f) &= +1 \\
L_{L_2 \oplus L_4}(f) &= -1 \\
L_{L_3 \oplus L_4}(f) &= +1
\end{align*}
\]

Therefore,

\[(28.69) \quad \chi(\text{Gr}_2(\mathbb{R}^4)) = L(f) = 2.\]

Lecture 29: More on Lefschetz numbers

In the first part of this lecture we develop a formula for computing the global Lefschetz number of a self map with isolated fixed points which need not be Lefschetz. Then we introduce de Rham cohomology and state a theorem which identifies the global Lefschetz number. We conclude with a brief discussion of fixed point theorems.

Isolated fixed points

\[(29.1) \quad \text{An example.} \quad \text{We begin with an echo of (26.45) and (26.55). Fix} \; m \in \mathbb{Z}^{>0} \quad \text{and consider the function}\]

\[(29.2) \quad f : \mathbb{C} \to \mathbb{C} \quad z \mapsto z + z^m\]
Then \( \text{Fix}(f) = \{0\} \). Then

\[
(29.3) \quad 1 - df_0 = \begin{cases} -1, & m = 1; \\ 0, & m > 1, \end{cases}
\]

so 0 is a Lefschetz fixed point iff \( m = 1 \). For \( m > 1 \) we make the perturbation

\[
(29.4) \quad f_\epsilon(z) = z + z^m - \epsilon^m, \quad \epsilon \in \mathbb{R}.
\]

If \( \epsilon \neq 0 \), then \( \text{Fix}(f_\epsilon) = \{\epsilon, \epsilon\omega, \ldots, \epsilon\omega^{m-1}\} \), where \( \omega = e^{2\pi i/m} \) is a primitive \( n \)th root of unity. Each of these fixed points is Lefschetz: at \( \epsilon z \) we compute \( 1 - df_\epsilon \) is multiplication by \(-me^{m-1}\omega^{k(m-1)}\), which is invertible. This complex linear transformation on \( \mathbb{C} \) preserves the orientation of the underlying real vector space \( \mathbb{R}^2 \), so each local Lefschetz number is \( 1 \) and the total Lefschetz number is \( m \).

**Remark 29.5.** The intuition is that under a generic perturbation, an isolated fixed point of a self-map breaks up (explodes) into a constellation of Lefschetz fixed points. The sum of the local Lefschetz numbers of those fixed points is an invariant we attach to the unperturbed fixed point, which may not be Lefschetz. We develop this idea in general.

**(29.6)** *A linear algebra lemma.* The proof which follows is based on (and proves) the “polar decomposition” of a linear transformation.

**Lemma 29.7.** Let \( V \) be a finite dimensional real inner product space and \( S : V \to V \) an invertible linear transformation. Then \( S \) is homotopic to an orthogonal transformation.

**Proof.** The endomorphism \( P = S^*S \) is positive and self-adjoint, hence it is diagonalizable. Decompose \( V = V_1 \oplus \cdots \oplus V_r \) into the orthogonal eigenspaces of \( P \), so that \( P \) acts as multiplication by a scalar \( \lambda_i > 0 \) on \( V_i \). Define \( Q : V \to V \) to act as \( \sqrt{\lambda_i} \) on \( V_i \). Then \( O := SQ^{-1} \) is orthogonal:

\[
(29.8) \quad (SQ^{-1})^* (SQ^{-1}) = Q^{-1} S^* SQ^{-1} = Q^{-1} Q^2 Q^{-1} = \text{id}_V.
\]

Define a homotopy \( Q_t : V \to V, t \in [0,1] \), in which \( Q_t \) acts as multiplication by \((1-t) + t\sqrt{\lambda_i} \) on \( V_i \); the desired homotopy of \( S \) to \( O \) is \( S_t = OQ_t \). \(
\)

**(29.9)** *Isolated fixed point in affine space.* We work locally so on an open subset of affine space.

**Theorem 29.10.** Let \( V \) be a finite dimensional real inner product space, \( A \) an affine space over \( V \), \( U \subset A \) an open subset, \( f : U \to A \) a smooth map, \( p \in U \) a fixed point of \( f \), and \( \epsilon \) a positive number such that \( \text{Fix}(f) \cap B_\epsilon(p) = \{p\} \). Let \( S_\epsilon(p) \) be the sphere of radius \( \epsilon \) about \( p \) and define

\[
(29.11) \quad \varphi_{p,\epsilon}(f) : S_\epsilon(p) \to S(V), \quad q \mapsto \frac{q - f(q)}{\|q - f(q)\|},
\]

where \( S(V) \subset V \) is the unit sphere. Then
There exists a homotopy \( f_t : U \to A \), \( t \in [0,1] \), such that \( f_0 = f \) on \( U \), \( f_t = f \) on \( U \setminus B_\epsilon(p) \) for all \( t \in [0,1] \), and \( g = f_1 \)|_{B_\epsilon(p)} is Lefschetz.

