Classical Field Theory and Supersymmetry

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Introduction

These notes are expanded versions of seven lectures given at the IAS/Park City Mathematics Institute. I had the impossible task of beginning with some basic formal structure of mechanics and ending with sufficient background about supersymmetric field theory for Ronan Plesser's lectures on mirror symmetry. As a result the details may be hard to follow, but hopefully some useful pictures emerge nonetheless. My goal is to explain some parts of field theory which only require fairly standard differential geometry and representation theory. (I also rely on basics of supermanifolds to treat fermionic fields; see the lectures by John Morgan.)

The formal aspects of lagrangian mechanics and field theory, including symmetries, are treated in Lectures 1 and 2; fermionic fields and supersymmetries are introduced in Lecture 4. The key examples and some basic concepts are discussed in Lecture 3. In some sense the heart of the material occurs in Lecture 5. It includes the relation between classical free fields and quantum particles, and also a discussion of free approximations to nonfree lagrangian theories. Physicists use these ideas without comment, so they are important for the mathematician to master. The final two lectures introduce some supersymmetric field theories in low dimensions. Each lecture concludes with exercises for the industrious reader.

I have written about this material elsewhere, most completely in joint articles with Pierre Deligne in *Quantum Fields and Strings: A Course for Mathematicians, Volume 1*, American Mathematics Society, 1999. My briefer *Five Lectures on Supersymmetry*, also published by the American Mathematics Society in 1999, provides a lighter introduction to some of the topics covered here. I have shamelessly lifted text from these other writings. At the same time I have given shamefully few references. The volumes *Quantum Fields and Strings: A Course for Mathematicians* have plenty of material for the mathematician who would like to pursue this subject further. Standard physics texts on quantum field theory and supersymmetry cover much of this material, but from a somewhat different point of view.

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LECTURE 1
Classical Mechanics

Particle motion

We begin with the most basic example of a mechanical system, a single particle moving in space. Mathematically we model this by a function

$$x\colon \text{time} \longrightarrow \text{space}; \tag{1.1}$$

$x(t)$ is the position of the particle at time t. Time is usually taken to be the real line $\mathbb{R}$, but we will take a moment to discuss what precise structure on $\mathbb{R}$ we use. For example, it does not make physical sense to add times, so we do not want to use the vector space structure of the reals. Nor do we have a distinguished time, a zero of time, unless perhaps we consider some particular cosmological or religious model. Rather, the measurements one makes are of differences of time. Furthermore, time is homogeneous in the sense that measurements of time differences are independent of the absolute time. Finally, time differences are given by a single real number. Mathematically we model this by asking that time be a *torsor*[2] for a one-dimensional vector space V; that is, an affine space M^1 whose underlying vector space is V. There is one more piece of structure to include, as we have not yet specified the choice of *units*—seconds, hours, etc.—in which to measure time. For that we ask that M^1 carry a translation-invariant Riemannian metric, or equivalently that V be an inner product space. Then the interval between $t, t' \in M^1$ is given by the distance from t to t'. Of course, we identify $V \cong \mathbb{R}$ by choosing a unit vector in V, but notice that there are two choices, corresponding to two distinct "arrows of time." The superscript '1' reminds us that time is one-dimensional, but also indicates that soon we will encounter a generalization: Minkowski spacetime M^n of dimension n. We fix an affine coordinate t on M^1 so that $|dt| = 1$.

In general one benefit of specifying a geometric structure carefully, for example without arbitrary choices, is that the group of symmetries is then defined. The group of symmetries of $M = M^1$ is the *Euclidean group* $\mathrm{Euc}(M)$ in one dimension, i.e., the group of affine transformations which preserve the metric. An affine transformation induces a linear transformation of the underlying vector space, and this leads to an exact sequence

$$1 \longrightarrow V \longrightarrow \mathrm{Euc}(M) \longrightarrow O(V) \longrightarrow 1. \tag{1.2}$$

[2]A *torsor* for a group G is a space T on which G acts simply transitively. In other words, given $t, t' \in T$ there is a unique $g \in G$ so that g acting on t gives t'. We could call T a principal G-bundle over a point.

The kernel V consists of time translations. Of course, $O(V) \cong \{\pm 1\}$, and correspondingly the Euclidean group is divided into two components: symmetries which project to -1 are *time-reversing*. Some systems are not invariant under such time-reversing symmetries; some are not invariant under time translations. Notice that the metric eliminates scalings of time, which would correspond to a change of units.

We take the space X in which our particle (1.1) moves to be a smooth manifold. One could wonder about the smoothness condition, and indeed there are situations in which this is relaxed, but in order that we may express mechanics and field theory using calculus we will always assume that spaces and maps are smooth (C^∞). Again we want a Riemannian metric in order that we may measure distances, and the precise scale of that metric reflects our choice of units (inches, centimeters, etc.). Usually that metric is taken to be complete; otherwise the particle might fall off the space. Physically an incomplete metric would mean that an incomplete description of the system, perhaps only a local one. Often we will take space to be the Euclidean space $\mathbb{E}^d$, that is, standard d-dimensional affine space together with the standard translation-invariant metric.

There is one more piece of data we need to specify particle motion on X: a *potential energy* function

$$V\colon X \longrightarrow \mathbb{R}. \tag{1.3}$$

Hopefully, you have some experience with potential energy in simple mechanics problems where it appears for example as the energy stored in a spring ($V(x) = kx^2/2$, where x is the displacement from equilibrium and k is the spring constant), or the potential energy due to gravity ($V(x) = mgx$, where m is the mass, g is the gravitational constant, and x is the height).

There is much to say from a physics point of view about time, length, and energy (see [**F**], for example), but in these lectures we move forward with our mathematical formulation. For the point particle we need to specify one more piece of data: the *mass* of the particle, which is a real number $m > 0$ whose value depends on the choice of units. Note that whereas the units of mass (M), length (L), and time (T) are fundamental, the units of energy (ML^2/T^2) may be expressed in terms of these.

So far we have described all potential particle motions in X as the infinite-dimensional function space $\mathcal{F} = \mathrm{Map}(M^1, X)$ of smooth maps (1.1). Actual particle trajectories are those which satisfy the differential equation

$$m\ddot{x}(t) = -V'\bigl(x(t)\bigr). \tag{1.4}$$

The dots over x denote time derivatives. As written the equation makes sense for $X = \mathbb{E}^1$; on a general Riemannian manifold the second time derivative is replaced by a covariant derivative and the spatial derivative of V by the Riemannian gradient; see (1.32). Equation (1.4) is *Newton's second law*: mass times acceleration equals force. Notice that we do not describe force as a fundamental quantity, but rather express Newton's law in terms of energy. This is an approach which generalizes to more complicated systems. Let $\mathcal{M}$ denote the space of all solutions to (1.4). It is the space of *states* of our classical system, often called the *phase space*. What structure does it have? First, it is a smooth manifold. Again, we will always treat this state space as if it is smooth, and write formulas with calculus, even though in some examples smoothness fails. Next, symmetries (1.2) of time act on $\mathcal{M}$ by composition on the right. For example, the time translation $T_s,\ s \in \mathbb{R}$ acts by

$$(T_s x)(t) = x(t-s), \qquad x \in \mathcal{M}. \tag{1.5}$$

You should check that (1.4) is invariant under time-reversing symmetries. (Velocity $\dot{x}$ changes sign under such a symmetry, but acceleration $\ddot{x}$ is preserved.) We could consider a particle moving in a time-varying potential, which would *break*[3] these symmetries. Also, isometries of X act as symmetries via composition on the left. For a particle moving in Euclidean space, there is a large group of such isometries and they play an important role, as we describe shortly. On the other hand, a general Riemannian manifold may have no isometries. These global symmetries are not required in the structure of a "classical mechanical system."

