Modularity and Invertibility

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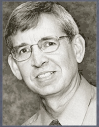
April 16, 2011

Work in progress with Constantin Teleman

Happy Birthday, Mike!



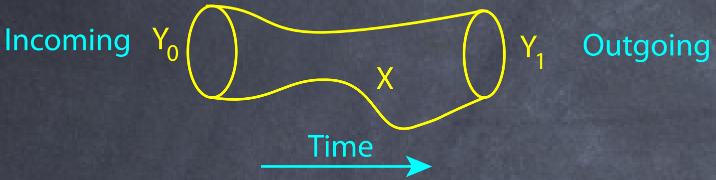
Quantum field theories are traditionally studied on flat Minkowski spacetime of dimension 4. Wick rotation to Euclidean field theory brings us to flat Euclidean space, and from there we can "couple to background gravity" to study theories on Riemannian 4-dimensional curved manifolds. Call the field theory F.

Canonical Quantization: Assigns a quantum Hilbert space to a Riemannian 3-manifold Y:

$$Y^3 \longmapsto F(Y)$$

Path Integral: Assigns a path integral to a Riemannian 4-manifold X with incoming and outgoing 3-manifold boundary:

$$(X^4, Y_0, Y_1) \longmapsto \big(F(X)\colon F(Y_0) \to F(Y_1)\big)$$

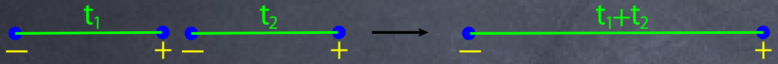


We can imagine QFTs in any (spacetime) dimension n, though examples are hard to come by as n increases.

$n = 1$ is the Euclidean version of quantum mechanics. To a point we attach a (usually ∞-dimensional) Hilbert space $\mathcal{H}$ and to an interval of length $t > 0$ the operator $e^{-tH/\hbar}$, where $H\colon \mathcal{H} \to \mathcal{H}$ is the Hamiltonian.

$$\mathcal{H} \xrightarrow{e^{-tH/\hbar}} \mathcal{H}$$

Intervals of lengths t_1 and t_2 *compose* by gluing to a single interval of length $t_1 + t_2$, which goes over into the semigroup law of quantum mechanics:

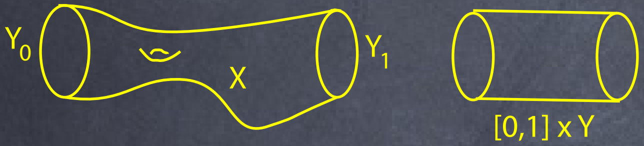


$$e^{-(t_1+t_2)H/\hbar} = e^{-t_2 H/\hbar} \circ e^{-t_1 H/\hbar}$$

Topological Quantum Field Theory (TQFT)

Abandon all geometric structure—metrics, conformal structures, etc. Only keep topological structures, such as orientations. This retains the algebraic structure of quantum field theory while leaving behind many analytical difficulties. We also assume all manifolds are compact.

In dimension n we still attach a vector space $F(Y)$ to a closed $(n-1)$-manifold and a linear map $F(X)\colon F(Y_0) \to F(Y_1)$ to a compact n-manifold with boundary $\partial X = Y_0 \cup Y_1$. Observe that the cylinder $[0,1] \times Y$ is the identity map under gluing.

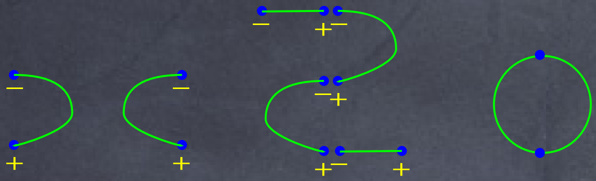


In particular, for $n = 1$ we simply have a vector space $\mathcal{H} = F(\mathrm{pt})$ attached to a point and the identity map $\mathrm{id}\colon \mathcal{H} \to \mathcal{H}$ attached to the interval. F is a homomorphism: disjoint union $\longrightarrow$ tensor product

gluing $\longrightarrow$ composition

Finite Dimensionality in TQFT

Continuing with $n = 1$, if we include orientations then $F(pt_+) = \mathcal{H}$ is a vector space and $F(\text{pt}\_) = \mathcal{H}'$ another vector space. If we consider carefully the "elbows" and "S-diagram" then we deduce the following:



- The right elbow gives a form $b\colon \mathcal{H} \otimes \mathcal{H}' \longrightarrow \mathbb{C}$
- The left elbow gives a co-form $b^*\colon \mathbb{C} \longrightarrow \mathcal{H}' \otimes \mathcal{H}$
- As the S-diagram is a cylinder, so the identity map, we deduce

$$\mathcal{H} \xrightarrow{\text{id} \otimes b^*} \mathcal{H} \otimes \mathcal{H}' \otimes \mathcal{H} \xrightarrow{b \otimes \text{id}} \mathcal{H}$$

  is the identity, and it follows that $\mathcal{H}$ is finite dimensional
- S^1 is left elbow followed by right elbow, so equals $\dim \mathcal{H}\colon \mathbb{C} \to \mathbb{C}$

Nondegeneracy and Algebra Structures in TQFT

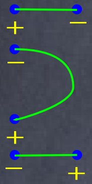
For an $n = 1$ *unoriented* theory there is an additional piece of data. Now there is only one point, so one finite dimensional vector space $F(\mathrm{pt}) = \mathcal{H}$, and the right elbow gives a bilinear form

$$: \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$$

The S-diagram now implies that this form is nondegenerate.

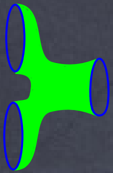
In any dimension, if F is a TQFT and S^k the k-dimensional sphere, then $F(S^k)$ is an algebra of some sort. For our $n = 1$ dimensional oriented theory we have the E_1-algebra (associative algebra)

$$F(S^0) = F(\mathrm{pt}_+ \cup \mathrm{pt}_-) = \mathcal{H} \otimes \mathcal{H}' = \mathrm{End}\,\mathcal{H}$$



This is the 1-ball D^1 with two sub-1-balls removed. The inner 0-spheres are incoming; the outer 0-sphere is outgoing.

This algebra structure and nondegeneracy is familiar in 2-dimensional topological field theories. Now there is a vector space $\mathcal{H}$ associated to the circle S^1, and by the pair of pants diagram it obtains a multiplication

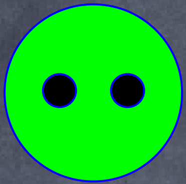


$$: \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H}$$

The incoming and outgoing disks define an identity element and a trace; the associated bilinear form is nondegenerate: $\mathcal{H}$ is a Frobenius algebra.

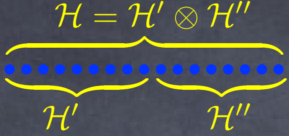
This algebra is commutative, and the pair of pants can be understood as the 2-disk D^2 with 2 sub-2-disks removed. The inner 1-spheres (circles) are incoming and the outer 1-spheres are outgoing. This is called an E_2-algebra, the '2' referring to 2-disk.

The algebra structure is the topological version of the operator product algebra in quantum field theory.

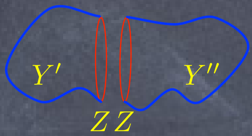


Locality of the Quantum Hilbert Space

The gluing law—which in $n = 1$ is the group law of quantum mechanics—encodes the locality of the path integral in spacetime. Now we go further and require that the Hilbert space be local in space. This is most clear in lattice models, where the global Hilbert space is a tensor product of local Hilbert spaces at the lattice points.



When space is not discrete the equation $\mathcal{H} = \mathcal{H}' \otimes \mathcal{H}''$ is more complicated: there is a linear category $F(Z)$ associated to an $(n-2)$-dimensional manifold Z, and the gluing leads to the composition



$$F(Y) : F(\varnothing^{n-2}) \xrightarrow{F(Y')} F(Z) \xrightarrow{F(Y'')} F(\varnothing^{n-2})$$

$F(\varnothing)$ is the "tensor unit", the linear category $\mathbf{Vect}_{\mathbb{C}}$ of $\mathbb{C}$-vector spaces.

This idea of extended quantum field theory was explored in various guises in the early '90s by several mathematicians, including Kazhdan, Segal, Lawrence, Reshetikhin, Turaev, Walker, Kapranov, Voevodsky, Crane, and Yetter. It is a topic of current interest for both mathematicians and physicists.

Consider now an $n = 3$-dimensional topological quantum field theory F, such as Chern-Simons theory. We assume it is a 1-2-3 theory, that is:

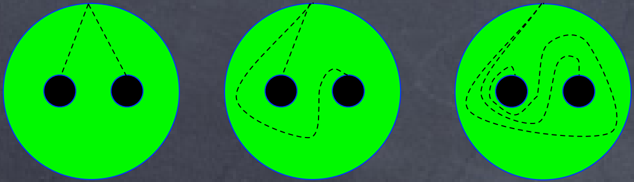
$$X^3 \longmapsto F(X) \qquad \text{complex number}$$

$$Y^2 \longmapsto F(Y) \qquad \text{complex vector space}$$

$$Z^1 \longmapsto F(Z) \qquad \text{complex linear category}$$

Our argument that $F(\text{sphere})$ is a commutative algebra now implies that $F(S^1)$ is an E_2-algebra in the 2-category $\mathbf{Cat}_{\mathbb{C}}$ of $\mathbb{C}$-linear categories. This appears with different terminology in papers of the early '90s.