(2) The sum of Lefschetz numbers of the fixed points of \( f_1 \) in \( B_\epsilon(p) \) is

\[
\deg \varphi_{p,\epsilon}(f) = \sum_{p' \in \text{Fix}(g)} L_{p'}(g). \tag{29.12}
\]

The left hand side of (29.12) measures the local contribution of the isolated fixed point \( p \) of \( f \) to the global Lefschetz number, once we transfer to a compact manifold.

**Proof.** Let \( \rho : U \to [0,1] \) be a smooth function such that \( \rho = 1 \) on \( B_\epsilon/2(p) \) and \( \text{supp}(\rho) \subset B_\epsilon(p) \). For \( \xi \in V \) set

\[
f_\xi^t(q) = f(q) + t\rho(q)\xi, \quad t \in [0,1]. \tag{29.13}
\]

Then \( f_0^\xi = f \) on \( U \) and \( f_1^\xi = f \) on \( U \setminus B_\epsilon(p) \) for all \( t \in [0,1] \). If \( q \in B_\epsilon(p) \setminus B_\epsilon/2(p) \), then

\[
\|q - f_1^\xi(q)\| \geq \|q - f(q)\| - |t|\|
\]

Since \( f \) has no fixed points on \( \overline{B_\epsilon(p)} \) other than \( p \), we can and do choose \( \delta > 0 \) such that \( \|q - f(q)\| > \delta \) on \( \overline{B_\epsilon(p)} \setminus B_\epsilon/2(p) \). Then from (29.14), if \( \|\xi\| < \delta/2 \) and \( q \in B_\epsilon(p) \setminus B_\epsilon/2(p) \) we have \( q \neq f_1^\xi(q) \).

Therefore, \( \text{Fix}(f_1^\xi(q)) \cap B_\epsilon(p) \subset B_\epsilon/2(p) \). Observe\(^{33}\) that \( f_1^\xi(q) = f(q) + t\xi \) on \( B_\epsilon/2(p) \). Choose \( \xi \in B_{\delta/2}(0) \subset V \) to be a regular value of

\[
B_\epsilon/2(p) \to V
q \mapsto q - f(q) \tag{29.15}
\]

and set \( f_t = f_1^\xi \). Then if \( p' \in \text{Fix}(g) \cap B_\epsilon/2(p) \), the map \( 1 - dg_{p'} : V \to V \) is an isomorphism, i.e., \( p' \) is a Lefschetz fixed point of \( g \). This proves (1). Note that \( \text{Fix}(g) \cap \overline{B_\epsilon(p)} = \{p_1, \ldots, p_N\} \) is a finite set since Lefschetz fixed points are isolated.

\(^{33}\)A translation deformation is also what we used in (29.4).
Figure 98. Reduction of the computation to Lefschetz fixed points

By the homotopy invariance of degree, $\deg \varphi_{p,\epsilon}(f_t)$ is constant in $t$ and equals the left hand side of (29.12) at $t = 0$. (Note that there are no fixed points of $f_t$ on $S_\epsilon(p)$, so $\varphi_{p,\epsilon}(f_t)$ is well-defined.) Choose an open ball $B_i \subset B_\epsilon(p)$ about $p_i$ and arrange that they are pairwise disjoint. The bordism invariance of degree (Proposition 25.35(3)) applied to $B_\epsilon(p) \setminus \bigsqcup_i B_i$ (see Figure 98) implies

\[
\deg \varphi_{p,\epsilon}(g) = \sum_i \deg_{B_i} \left( q \longmapsto \frac{q - g(q)}{\|q - g(q)\|} \right).
\]

Figure 99. Reduction to an orthogonal map

Fix $i$ and define the homotopy

\[
g_t(p_i + \eta) = \begin{cases} 
g(p_i + t\eta) & t \in (0, 1]; \\
 dg_{p_i} & t = 1. 
\end{cases}
\]