So far, then, the state space $\mathcal{M}$ is a smooth manifold with an action of time translations (and possible global symmetries). There is one more piece of data, and it is not apparent from Newton's law: $\mathcal{M}$ carries a symplectic structure for which time translations act as symplectic diffeomorphisms. We can describe it by choosing a particular time t_0, thus breaking the time-translation invariance, and consider the map

$$\begin{aligned} \mathcal{M} &\longrightarrow TX \\ x &\longmapsto \big(x(t_0), \dot{x}(t_0)\big) \end{aligned} \tag{1.6}$$

If X is a complete Riemannian manifold, then this map is a diffeomorphism. Now the Riemannian structure gives an isomorphism of vector bundles $TX \cong T^*X$, so composing with (1.6) a diffeomorphism $\mathcal{M} \to T^*X$. The symplectic structure on $\mathcal{M}$ is the pullback of the natural symplectic structure on T^*X. When we turn to the lagrangian description below, we will give a more intrinsic description of this symplectic structure.

There is an abstract framework for mathematical descriptions of physical systems in which the basic objects are *states* and *observables*. (See [**Fa**] for one account.) These spaces are in duality in that we can evaluate observables on states. Both classical and quantum systems—including statistical systems—fit into this general framework. We have already indicated that in classical mechanics the space $\mathcal{M}$ of states is a symplectic manifold, and that motion is described by a particular one-parameter group of symplectic diffeomorphisms called time translation. What are the observables? The observables of our classical system are simply functions on $\mathcal{M}$. The pairing of observables and states is then the evaluation of functions. We term this pairing "the expectation value of an observable" in a given state, though the classical theory is deterministic and the expectation value is the actual value. A typical family of real-valued observables $\mathcal{O}_{(t,f)}$ is parametrized by pairs (t,f) consisting of a time $t \in M$ and a function $f\colon X \to \mathbb{R}$; then

$$\mathcal{O}_{(t,f)}(x) = f\big(x(t)\big), \qquad x \in \mathcal{M}. \tag{1.7}$$

For example, if $X = \mathbb{E}^d$ we can take f to be the i^{th} coordinate function. Then $\mathcal{O}_{(t,f)}$ is the i^{th} coordinate of the particle at time t. Sums and products of real-valued

[3]That is, the system would not have symmetries induced from (1.2).

observables are also observables. In the classical theory we can consider observables with values in other spaces.[4] The observables (1.7) are *local* in time.

This basic structure persists in classical field theory, with the understanding that the space $\mathcal{M}$ is typically infinite-dimensional.

Some differential geometry

We quickly review some standard notions, in part to set notation and sign conventions.

Let Y be a smooth manifold. The set of differential forms on Y is a graded algebra with a differential, called the *de Rham complex*:

$$0 \to \Omega^0(Y) \xrightarrow{d} \Omega^1(Y) \xrightarrow{d} \Omega^2(Y) \xrightarrow{d} \cdots \tag{1.8}$$

A vector field ξ induces a contraction, or interior product

$$\iota(\xi)\colon \Omega^q(Y) \longrightarrow \Omega^{q-1}(Y) \tag{1.9}$$

and also a Lie derivative

$$\mathrm{Lie}(\xi)\colon \Omega^q(Y) \longrightarrow \Omega^q(Y). \tag{1.10}$$

The exterior derivative d, interior product ι, and Lie derivative are related by Cartan's formula

$$\mathrm{Lie}(\xi) = d\iota(\xi) + \iota(\xi)d. \tag{1.11}$$

Now if $E \to Y$ is a vector bundle with connection (covariant derivative) ∇, then there is an extension of the de Rham complex to differential forms with coefficients in E:

$$\Omega^0(Y;E) \xrightarrow{d_\nabla = \nabla} \Omega^1(Y;E) \xrightarrow{d_\nabla} \Omega^2(Y;E) \xrightarrow{d_\nabla} \cdots . \tag{1.12}$$

It is not a complex in general, but rather

$$d_\nabla^2 = R_\nabla \quad \in \Omega^2(Y;\operatorname{End} E), \tag{1.13}$$

where R_∇ is the curvature of ∇.

All of this applies to appropriate infinite-dimensional manifolds as well as finite-dimensional manifolds. In these notes we do not discuss the differential topology of infinite-dimensional manifolds. The function spaces we use are assumed to consist of smooth functions, though in a more detailed treatment we would often use completions of smooth functions.

Next, if $\phi\colon Y \to X$ is a map between manifolds, then $d\phi$ is a section of $\mathrm{Hom}(TY, \phi^*TX) \to Y$, i.e., $d\phi \in \Omega^1(Y;\phi^*TX)$. If now TX has a connection ∇, then

$$d_\nabla d\phi = \phi^* T_\nabla \quad \in \Omega^2(Y;\phi^*TX), \tag{1.14}$$

[4] In the quantum theory the values should lie in a fixed space, since the observables are integrated over the space of fields.

where T_∇ is the torsion of the connection ∇. Often ∇ is the Levi-Civita connection of a Riemannian manifold, in which case the torsion vanishes. More concretely, if $\phi\colon (-\epsilon,\epsilon)_u \times (-\epsilon,\epsilon)_t \to X$ is a smooth two-parameter map, then the partial derivatives ϕ_u and ϕ_t are sections of the pullback bundle ϕ^*TX. In the case of vanishing torsion equation (1.14) asserts

$$\nabla_{\phi_u}\phi_t = \nabla_{\phi_t}\phi_u. \tag{1.15}$$

The complex of differential forms on a product manifold $Y = Y'' \times Y'$ is naturally bigraded, and the exterior derivative d on Y is the sum of the exterior derivative d'' on Y'' and the exterior derivative d' on Y'. In computations we use the usual sign conventions for differential forms: if $\alpha = \alpha'' \wedge \alpha'$ for $\alpha'' \in \Omega^{q''}(Y'')$ and $\alpha' \in \Omega^{q'}(Y')$, then

$$d\alpha = d''\alpha'' \wedge \alpha + (-1)^{q''}\alpha'' \wedge d'\alpha'. \tag{1.16}$$

We often apply this for Y'' a function space.

We follow the standard convention in geometry that symmetry groups act on the left. For example, if $Y = \mathrm{Map}(M,X)$ and $\varphi\colon M \to M$ is a diffeomorphism, then the induced diffeomorphism of Y acts on $x \in Y$ by $x \mapsto x\circ\varphi^{-1}$. On the other hand, if $\psi\colon X \to X$ is a diffeomorphism, then the induced diffeomorphism of Y is $x \mapsto \psi\circ x$. There is an unfortunate sign to remember when working with left group actions. If $G \to \mathrm{Diff}(Y)$ is such an action, then the induced map on Lie algebras $\mathfrak{g} \to \mathfrak{X}(Y)$ is an *anti*homomorphisms. In other words, if ξ_ζ is the vector field which corresponds to $\zeta \in \mathfrak{g}$, then

$$[\xi_\zeta, \xi_{\zeta'}] = -\xi_{[\zeta,\zeta']}. \tag{1.17}$$

Hamiltonian mechanics

We begin by recalling a basic piece of symplectic geometry. Namely, if $\mathcal{M}$ is a symplectic manifold with symplectic form Ω, then there is an exact sequence

$$0 \longrightarrow H^0(\mathcal{M};\mathbb{R}) \longrightarrow \Omega^0(\mathcal{M}) \xrightarrow{\text{grad}} \mathfrak{X}_\Omega(M) \longrightarrow H^1(\mathcal{M};\mathbb{R}) \longrightarrow 0. \tag{1.18}$$

Here $\mathfrak{X}_\Omega(X)$ is the space of vector fields ξ on X which (infinitesimally) preserve Ω: $\mathrm{Lie}(\xi)\Omega = 0$. The symplectic gradient "grad" of a function $\mathcal{O}$ is the unique vector field $\xi_\mathcal{O}$ such that

$$d\mathcal{O} = -\iota(\xi_\mathcal{O})\Omega. \tag{1.19}$$

Thus any observable determines an infinitesimal group of symplectic automorphisms (so in good cases a one-parameter group of symplectic diffeomorphisms), and conversely an infinitesimal group of symplectic automorphisms which satisfies a certain cohomological constraint determines a set of observables any two elements of which differ by a locally constant function.