An E_2-algebra in $\mathbf{Cat}_{\mathbb{C}}$ is a braided tensor category. (Compare: an E_1-algebra in $\mathbf{Cat}_{\mathbb{C}}$ is a tensor category and an E_3-algebra in $\mathbf{Cat}_{\mathbb{C}}$ is a symmetric tensor category. The latter is attached to S^2 in a 4d theory.)



The figure illustrates that the square of the diffeomorphism which braids the inner disks is not isotopic to the identity.

There is also a nondegeneracy condition which holds here. It implies that the braided tensor category $F(S^1)$ is “modular”, a moniker arising from the Moore-Seiberg work on rational conformal field theory. I prefer the simpler adjective “nondegenerate”. The new result I will discuss pertains to this nondegeneracy condition.

Modular tensor categories play an important role in topological quantum computation and conformal field theory. They have also been investigated in representation theory. A theorem of Reshetikhin-Turaev asserts that if B is a modular tensor category, then there is a 1-2-3 topological field theory F with $F(S^1) = B$.

Cobordism Hypothesis

Factorization of the path integral, then quantum Hilbert space, can extend further downward, at least in topological theories. This leads inevitably to higher categories, an algebraic structure which controls such factorizations. Intuitively these factorizations encode higher locality in topological quantum field theories.

Cobordism Hypothesis: If an n-dimensional TQFT "extends to points", i.e. to a 0-1-2-$\cdots$-n theory F, then F is determined by $F(\mathrm{pt})$. Furthermore, $F(\mathrm{pt})$ obeys strict finiteness and nondegeneracy conditions (called "full dualizability"). This was conjectured by Baez and Dolan and proved by Hopkins-Lurie ($n = 2$) and Lurie (in general).
Warning: We must specify the type (framed, oriented, etc.) of manifold.

Longstanding Problem: Do 1-2-3 theories based on modular tensor categories, such as Chern-Simons theories, extend to 0-1-2-3 theories?

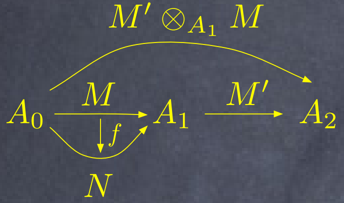
Done for CS with finite gauge group, CS with torus gauge group. Bartels-Douglas-Henriques in progress for simple gauge groups. We will make contact with Walker's work on these theories.

Morita Madness

Principle: Collections of algebras have an extra layer of structure.

For example, the collection **Vect**$_{\mathbb{C}}$ of vector spaces has two layers: vector spaces and linear maps. Thus **Vect**$_{\mathbb{C}}$ is a 1-category. But the collection of algebras $\mathrm{Alg}_{E_1}(\mathbf{Vect}_{\mathbb{C}})$ in **Vect**$_{\mathbb{C}}$ has three layers—algebras, bimodules, and intertwiners—so is a 2-category.

Objects in $\mathrm{Alg}_{E_1}(\mathbf{Vect}_{\mathbb{C}})$ are associative algebras A. Then a 1-morphism $M\colon A_0 \to A_1$ is an (A_1, A_0)-*bimodule*. Bimodules $M\colon A_0 \to A_1$ and $M'\colon A_1 \to A_2$ compose to the bimodule $M' \otimes_{A_1} M\colon A_0 \to A_2$.



A 2-morphism $f\colon M \to N$ is a linear map which commutes with the algebra actions.

Now we bump up by 1 twice: we go from $\mathbf{Vect}_{\mathbb{C}}$ to $\mathbf{Cat}_{\mathbb{C}}$ and from E_1 to E_2. So whereas $\mathrm{Alg}_{E_1}(\mathbf{Vect}_{\mathbb{C}})$ is a $(1+1)$-category, it follows that $\mathrm{Alg}_{E_2}(\mathbf{Cat}_{\mathbb{C}})$ is a $(2+2)$-category! In other words, there is 4-category

$$\beta\otimes\mathbf{Cat}_{\mathbb{C}} = \mathrm{Alg}_{E_2}(\mathbf{Cat}_{\mathbb{C}})$$

whose objects are braided tensor categories. Furthermore, it is symmetric monoidal: $B_1 \otimes B_2$ is defined for $B_1, B_2 \in \beta\otimes\mathbf{Cat}_{\mathbb{C}}$.