Under this homotopy the degree in the $i^{th}$ term on the right hand side of (29.16) does not change. Set $S = 1 - dg_{p_i} : V \to V$. Apply Lemma 29.7 to see that the degree in question equals

\[
\deg(\eta \longmapsto O(\eta)), \quad \eta \in S(V) \subset V
\]
for an orthogonal transformation $O : V \to V$. We must prove that this degree equals $\det O$. By composition with a rotation, which does not change the degree (Proposition 25.35(4)), we may assume that $O$ has a fixed vector $\eta_0$. The degree in (29.18) is $\pm 1$ according to whether the differential of $O |_{S(V)}$ at $\eta_0$ preserves or reverses orientation on $T_{\eta_0}S(V)$.

There is an orthogonal decomposition $V = \mathbb{R} \cdot \eta_0 \oplus T_{\eta_0}S(V)$ which is invariant under $O$, and since $O |_{S(V)} = +1$, it follows that $\det O = \det O |_{T_{\eta_0}S(V)}$. It remains to observe that the differential of $O |_{S(V)}$ at $\eta_0$ equals $O |_{T_{\eta_0}S(V)}$. □

**Corollary 29.19.** In the situation of Theorem 29.10, if $p$ is a Lefschetz fixed point of $f$, then $L_p(f) = \deg \varphi_{p,\epsilon}(f)$.

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**Figure 100.** Exploding an isolated fixed point on a manifold

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**Isolated fixed points on a smooth manifold.** Let $X$ be a compact manifold, assumed oriented for convenience, and suppose $f : X \to X$ is a smooth map with isolated fixed points. (A fixed point $p \in X$ is isolated if there exists an open neighborhood $N \subset X$ of $p$ such that $\text{Fix}(f) \cap N = \{p\}$.) For each $p \in \text{Fix}(f)$ we want to define a local Lefschetz number $L_p(f) \in \mathbb{Z}$ so that the global Lefschetz number $L(f)$ is the sum of the local Lefschetz numbers. We sketch the construction now.

Choose a coordinate chart $(U, x)$ such that $p \in U$, say $x : U \to A$ for an affine space $A$ over an inner product space, set $\tilde{p} = \tilde{p}$, and fix $\epsilon > 0$ such that $x^{-1}(B_\epsilon(\tilde{p}))$ contains no fixed points other than $p$ and $f(x^{-1}(B_\epsilon(\tilde{p}))) \subset U$. Then $f$ transports via $x$ to a map $\tilde{f} : B_\epsilon(\tilde{p}) \to A$ with $\tilde{p}$ as its unique fixed point. Define

$$L_p(f) = \deg \varphi_{p,\epsilon}(\tilde{f}),$$

where $\varphi_{p,\epsilon}(\tilde{f})$ is the map (29.11). Then Theorem 29.10 identifies $L_p(f)$ with the sum of the local Lefschetz numbers at Lefschetz fixed points of a perturbation of $f$ supported in $x^{-1}(B_\epsilon(\tilde{p}))$. Since these local Lefschetz numbers are defined intrinsically on $X$, without reference to a coordinate chart, it follows that (29.21) is independent of the choice of coordinate chart and of the choice of $\epsilon$. The assertion about the global Lefschetz number follows from (29.12).
Introducing de Rham cohomology

*(29.22)* Definition of de Rham cohomology. Let \( X \) be a smooth \( n \)-dimensional manifold. Recall the de Rham complex *(19.23)*

\[
0 \longrightarrow \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(X) \longrightarrow 0,
\]

in which \( d^2 = 0 \).

**Definition 29.24.**

1. A differential form \( \alpha \in \Omega^*(X) \) is *closed* if \( d\alpha = 0 \).
2. A differential form \( \alpha \in \Omega^*(X) \) is *exact* if there exists \( \beta \in \Omega^*(X) \) such that \( \alpha = d\beta \).

Since \( d^2 = 0 \) we have for each \( k \in \{0, 1, \ldots, n\} \) the inclusions

\[
d\Omega^{k-1}(X) \subset \Omega^k_{\text{closed}}(X) \subset \Omega^k(X),
\]

where \( d\Omega^{k-1}(X) \) is the space of exact \( k \)-forms.

**Definition 29.26.** The *de Rham cohomology* in degree \( k \) is the vector space

\[
H^k_{\text{dR}}(X) = \frac{\Omega^k_{\text{closed}}(X)}{d\Omega^{k-1}(X)}.
\]

**Example 29.28.** A function \( f \in \Omega^0(X) \) is closed iff \( f \) is locally constant, and it is exact iff it vanishes. Hence \( H^0_{\text{dR}}(X) \) is the vector space of locally constant functions on \( X \).