Now in a classical mechanical system the state space $\mathcal{M}$ carries a one-parameter group of time translations, which we assume defines a vanishing cohomology class in $\Omega^2(\mathcal{M})$, i.e., if ξ is the infinitesimal generator, we assume that $\iota(\xi)\Omega$ is exact. A choice of corresponding observable is the *negative* of a quantity we call the *energy* or *Hamiltonian* of the system. Put differently, the total energy generates a motion which is the minus time translation. For example, the energy H of a particle $x\colon M \to X$ satisfying (1.4) with the symplectic structure defined after (1.6) is[5]

$$H(x) = \frac{m}{2}|\dot{x}(t)|^2 + V\big(x(t)\big). \tag{1.20}$$

The right hand side is independent of t, so in fact defines a function of $x \in \mathcal{M}$. It is the sum of the kinetic and potential energy of the particle.

A classical system $(\mathcal{M}, H)$ consisting of a symplectic manifold $\mathcal{M}$ and a Hamiltonian H is *free* if $\mathcal{M}$ is a symplectic affine space and the motion generated by H is a one-parameter group of affine symplectic transformations, or equivalently H is at most quadratic.

Another general piece of symplectic geometry may also be expressed in terms of the exact sequence (1.18). Namely, there is a Lie algebra structure on $\Omega^0(\mathcal{M})$ so that the symplectic gradient is a Lie algebra homomorphism to the Lie algebra of vector fields (with Lie bracket). In other words, this *Poisson bracket* on functions $\mathcal{O}, \mathcal{O}'$ satisfies

$$[\xi_{\mathcal{O}}, \xi_{\mathcal{O}'}] = \xi_{\{\mathcal{O},\mathcal{O}'\}}. \tag{1.21}$$

Because the symplectic gradient has a kernel, (1.21) does not quite determine the Poisson bracket; rather we *define* it by the formula

$$\{\mathcal{O}, \mathcal{O}'\} = \xi_{\mathcal{O}}\mathcal{O}'. \tag{1.22}$$

We note that any physical system has a similar bracket on observables; e.g., in quantum mechanics it is the commutator of operators on Hilbert space.

Let $\mathcal{M}$ be a classical state space with Hamiltonian H. A *global symmetry* of the system is a symplectic diffeomorphism of $\mathcal{M}$ which preserves H. An *infinitesimal symmetry* is a vector field ξ on $\mathcal{M}$ which preserves both the symplectic form Ω and the Hamiltonian H: $\operatorname{Lie}(\xi)\Omega = \operatorname{Lie}(\xi)H = 0$. Then the observable Q which corresponds to an infinitesimal symmetry satisfies

$$\{H, Q\} = 0. \tag{1.23}$$

Quite generally, the time-translation flow on $\mathcal{M}$ induces a flow on observables which may be expressed by[6]

$$\dot{\mathcal{O}} = \{H, \mathcal{O}\}, \tag{1.24}$$

so that (1.23) is equivalent to a conservation law for Q:

$$\dot{Q} = 0. \tag{1.25}$$

[5]We compute this in Lecture 2. Here m is the mass of the particle.

[6]To understand the sign, recall that a diffeomorphism of $\mathcal{M}$ induces an action of functions on $\mathcal{M}$ using pullback by the *inverse*. On the infinitesimal level this introduces a minus sign, and it cancels the minus sign which relates H to infinitesimal time translation.

For this reason the observable Q is called the *conserved charge* associated to an infinitesimal symmetry. Again: *Symmetries lead to conservation laws.*

The symplectic point of view is fundamental in classical mechanics, but we do not emphasize it in these lectures. Rather, many of the most interesting field theories—including most of those of interest in geometry and topology—have a lagrangian description. From a lagrangian description one recovers the symplectic story, but the lagrangian description can be used in field theory when there is no distinguished time direction (which is usually the situation in geometry: an arbitrary manifold does not come equipped with a time function).

Lagrangian mechanics

Recall our description of the point particle. Its state space $\mathcal{M}$ is defined to be the submanifold of the function space $\mathcal{F} = \operatorname{Map}(M^1, X)$ cut out by equation (1.4). Ideally, we would like to describe this submanifold as the critical manifold of a function

$$S\colon \mathcal{F} \longrightarrow \mathbb{R}. \tag{1.26}$$

In that ideal world $\mathcal{M}$ would be the space of solutions to $dS = 0$. Such a function would be called the *action* of the theory, and the critical point equation the *Euler-Lagrange equation*. This is a typical situation in geometry, where we often derive interesting differential equations from variational principles, especially on compact manifolds. In our situation the function S we would like to write down is infinite on typical elements of $\mathcal{F}$, including elements of $\mathcal{M}$, due to the noncompactness of M^1. Rather, the more basic object is the *lagrangian density*, or simply *lagrangian*,

$$L\colon \mathcal{F} \longrightarrow \operatorname{Densities}(M^1) \tag{1.27}$$

which attaches to each potential particle motion $x \in \operatorname{Map}(M^1, X)$ a density[7] $L(x)$ on the line M^1. The lagrangian density is well-defined on all of $\mathcal{F}$, but its integral over the whole line may well be infinite. In this section we study its integrals over finite intervals of time, which are finite, and derive appropriate Euler-Lagrange equations. We will see that these equations can be deduced from L without consideration of S. In addition, we will construct the symplectic structure on $\mathcal{M}$. In Lecture 2 we systematize these constructions in the context of general field theories.

The lagrangian for the particle is

$$L(x) = \Big[\frac{m}{2}|\dot{x}(t)|^2 - V\big(x(t)\big)\Big]\,|dt|. \tag{1.28}$$

It is the *difference* of kinetic energy and potential energy. Also, the dependence of the lagrangian on the path x is *local* in the time variable t. (We formalize the notion of locality in the next lecture.) For each $x\colon M^1 \to X$ the right hand side is a density on M^1. For (finite) times $t_0 < t_1$ the action

$$S_{[t_0,t_1]}(x) = \int_{t_0}^{t_1} L(x) \tag{1.29}$$

[7] A density on M^1 has the form $g(t)\,|dt|$ for some function $g\colon M^1 \to \mathbb{R}$. Densities have a transformation law which corresponds to the change of variables formula for integrals.

is well-defined, whereas the integral over the whole line may be infinite. Nonetheless, we may deduce Euler-Lagrange by asking that S be stationary to first order for variations of x which are compactly supported in time.[8] Thus we choose t_0, t_1 so that a particular variation has support in $[t_0, t_1]$. Now a "variation of x" simply means a tangent vector to $\mathcal{F}$ at x. We have

$$T_x\mathcal{F} \cong C^\infty(M^1; x^*TX), \tag{1.30}$$

i.e., a tangent vector to $\mathcal{F}$ is a section over M^1 of the pullback via x of the tangent bundle TX. To see this, consider a small curve x_u, $-\epsilon < u < \epsilon$, in $\mathcal{F}$ with $x_0 = x$. It is simply a map $x\colon (-\epsilon,\epsilon)\times M^1 \to X$, and differentiating with respect to u at $u = 0$ we obtain precisely an element ζ of the right hand side of (1.30). For the moment do not impose any support condition on ζ, plug $x = x_u$ into (1.29), and differentiate at $u = 0$. We find

(1.31)
$$\begin{aligned}\frac{d}{du}\Big|_{u=0} S_{[t_0,t_1]}(x_u) &= \frac{d}{du}\Big|_{u=0}\int_{t_0}^{t_1}\Big[\frac{m}{2}|\dot x_u(t)|^2 - V\big(x_u(t)\big)\Big]\,|dt| \\ &= \int_{t_0}^{t_1}\Big[m\langle \dot x(t), \nabla_{\zeta(t)}\dot x(t)\rangle - \langle \operatorname{grad} V\big(x(t)\big), \zeta(t)\rangle\Big]\,|dt| \\ &= \int_{t_0}^{t_1}\Big[m\langle \dot x(t), \nabla_{\dot x(t)}\zeta(t)\rangle - \langle \operatorname{grad} V\big(x(t)\big), \zeta(t)\rangle\Big]\,|dt| \\ &= \int_{t_0}^{t_1} -\langle m\nabla_{\dot x(t)}\dot x(t) + \operatorname{grad} V\big(x(t)\big), \zeta(t)\rangle\,|dt| \\ &\qquad\qquad + m\langle \dot x(t_1), \zeta(t_1)\rangle - m\langle \dot x(t_0), \zeta(t_0)\rangle.\end{aligned}$$