Because of the algebra structure, we have a 4-category whose objects are 1-categories.

Definition: A braided tensor category $B \in \beta\otimes\mathbf{Cat}_{\mathbb{C}}$ is invertible if there exists $\tilde{B} \in \beta\otimes\mathbf{Cat}_{\mathbb{C}}$ and an isomorphism $B \otimes \tilde{B} \cong \mathbb{1}$.

Compare: An invertible object in the category $\mathbf{Vect}_{\mathbb{C}}$ is a vector space V for which there exists V' with $V \otimes V' \cong \mathbb{C}$. It follows that $\dim V = 1$, i.e., V is a line.

Nondegeneracy and Invertibility in $\beta\otimes\mathbf{Cat}_{\mathbb{C}}$

The cobordism hypothesis implies that if $B \in \beta\otimes\mathbf{Cat}_{\mathbb{C}}$ satisfies sufficient finiteness and nondegeneracy conditions, then there is a 4d TQFT

$$A_B\colon\ \mathrm{Bord}_4 \longrightarrow \beta\otimes\mathbf{Cat}_{\mathbb{C}}$$

with $A_B(\mathrm{pt}) = B$. For objects $x, y \in B$ let $\beta_{x,y}\colon x \otimes y \to y \otimes x$ be the braiding. We compute the category

$$A_B(S^2) = \{x \in B : \beta_{y,x} \circ \beta_{x,y} = \mathrm{id}_{x\otimes y} \ \text{ for all } y \in B\}$$

Note that $A_B(S^2)$ is an E_3-category, i.e., is symmetric monoidal.

Theorem (Müger): B is nondegenerate=modular iff $A_B(S^2) = \mathbb{1}$ is the category $\mathbf{Vect}_{\mathbb{C}}$. We take this as a definition of 'nondegenerate'.

Assertion (F-Teleman): B is nondegenerate iff B is invertible as an object in $\beta\otimes\mathbf{Cat}_{\mathbb{C}}$.

Warning: We have not yet written down a complete proof.

A More General Conjecture

The cobordism hypothesis allows us to apply theorems in topology to algebra using a field theory $\mathrm{Bord}_n \to \mathcal{C}$ for some multi-category $\mathcal{C}$.

A takeaway: Braided tensor categories are often studied using 3-dimensional pictures (braids). Here we use manifolds of dimension $\leqslant 4$ to study braided tensor categories. New pictures!

Conjecture: If $A\colon \mathrm{Bord}_{2m} \to \mathcal{C}$ is a TQFT and $A(S^m)$ is invertible, then $A(\mathrm{pt})$ is invertible.

- If $A(\mathrm{pt})$ is invertible, then it is easy to see that $A(X)$ is invertible for all X and so A is an invertible field theory.
- Intuitively, the conjecture states that if we invert the m-morphism S^m in Bord_{2m}, then everything in Bord_{2m} is inverted.
- We have sketched out the proof for $m = 1$ and $m = 2$ (which should imply the Assertion on the previous slide).
- The proof uses simple surgeries, working both up and down from the middle dimension.

Application to 1-2-3 Theories

This suggests the following. If B is a nondegenerate (modular) braided tensor category, then B defines an invertible field theory

$$A_B\colon \mathrm{Bord}_4 \longrightarrow \beta^{\otimes}\mathbf{Cat}_{\mathbb{C}}$$

It is a 0-1-2-3-4 theory.

B is a left module over B, so a 1-morphism $\mathbb{1} \to B$ in $\beta^{\otimes}\mathbf{Cat}_{\mathbb{C}}$. An extension of the cobordism hypothesis, assuming enough finiteness and nondegeneracy, gives a 0-1-2-3 theory F_B with values in A_B.

- A k-dimensional field theory F with values in an invertible $(k+1)$-dimensional field theory A is called anomalous with anomaly A.
- If $\dim X = 1, 2$ or 3 then $A_B(X) \cong \mathbb{1}$. We hope that, at least with framings, there is a *canonical* trivialization.
- If so, then the 0-1-2-3 theory F_B restricts to a non-anomalous 1-2-3 theory. We believe this is the 1-2-3 theory defined by B (Reshetikhin-Turaev).
- This is very close to Walker's picture of these theories.