**Example 29.29.** We indicate an isomorphism

\[
H^1_{\text{dR}}(\mathbb{R}/\mathbb{Z}) \xrightarrow{\cong} \mathbb{R}
\]

Namely, a function on \( \mathbb{R}/\mathbb{Z} \) lifts under the covering \( \mathbb{R} \to \mathbb{R}/\mathbb{Z} \) to a periodic function \( f : \mathbb{R} \to \mathbb{R} \), i.e., one for which \( f(x + 1) = f(x) \) for all \( x \in \mathbb{R} \). Similarly, a 1-form \( \alpha \in \Omega^1(\mathbb{R}/\mathbb{Z}) \) lifts to a 1-form \( \alpha = g(x)dx \in \Omega^1(\mathbb{R}) \) in which \( g \) is a periodic function. Every 1-form on a 1-manifold is closed, and we claim \( \alpha \) is exact iff

\[
\int_{\mathbb{R}/\mathbb{Z}} \alpha = \int_0^1 g(x)dx = 0.
\]

The isomorphism *(29.30)* is integration over \( \mathbb{R}/\mathbb{Z} \). We leave the reader to fill in the details.
(29.32) **Pullbacks.** Let \( f : X' \to X \) be a smooth map of smooth manifolds. As noted in Theorem 19.28 there is an induced pullback map of differential forms

\[
(29.33) \quad f^* : \Omega^k(X) \to \Omega^k(X'),
\]

and \( df^* = f^*d \). This latter implies that \( f^* \) maps closed forms to closed forms and exact forms to exact forms, hence induces a map

\[
(29.34) \quad f^* : H^k_{\text{dR}}(X) \to H^k_{\text{dR}}(X')
\]
on de Rham cohomology.

(29.35) **The de Rham theorem.** The utility of de Rham cohomology arises from a comparison with other cohomology theories on a smooth manifold, say the singular theory. The book by Frank Warner has a very nice treatment of the following.

**Theorem 29.36.** Let \( X \) be a smooth manifold. Then there exists a natural isomorphism

\[
(29.37) \quad H^k_{\text{dR}}(X) \to H^k(X; \mathbb{R}).
\]

The codomain of \( (29.37) \) is the singular cohomology with real coefficients, which is isomorphic to \( \text{Hom}(H_k(X), \mathbb{R}) \), where \( H_k(X) \) is the singular homology group. The map \( (29.37) \) is constructed by integration over “smooth singular chains”.

**Fixed point theorems**

(29.38) **The Lefschetz fixed point theorem.** We have defined the Lefschetz number of a self map \( f : X \to X \) in (28.39) as an intersection number, and in (29.20) have a formula in terms of local data at fixed points in case every fixed point of \( f \) is isolated. This has more power if we can compute the global intersection number effectively. The following theorem does this in terms of de Rham cohomology and the induced map \( (29.34) \). Implicit is the assertion that the de Rham cohomology vector spaces of a compact manifold are finite dimensional.

**Theorem 29.39.** Let \( f : X \to X \) be a self map of a compact oriented manifold. Then

\[
(29.40) \quad L(f) = \sum_{k=0}^{\dim X} (-1)^k \text{Tr}\left(f^*\left|_{H^k_{\text{dR}}(X)}\right.\right).
\]

Sadly, it is beyond the scope of this course to prove Theorem 29.39.
Remark 29.41. Fixed point theorems are important in many parts of geometry and beyond. For example, recall that we use the contraction fixed point theorem to prove the inverse function theorem and to construct solutions to ordinary differential equations, so more geometrically to construct integral curves of vector fields. There are also infinite dimensional analogs of degree (Leray-Schauder) and of fixed point theorems (Schauder) with many applications to integral and partial differential equations; see *Topics in Nonlinear Functional Analysis* by Louis Nirenberg.

Remark 29.42. The relationship between fixed points and cohomology expressed in the Lefschetz fixed point theorem has an arithmetic cousin which is very powerful. It uses the Frobenius map of a variety defined over a finite field.

Remark 29.43. Atiyah-Bott prove a generalization of Theorem 29.39 for linear elliptic differential operators, a theorem with diverse applications in topology, geometry, and representation theory.