In the last step we integrate by parts, and in the third we use the fact that the Levi-Civita connection ∇ is torsionfree (in the form of (1.15)). If we impose the condition that the support of ζ is compactly contained in $[t_0, t_1]$, then the boundary terms on the last line vanish. The integral that remains vanishes for all such ζ if and only if the Euler-Lagrange equation

$$m\nabla_{\dot x(t)}\dot x(t) + \operatorname{grad} V\big(x(t)\big) = 0 \tag{1.32}$$

is satisfied. Thus we have recovered Newton's law (1.4), here written for maps into a Riemannian manifold. Now let us work *on-shell*, that is, on the space $\mathcal{M} \subset \mathcal{F}$ of solutions to the Euler-Lagrange equations. Denote the differential on $\mathcal{F}$, and so the differential on the submanifold $\mathcal{M}$, as 'δ'. For each $t \in M^1$ define the 1-form $\gamma_t \in \Omega^1(\mathcal{M})$ by

$$\gamma_t(\zeta) = m\langle \dot x(t), \zeta(t)\rangle. \tag{1.33}$$

Then equation (1.31) implies that the differential of the action on-shell is

$$\delta S_{[t_0,t_1]} = \gamma_{t_1} - \gamma_{t_0} \qquad \text{on } \mathcal{M}. \tag{1.34}$$

[8]This has an analog in the quantum theory. Quantization is done around a classical solution x_0, which may or may not have finite action, but the path integral is done over field configurations x for which the "difference" of the lagrangians $L(x) - L(x_0)$ has a finite integral over M^1. The same remark applies in general field theories.

We draw two conclusions from this equation:

- The 2-form on $\mathcal{M}$

$$\Omega_t := \delta\gamma_t \tag{1.35}$$

  is independent of t.
- The 1-forms γ_t determine a principal $\mathbb{R}$-bundle $P \to \mathcal{M}$ with connection whose curvature is Ω_t.

The first statement follows simply by differentiating (1.34). Turning to the second, for each $t \in M^1$ we define $P_t \to \mathcal{M}$ to be the trivial principal $\mathbb{R}$-bundle with connection 1-form γ_t. Then given $t_0 < t_1$ equation (1.34) asserts that addition of $-S_{[t_0,t_1]}$ is an isomorphism $P_{t_0} \to P_{t_1}$ which preserves the connection. The crucial property of S which makes these trivializations consistent is its *locality* in time: for $t_0 < t_1 < t_2$ we have

$$S_{[t_0,t_2]} = S_{[t_0,t_1]} + S_{[t_1,t_2]}. \tag{1.36}$$

Finally, we remark that one can see from (1.33) that the symplectic form (1.35) has the description given after (1.6).

The Euler-Lagrange equations (1.32), the connection form (1.33), and the symplectic form (1.35) may all be defined directly in terms of the lagrangian density L; the action (1.29) is not needed. In the next lecture we develop a systematic calculus for these manipulations. In particular, we will develop formulas to compute the conserved charges associated to infinitesimal symmetries. Thus a single quantity—the lagrangian density—contains all of the information of the classical system. That is one reason why the lagrangian formulation of a classical system, if it exists, is so powerful. At a less formal level, the lagrangian formulation is a practical way to encode the physics of a complicated system. For example, it is not difficult to write the lagrangian corresponding to a mechanical system with pendulums, springs, etc: it is the total kinetic energy minus the total potential energy. (See the exercises at the end of the lecture.) Lagrangians are also practical for systems which include other physical objects: strings, membranes, fields, etc. In these lectures we focus mainly on fields.

Classical electromagnetism

Let space X be an oriented 3-dimensional Riemannian manifold. For example, we might consider $X = \mathbb{E}^3$. Then X has a *star operator* which in particular gives an isomorphism

$$*_X : \Omega^1(X) \longrightarrow \Omega^2(X) \tag{1.37}$$

whose square is the identity map. Also, we can use the metric and orientation to identify each of these spaces with the space of vector fields on X. For example, on $\mathbb{E}^3$ with standard coordinates x, y, z we have the identifications

$$a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \longleftrightarrow a\,dx + b\,dy + c\,dz \longleftrightarrow a\,dy\wedge dz - b\,dx\wedge dz + c\,dx\wedge dy \tag{1.38}$$

for any functions a, b, c.

The electric field E is usually taken to be a time-varying vector field on X, but using (1.38) we take it instead to be a time-varying 1-form. Similarly, the magnetic field B, which is usually a time-varying vector field on X, is taken to be a time-varying 2-form instead. From one point of view these fields are maps of time M^1 into an infinite-dimensional function space, so comprise an infinite-dimensional mechanical system. A better point of view is that they are fields defined on the *spacetime* $M^1 \times X$, and map this space into a finite-dimensional one.[9] However we view this, Maxwell's law in empty space asserts that

$$\begin{aligned} dB &= 0 & \qquad dE &= -\frac{\partial B}{\partial t} \\ d{*_X}E &= 0 & \qquad c^2\, d{*_X}B &= {*_X}\frac{\partial E}{\partial t} \end{aligned} \tag{1.39}$$

where c is the speed of light. One can think of the right hand equations as evolution equations, analogous to Newton's law, and the left-hand equations as telling that we are really studying motion in a subspace of $\Omega^1(X) \oplus \Omega^2(X)$. (Note that the right hand equations preserve the subspaces defined by the left-hand equations.) To complete the description of a mechanical system we should specify a symplectic structure on the (infinite-dimensional) space of solutions $\mathcal{M}$ to (1.39).

The question arises as to whether or not there is a lagrangian description of this system. In fact, there is. Why would we like to have one? As explained earlier, the lagrangian encodes not only Maxwell's equations (1.39), but also the symplectic structure on the space of solutions. But in addition we could now try to study the combined system consisting of a particle together with an electric and magnetic field. In the combined system there is an *interaction*, which is encoded as a potential energy term in the action. Once this total lagrangian is described, we are off to the races. For example, we can fix an electric and/or magnetic field and study the motion of a *charged* particle in that background. The resulting equation of motion is called the Lorentz force law. On the other hand, one could deduce the equation for the electric and magnetic fields created by a charged particle, or system of charged particles. At a more sophisticated level one can determine the energy, momentum, etc. of the electric and magnetic fields. It is clear that lots of physics is summarized by the lagrangian! We will give the lagrangian formulation of Maxwell's equations in Lecture 3, and now conclude this first lecture by introducing a more natural setting for field theory: spacetime.

Minkowski spacetime

The electric and magnetic fields are naturally functions of four variables: $E = E(t, x, y, z)$ and $B = B(t, x, y, z)$. The domain of that function is *spacetime*[10]. See Clifford Johnson's lectures on general relativity for the physics behind the geometrization of spacetime; here we content ourselves with a mathematical description.

So far we have modeled time as the one-dimensional Euclidean space M^1 and space as a complete Riemannian manifold X. At first glance it would be natural, then, to consider spacetime as the product $M^1 \times X$ with a "partial metric" and an

[9] More precisely, they are sections of a finite-dimensional vector bundle over $M^1 \times X$.

[10] or perhaps *timespace*, since we write the time coordinate first.

isometric action of the additive group $\mathbb{R}$. (By "partial metric" we mean that we know the inner product of two vectors tangent to X or two vectors tangent to M^1, but we do not say anything about other inner products.) But partial metrics do not pull back under diffeomorphisms to partial metrics, so it not a very good geometric structure. A more physical problem is that the units are mixed: along M^1 we use time but along X we use length. We now describe two alternatives.

For a nonrelativistic spacetime we replace the metric along M^1 with a 1-form dt (or density $|dt|$) on $M^1 \times X$ which evaluates to 1 on the vector field generating the $\mathbb{R}$ action and vanishes on tangents to X. Note that dt determines a codimension one foliation of *simultaneous events* in spacetime. Putting dt as part of the structure requires that symmetries preserve this notion of simultaneity. The other piece of structure is a metric on the foliation, which is simply the Riemannian metric on X. In case $X = \mathbb{E}^{n-1}$ we obtain the *Galilean spacetime* of dimension n; its group of symmetries is the *Galilean group*.

We will focus instead on relativistic theories, in which case we extend the metrics on M^1 and X to a metric on the spacetime $M^1 \times X$. To do this we need to reconcile the disparity in units. This is accomplished by a universal constant c with units of length/time, i.e., a speed. Physically it is the speed of light. However, the spacetime metric is not positive definite; rather it is the Lorentz metric

$$c^2 ds^2_{M^1} - ds^2_X. \tag{1.40}$$

If $\dim X = n-1$, then the metric has signature $(1, n-1)$. For $X = \mathbb{E}^{n-1}$ we obtain *Minkowski spacetime* M^n.

So Minkowski spacetime M^n is a real n-dimensional affine space with underlying translation group V a vector space endowed with a nondegenerate symmetric bilinear form of signature $(1, n-1)$. To compute we often fix an affine coordinate system $x^0, x^1, \dots, x^{n-1}$ with respect to which the metric is

$$g = (dx^0)^2 - (dx^1)^2 - \dots - (dx^{n-1})^2. \tag{1.41}$$

Note that $x^0 = ct$ for t the standard time coordinate. Minkowski spacetime carries a positive density defined from the metric

$$|d^n x| = |\, dx^0 \wedge dx^1 \wedge \dots \wedge dx^{n-1} \,|, \tag{1.42}$$

and each time slice $x^0 = \text{constant}$ also carries a canonical positive density

$$|d^{n-1} x| = |\, dx^1 \wedge \dots \wedge dx^{n-1} \,|. \tag{1.43}$$

The vector space V has a distinguished cone, called the *lightcone*, consisting of vectors with norm zero. For $n = 1$ it degenerates to the origin, for $n = 2$ it is the union of two lines, and for $n > 2$ the set of nonzero elements on the lightcone has two components.

The group $\mathrm{Iso}(M)$ of isometries of M^n is a subgroup of the affine group, and it contains all translations by vectors in V. It maps onto the group $O(V)$ of linear transformations of V which preserve the Lorentz inner product:

$$1 \longrightarrow V \longrightarrow \mathrm{Iso}(M) \longrightarrow O(V) \longrightarrow 1. \tag{1.44}$$

The group $O(V)$ has four components if $n \geq 2$. (For $n = 1$ it is cyclic of order two.) To determine which component a transformation $T \in O(V)$ lies in we ask two questions: Is T orientation-preserving? Does T preserve or exchange the components of the nonzero elements on the lightcone? (This question needs to be refined for $n = 2$.) For elements in the identity component the answer to both questions is "yes". There is a nontrivial double cover of the identity component, called the *Lorentz group* $\mathrm{Spin}(V)$, and correspondingly a double cover of the identity component of the isometry group of M^n called the *Poincaré group* P^n:

$$1 \longrightarrow V \longrightarrow P^n \longrightarrow \mathrm{Spin}(V) \longrightarrow 1. \tag{1.45}$$

The Lorentz spin groups have some beautiful properties, some of which we will explore when we study supersymmetry. In any case this double cover is not relevant until when we introduce fermionic fields in Lecture 4.

Exercises

1. In this problem use standard affine coordinates $x^0, x^1, \dots, x^{n-1}$ on Minkowski spacetime M^n and on Euclidean space $\mathbb{E}^n$. Set the speed of light to be one, so that in the Minkowski case we can identify x^0 with a time coordinate t. Let $\partial_\mu = \partial/\partial x^\mu$ be the coordinate vector field. It is the infinitesimal generator of a translation.
   (a) Write a basis of infinitesimal generators for the orthogonal group of isometries of $\mathbb{E}^n$ which fix the origin $(0, 0, \dots, 0)$. Compute the Lie brackets.
   (b) Similarly, write a basis of infinitesimal generators for the orthogonal group of isometries of M^n which fix the origin $(0, 0, \dots, 0)$. Compute the Lie brackets. Write your formulas with 't' in place of 'x^0', and use Roman letters $i, j, \dots$ for spatial indices (which run from 1 to $n - 1$).

2. In this problem we review some standard facts in symplectic geometry. Let M be a symplectic manifold.
   (a) Verify that (1.19) determines a unique vector field $\xi_{\mathcal{O}}$. Prove that (1.18) is an exact sequence.
   (b) Using (1.22) as the definition of the Poisson bracket, verify (1.21). Also, prove the Jacobi identity:
   $$\{\mathcal{O}, \{\mathcal{O}', \mathcal{O}''\}\} + \{\mathcal{O}'', \{\mathcal{O}, \mathcal{O}'\}\} + \{\mathcal{O}', \{\mathcal{O}'', \mathcal{O}\}\} = 0.$$
   (c) On a symplectic manifold of dimension $2m$ one can always find local coordinates q^i, p_j, $i, j = 1, \dots, m$, such that
   $$\omega = dp_i \wedge dq^i.$$
   (We use the summation convention.) Compute the Poisson brackets of the coordinate functions. Verify that on a symplectic affine space (define) functions of degree ≤ 2 form a Lie algebra under the Poisson bracket. Identify this Lie algebra. (Hint: Consider the image under the symplectic gradient.) Start with $m = 1$.

3. Let M be a smooth manifold of dimension m.
   (a) Define a canonical 1-form θ on the cotangent bundle $\pi\colon T^*M \to M$ as follows. Let $\alpha \in T^*M_m$ and $\xi \in T_\alpha(T^*M)$. Then $\theta_\alpha(\xi) = \alpha(\pi_*\xi)$. Check that $\omega := d\theta$ is a symplectic form.
   (b) Local coordinates $q^1, \dots, q^m$ on M induce local coordinates q^i, p_j on T^*M (the indices runs from 1 to m) by writing an element $\alpha \in T^*X$ as $\alpha = p_i dq^i$. Make sense of this procedure, then write θ and ω in this coordinate system.

4. (a) Check that (1.13) reproduces any other definition of curvature you may have learned. For example, if ξ, η, ζ are vector fields on Y, show that
$$R_\nabla(\xi,\eta)\zeta = \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi,\eta]} \zeta.$$
   (b) Demonstrate that (1.14) and (1.15) are equivalent (in the case of vanishing torsion).

5. Let G be a Lie group. Consider the action of G on itself by left multiplication. Verify (1.17) in this case. What happens for right multiplication?

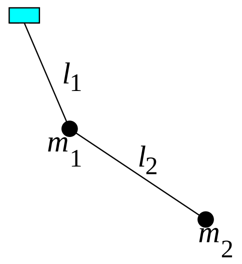
6. Let V be a finite-dimensional real vector space endowed with a nondegenerate symmetric real-valued bilinear form. We denote the pairing of vectors v, w as $\langle v, w\rangle$. Let $n = \dim V$.
   (a) Define an induced nondegenerate symmetric form on the dual V^* and on all exterior powers of both V and V^*.
   (b) The highest exterior power of a vector space is one-dimensional, and is called the *determinant line* of the vector space. An element in $\operatorname{Det} V^*$ is a *volume form* on V. There are precisely two such forms ω such that $\langle \omega, \omega\rangle = \pm 1$; they are opposite. Choose one. This amounts to fixing an *orientation* of V, which is a choice of component of $\operatorname{Det} V \setminus \{0\}$. Then the *Hodge $*$ operator*, which is a map
$$*\colon {\textstyle\bigwedge}^q V^* \longrightarrow {\textstyle\bigwedge}^{n-q} V^*$$
   is defined implicitly by the equation
$$\alpha \wedge *\beta = \langle \alpha, \beta\rangle\, \omega, \qquad \alpha, \beta \in {\textstyle\bigwedge}^q V^*.$$
   Verify that the $*$ operator is well-defined.
   (c) Compute $**$. The answer depends on the signature of the quadratic form and the degree q.
   (d) Compute $*$ on a 3 dimensional vector space with a positive definite inner product (choose a basis!) and on a 4 dimensional vector space with a Lorentz inner product.

7. (a) Determine all particle motions on $X = \mathbb{E}^1$ which lead to free mechanical systems. (What are the possible potential energy functions?) For each determine the state space and one-parameter group of time translations. Generalize to $X = \mathbb{E}^n$.
   (b) A coupled system of n one-dimensional harmonic oscillators may be modeled as motion in an n-dimensional inner product space X with potential $V(x) = \frac{1}{2}|x|^2$. Investigate the equations of motion of this system. Find the state space and one-parameter group of time translations. Is this system free?

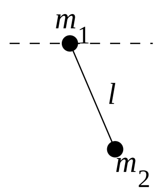
8. Write the lagrangians for the following mechanical systems with one or more particles. These systems are placed in a uniform gravitational field. The potential energy (determined only up to a constant) for a particle of mass m at height h in the gravitational field is mgh for some universal constant g. These problems are taken from *Mechanics*, by Landau and Lifshitz, a highly recommended text.

   (a) A simple pendulum of mass m and length ℓ.

   (b) A coplanar double pendulum.



   (c) A simple pendulum of mass m_2, with a mass m_1 at the point of support which can move on a horizontal line lying in the plain in which m_2 moves.



9. Rewrite equations (1.39) in terms of vector fields E, B, instead of forms, and verify that you get the standard version of Maxwell's equations.

LECTURE 2
Lagrangian Field Theory and Symmetries

The differential geometry of function spaces

We begin with a word about densities. Recall from Lecture 1 that the particle lagrangian gives a density on the affine time line M^1 for each path of the particle, and we investigated integrals of that density, called the action. Our motivation for using densities, rather than 1-forms, is that the many systems are invariant under time-reversing symmetries. In general, a *density* on a finite-dimensional manifold M is a tensor field which in a local coordinate system $\{x^\mu\}$ is represented by

$$\ell(x)\,|dx^1\cdots dx^n| \tag{2.1}$$

for some function ℓ. It transforms under change of coordinates by the absolute value of the Jacobian of the coordinate change. In other words, it is a twisted n-form—the twisting is by the orientation bundle. We denote the set of densities by $\Omega^{|0|}(M)$. Then we define $\Omega^{|-q|}(M)$ to be the set of twisted $(n-q)$-forms. A twisted $(n-q)$-form is the tensor product of a section of $\bigwedge^q TM$ and a density. For example, an element of $\Omega^{|-1|}(M)$ in local coordinates looks like

$$j^\mu(x)\,\frac{\partial}{\partial x^\mu}\otimes|dx^1\cdots dx^n|. \tag{2.2}$$

The twisted forms are graded as indicated; a twisted $(n-q)$-form, or $|-q|$-form, has degree $-q$. On the graded vector space of twisted forms $\Omega^{|\bullet|}(M)$ we have the usual operations of exterior differentiation d, Lie derivative $\mathrm{Lie}(\xi)$ by a vector field ξ, and interior product $\iota(\xi)$, with the Cartan formula (1.11) relating them. However, twisted forms do not form a ring; rather, twisted forms are a graded module over untwisted forms. Just as we can integrate densities over manifolds, we can integrate $|-q|$-forms over codimension q submanifolds whose normal bundle is oriented.

Continuing with a manifold M, consider the mapping space $\mathcal{F}=\mathrm{Map}(M,X)$ of smooth maps $\phi\colon M\to X$ into a manifold X. Everything we say generalizes to the case where we replace the single copy of X with a fiber bundle $E\to M$ (whose typical fiber is diffeomorphic to X, say) and $\mathcal{F}$ with its space of sections. First, note that the tangent space at ϕ to the mapping space $\mathcal{F}=\mathrm{Map}(M,X)$ is

$$T_\phi\mathcal{F}\cong\Omega^0(M;\phi^*TX). \tag{2.3}$$

To see this, consider a path ϕ_u, $-\epsilon < u < \epsilon$, in $\mathcal{F}$ such that ϕ_0 is the given map ϕ. Then the derivative in u at $u = 0$ is naturally a section of the pullback tangent bundle $\Omega^0(M; \phi^*TX)$. A crucial piece of structure is the evaluation map

$$\begin{aligned} e\colon \mathcal{F} \times M &\longrightarrow X \\ (\phi, m) &\longmapsto \phi(m). \end{aligned} \tag{2.4}$$

What distinguishes function spaces from arbitrary infinite-dimensional manifolds is this evaluation map. We obtain real-valued functions on $\mathcal{F} \times M$ by composing (2.4) with a real-valued function on X.

We express lagrangian field theory in terms of differential forms on $\mathcal{F} \times M$, except that we twist by the orientation bundle to use densities on M instead of forms. So we work in a double complex $\Omega^{\bullet,|\bullet|}(\mathcal{F} \times M)$ whose homogeneous subspace $\Omega^{p,|-q|}(\mathcal{F} \times M)$ is the space of p-forms on $\mathcal{F}$ with values in the space of twisted $(n-q)$-forms on M. Let δ be the exterior derivative on $\mathcal{F}$, d the exterior derivative[11] of forms on M, and $D = \delta + d$ the total exterior derivative. We have

$$D^2 = d^2 = \delta^2 = 0, \qquad d\delta = -\delta d. \tag{2.6}$$

We will use the following picture to depict elements in the double complex:

$$\begin{array}{c|ccc} & 0 & 1 & \cdots \\ \hline |0| & & & \\ & d\uparrow & & \\ |-1| & & \underset{\delta}{\rightarrow} & \\ \vdots & & & \end{array} \quad \mathcal{F}$$

$$M$$

For example, a lagrangian (1.27), which to each point of $\mathcal{F}$ attaches a density on M, is an element in $\Omega^{0,|0|}(\mathcal{F} \times M)$.

There is an important subcomplex $\Omega_{\text{loc}}^{\bullet,|\bullet|}(\mathcal{F} \times M)$ of *local* forms. The value of a form $\alpha \in \Omega^{p,|\bullet|}(\mathcal{F} \times M)$ at a point $m \in M$ and a field $\phi \in \mathcal{F}$ on tangent vectors $\xi_1, \dots, \xi_p$ to $\mathcal{F}$ is a twisted form at m. The form α is *local* if this twisted form depends only on the k-jet at m of ϕ and the ξ_i. For example, let ζ_1, ζ_2 be fixed vector fields on M and consider $X = \mathbb{R}$, so that $\mathcal{F}$ is the space of real-valued functions on M. Then the $(0, |0|)$-form

$$L = \zeta_1\zeta_2\phi(m) \cdot \phi(m)\ |d^nx| \tag{2.7}$$

is local: $L \in \Omega_{\text{loc}}^{0,|0|}(\mathcal{F} \times M)$. We explain the notation in detail. First, L is the product of a function $\ell = \zeta_1\zeta_2\phi(m) \cdot \phi(m)$ with the standard density $|d^nx|$. Now

[11] Our sign convention is that for $\alpha \in \Omega^p(\mathcal{F})$ and $\beta \in \Omega^{|\bullet|}(M)$,

$$d(\alpha \wedge \beta) = (-1)^p \alpha \wedge d\beta. \tag{2.5}$$

the function ℓ may be written in terms of the evaluation function e (2.4), which is local: $\ell = \zeta_1\zeta_2 e \cdot e$. Finite derivatives of local functions are local, as are products of local functions, so ℓ is local as claimed. In terms of the definition, at a point $m \in M$, the form L depends on the 2-jet of ϕ. On the other hand, if $m_0 \in M$ is fixed, then the function $\ell = \phi(m) \cdot \phi(m_0)$ is not local: its value at (ϕ, m) depends on the value of ϕ at m_0. Expressions similar to (2.7) are what appear in field theories; at first it is useful to rewrite them in terms of the evaluation map.

There is one nontrivial mathematical theorem which we use in our formal development. This theorem, due to F. Takens [**T**], asserts the vanishing of certain cohomology groups in the double complex of *local* forms with respect to the vertical differential d.

Theorem 2.8 (Takens). *For $p > 0$ the complex $\big(\Omega^{p,|\bullet|}_{\mathrm{loc}}(\mathcal{F} \times M), d\big)$ of local differential forms is exact except in the top degree $|\bullet| = 0$.*

A proof of this result may be found in [**DF**].

Basic notions

We study *fields* on a smooth finite-dimensional manifold M, which we usually think of as spacetime. For the most part we will specialize to $M = M^n$ Minkowski spacetime, but for the general discussion it can be arbitrary. Often the spacetime M is equipped with topological or geometric structures—an orientation, spin structure, metric, etc.—which are fixed throughout. Attached to M is a space of fields $\mathcal{F}$ which are the variables for the field theory.

What is a field? We will not try to pin down a definition which works in every situation. Roughly a field is some kind of "function" on M, and many types of fields are in fact sections of some fiber bundle over M. Thus a *scalar field* is a map $\phi\colon M \to X$ for some manifold X. We also have tensor fields, spinor fields, metrics, and so on. The only real requirement is that a field be *local* in the sense that it can be cut and pasted. So, as an example of fields somewhat different than those we will encounter, fix a finite group G and consider the *category* of principal G-bundles over M. (This is more interesting when M has nontrivial fundamental group.) Since these bundles (Galois covering spaces) can be cut and pasted, they are valid fields.[12] Connections on principal bundles for a fixed structure group also form a category—morphisms are equivalences of connections—and these are the basic fields of gauge theory for arbitrary gauge groups. In setting up the general theory we will treat fields as maps $\phi\colon M \to X$ to a fixed manifold X, though the reader can easily generalize to fields which are sections of a fiber bundle $E \to M$ (and to gauge fields as well). This covers most of the examples in these lectures.

In classical mechanics one meets systems with constraints. For example a particle moving in $\mathbb{E}^3$ may be constrained to lie on the surface of a sphere. Such *holonomic* constraints are easily dealt with in our setup: simply change the manifold X to accommodate the constraint. On the other hand, our formalism does not apply as presented to nonholonomic constraints. In supersymmetric gauge theories we meet such constraints (in the superspacetime formulation).

[12] That is, the objects in the category are the fields. All morphisms in the category are invertible—the category is a *groupoid*—and ultimately one is interested in the set of equivalence classes.

Fix a spacetime M and a space of fields $\mathcal{F}$. A classical field theory is specified by two pieces of data: a *lagrangian density*

$$L \in \Omega^{0,|0|}_{\mathrm{loc}}(\mathcal{F} \times M) \tag{2.9}$$

and a *variational 1-form*

$$\gamma \in \Omega^{1,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M). \tag{2.10}$$

These fit into the diagram:

$$\begin{array}{c|cc} & 0 & 1 \\ \hline |0| & L & \\ & & \\ |-1| & & \gamma \end{array} \qquad \mathcal{F}$$

$$M$$

It is natural to define the *total lagrangian* as the sum of the lagrangian and the variational 1-form:

$$\mathcal{L} = L + \gamma \in \Omega^{|0|}(\mathcal{F} \times M). \tag{2.11}$$

There is a compatibility condition for L and γ, which we specify shortly, and we will see that in many cases of interest L determines γ. But first we recast the point particle of Lecture 1 in this language.

Example 2.12 (point particle). Take "spacetime" to be simply time $M = M^1$; the space of fields to be the mapping space $\mathcal{F} = \mathrm{Map}(M, X)$ of paths in a fixed Riemannian manifold X equipped with a potential energy function $V\colon X \to \mathbb{R}$; the lagrangian to be the particle lagrangian (1.28), now written as an element of $\Omega^{0,|0|}_{\mathrm{loc}}(\mathcal{F} \times M)$:

$$L = \Big[\frac{m}{2}|\dot{x}|^2 - V(x)\Big]\ |dt|; \tag{2.13}$$

and the variational 1-form to be the expression (1.33), now written as an element of $\Omega^{1,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M)$:

$$\begin{aligned} \gamma &= \langle \dot{x}, \delta x\rangle\ \partial_t \otimes |dt| \\ &= \langle \iota(\partial_t) dx, \delta x\rangle\ \partial_t \otimes |dt|. \end{aligned} \tag{2.14}$$

In these expressions $x\colon \mathcal{F} \times M \to X$ is the evaluation map, so that dx and δx are one-forms on $\mathcal{F} \times M$ with values in the pullback of TX via x. The reader may prefer to consider first $X = \mathbb{E}^d$, in which case there are simplifications: the covariant derivative is the ordinary derivative and $dx, \delta x$ are 1-forms with values in the constant vector space $\mathbb{R}^d$. Nothing essential to our discussion is lost in this specialization. The reader should check that (2.13) is indeed local and agrees with (1.28). In both expressions we use the standard density $|dt|$ on M^1 determined by the Riemannian metric, and we have chosen a unit length translation-invariant vector field ∂_t on M^1; it is determined up to sign. Then $\dot{x} = \partial_t x$ changes sign if we change ∂_t for its opposite. Since ∂_t appears twice in both expressions, they are invariant under this change, hence well-defined.

Though we haven't developed the theory yet, as an illustration of calculus on function spaces let us compute the differential $D\mathcal{L}$ of the total lagrangian $\mathcal{L} = L+\gamma$. First, we consider the component in degree $(1, |0|)$:

$$(D\mathcal{L})^{1,|0|} = \delta L + d\gamma. \tag{2.15}$$

It is convenient for computation to (finally!) fix an orientation of M^1, and using it identify the density $|dt|$ with the 1-form dt. Since we have already expressed everything without the choice of orientation, the results of the computation are necessarily independent of this choice. The orientation allows us to identify

$$dx = \dot{x}\, dt = \dot{x}\, |dt|. \tag{2.16}$$

Then the variation of the lagrangian is

$$\begin{aligned} \delta L &= m\langle \delta_\nabla \dot{x}, \dot{x}\rangle \wedge |dt| - dV \circ \delta x \wedge |dt| \\ &= m\langle \delta_\nabla dx, \dot{x}\rangle - \langle \operatorname{grad} V, \delta x\rangle \wedge dt. \end{aligned} \tag{2.17}$$

Here, as explained in Lecture 1 (see the text preceding (1.14)), the first derivative dx is a section of x^*TX, so the second derivative uses the pullback of the Levi-Civita connection. Using (2.16) we write $\gamma = m\langle \dot{x}, \delta x\rangle$, and so

$$\begin{aligned} d\gamma &= m\langle d_\nabla \dot{x} \wedge \delta x\rangle + m\langle \dot{x}, d_\nabla \delta x\rangle \\ &= -m\langle \nabla_{\dot{x}} \dot{x}, \delta x\rangle \wedge dt + m\langle \dot{x}, d_\nabla \delta x\rangle. \end{aligned} \tag{2.5}$$

The minus sign in the second line comes from commuting the 1-form dt past the 1-form δx. Since the Levi-Civita connection has no torsion, it follows from (1.14) (applied to the total differential D), and the fact that d and δ anticommute, that $\delta_\nabla dx = -d_\nabla \delta x$, so there is a cancellation in the sum

$$\begin{aligned} (D\mathcal{L})^{1,|0|} &= \delta L + d\gamma \\ &= -m\langle \nabla_{\dot{x}} \dot{x} + \operatorname{grad} V, \delta x\rangle \wedge dt. \end{aligned} \tag{2.18}$$

Newton's equation (1.32) is embedded in this formula. We also compute

$$\begin{aligned} (D\mathcal{L})^{2,|-1|} &= \delta\gamma \\ &= m\langle \delta_\nabla \dot{x} \wedge \delta x\rangle, \end{aligned} \tag{2.19}$$

where we use the vanishing of torsion again: $\delta_\nabla \delta x = 0$. This last expression is a 2-form on $\mathcal{F}$; it restricts to the symplectic form on the space $\mathcal{M}$ of solutions to Newton's law.

The formulas simplify for $X = \mathbb{E}^1$. Then we write

$$L = \frac{m}{2}\dot{x}\,dx - V(x)\,dt,$$

and so

$$\begin{aligned}\delta L &= \quad m\dot{x}\,\delta dx - V'(x)\,\delta x \wedge dt \\ &= -m\dot{x}\,d\delta x - V'(x)\,\delta x \wedge dt.\end{aligned}$$

We then set

$$\gamma = m\dot{x}\,\delta x,$$

so that

$$d\gamma = m\ddot{x}\,dt \wedge \delta x + m\dot{x}\,d\delta x.$$

Newton's law is contained in the equation

$$\delta L + d\gamma = -\left[m\ddot{x} + V'(x)\right]\,\delta x \wedge dt.$$

The reader will recognize the computation of $(D\mathcal{L})^{1,|0|}$ as a version of the computation (1.31) in Lecture 1. It encodes the integration by parts we did there. This is precisely the role of the variational 1-form γ.

For a general lagrangian $\mathcal{L} = L + \gamma$ we encode the integration by parts in the following relationship between L and γ.

Definition 2.20. A *lagrangian field theory* on a spacetime M with fields $\mathcal{F}$ is a lagrangian density $L \in \Omega^{0,|0|}_{\text{loc}}(\mathcal{F} \times M)$ and a variational 1-form $\gamma \in \Omega^{1,|-1|}_{\text{loc}}(\mathcal{F} \times M)$ such that if $\mathcal{L} = L + \gamma$ is the total lagrangian, then $(D\mathcal{L})^{1,|0|}$ is *linear over functions* on M.

To explain "linear over functions," note first that $T_\phi\mathcal{F} \cong \Omega^0(M;\phi^*TX)$ (see (2.3)) is a module over $\Omega^0(M)$, the algebra of smooth functions on M. A form $\beta \in \Omega^{1,|\bullet|}(\mathcal{F} \times M)$ is linear over functions if for all $(\phi, m) \in \mathcal{F} \times M$ we have

$$\beta_{(\phi,m)}(f\hat{\xi}) = f(m)\beta_{(\phi,m)}(\hat{\xi}), \qquad f \in \Omega^0_M, \quad \hat{\xi} \in T_\phi\mathcal{F}. \tag{2.21}$$

The reader can verify that (2.18) satisfies this condition. More plainly: δx is linear over functions, whereas $\delta_\nabla dx$ is not. The variational 1-form γ is chosen precisely to cancel all terms where $\delta_\nabla dx$ appears. This is what the usual integration by parts accomplishes; it isolates δx in the variation of the action.

A few comments about the variational 1-form γ:

- If the lagrangian L depends only on the 1-jet of the fields, then there is a canonical choice for γ which is characterized as being linear over functions. We always choose this γ. For such systems, then, we need only specify the lagrangian. This is the case for all examples we will meet in this course.
- If the lagrangian depends on higher derivatives, then there is more than one choice for γ, but for local lagrangians the difference between any two choices is d-exact (by Takens' Theorem 2.8). In this sense the choice of γ is not crucial.
- In many mechanics texts you will find this canonical γ written as '$p_i dq^i$'.

Given a classical field theory—a total lagrangian $\mathcal{L} = L + \gamma$ such that $(D\mathcal{L})^{1,|0|}$ is linear over functions—we define the *space of classical solutions* $\mathcal{M} \subset \mathcal{F}$ to be the space of $\phi \in \mathcal{F}$ such that the restriction of $(D\mathcal{L})^{1,|0|}$ to $\{\phi\} \times M$ vanishes:

$$(D\mathcal{L})^{1,|0|} = \delta L + d\gamma = 0 \qquad \text{on } \mathcal{M} \times M. \tag{2.22}$$

This definition is motivated by the usual integration by parts manipulation in the calculus of variations. For the point particle we read off Newton's law (1.32) directly from (2.18). Notice in general that since $(D\mathcal{L})^{1,|0|}$ lies in the subcomplex of local forms, the Euler-Lagrange equations are local. Physicists term fields in $\mathcal{M} \subset \mathcal{F}$ *on-shell*, whereas the complement of $\mathcal{M}$—or sometimes all of $\mathcal{F}$—is referred to as *off-shell*.

For field theory on a general manifold M there is no Hamiltonian interpretation. The Hamiltonian story requires that we write spacetime M as time $\times$ space. In that case it is appropriate to call $\mathcal{M}$ the *state space* or *phase space*.

Definition 2.23. Let $\mathcal{L} = L + \gamma$ define a lagrangian field theory. Then the associated *local symplectic form* is

$$\omega := \delta\gamma \in \Omega^{2,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M). \tag{2.24}$$

On-shell we have

$$\omega = D\mathcal{L} \qquad \text{on } \mathcal{M} \times M, \tag{2.25}$$

and so

$$D\omega = 0 \qquad \text{on } \mathcal{M} \times M. \tag{2.26}$$

We represent the on-shell data in the diagram:

$$\begin{array}{c|ccccc} & 0 & & 1 & & 2 \\ \hline |0| & L & \to & 0 & & \\ & & & \uparrow & & \\ |-1| & & & \gamma & \to & \omega \end{array} \qquad \mathcal{M}$$

$$M$$

In mechanics ($n = 1$) for each time t the form $\omega(t)$ is a closed 2-form on the space of classical solutions $\mathcal{M}$, since $\delta\omega = 0$. It is independent of t, since $d\omega = 0$. A theory is *nondegenerate* if ω is a symplectic form. In classical mechanics texts the nondegeneracy amounts to the nondegeneracy of the Hessian matrix with entries $\partial^2 L/\partial\dot{q}^i\partial\dot{q}^j$; the symplectic form ω is usually denoted '$dp_i \wedge dq^i$' in appropriate coordinates. For field theories in higher dimensions, we obtain a symplectic structure on $\mathcal{M}$ only in the Hamiltonian situation where spacetime $=$ time $\times$ space. We discuss that later in this lecture.

The symplectic form for the point particle was computed in (2.19):

$$\omega = m\langle \delta_\nabla \dot{x} \wedge \delta x\rangle. \tag{2.27}$$

We conclude this general discussion with an example which involves fields. We write it quite explicitly in 2 dimensions; the reader can easily generalize to higher dimensions.

Example 2.28 (scalar field). Let $M = M^2$ denote two-dimensional Minkowski spacetime with coordinates x^0, x^1. Here x^0 is the speed of light times a standard time coordinate. For convenience fix the orientation $\{x^0, x^1\}$ and use it to identify twisted forms with forms. Note[13] that $*dx^0 = dx^1$ and $*dx^1 = dx^0$. Let $\mathcal{F} = \{\phi\colon M^{1,1} \to \mathbb{R}\}$ be the set of real scalar fields. We use the notation $\partial_\mu = \partial/\partial x^\mu$. The free (massless) lagrangian is

$$\begin{aligned} L &= \frac{1}{2} d\phi \wedge *d\phi \\ &= \frac{1}{2}|d\phi|^2\, dx^0 \wedge dx^1 \\ &= \frac{1}{2}\bigl\{(\partial_0\phi)^2 - (\partial_1\phi)^2\bigr\}\, dx^0 \wedge dx^1. \end{aligned} \tag{2.29}$$

From this we derive

$$\gamma = \partial_0\phi\; \delta\phi \wedge dx^1 + \partial_1\phi\; \delta\phi \wedge dx^0 \tag{2.30}$$

$$\omega = \partial_0\delta\phi \wedge \delta\phi \wedge dx^1 + \partial_1\delta\phi \wedge \delta\phi \wedge dx^0 \tag{2.31}$$

and the equation of motion

$$\partial_0^2\phi - \partial_1^2\phi = 0. \tag{2.32}$$

It is important to identify the equation of motion in this scalar field theory as a second order *wave equation*. Just as with particle motion, we can solve the wave equation—at least locally—by specifying an initial value for the field and its time derivative. The signature of the Lorentz metric manifests itself directly in this equation.

Symmetries and Noether's theorem

In lagrangian field theory there is a local version of the relationship one has in symplectic geometry between infinitesimal symmetries and conserved charges (see the text preceding (1.23)). We distinguish two types of symmetries: *manifest* and *nonmanifest*. To each symmetry we attach a local *Noether current*, and on-shell it satisfies a local version of the conservation law (1.25) in symplectic geometry. In mechanics the Noether current *is* the conserved (Noether) charge, but for general field theories we must work in the Hamiltonian framework and integrate over space to construct a conserved charge. What is important here is that we can write explicit and computable formulas for the conserved currents attached to infinitesimal symmetries.

We continue with our general setup: spacetime M, fields $\mathcal{F} = \operatorname{Map}(M, X)$, total lagrangian $\mathcal{L} = L + \gamma$, local symplectic form ω, and space of classical solutions $\mathcal{M}$. As a preliminary we extend the notion of *locality* to vector fields: a vector field $\hat{\xi}$ on $\mathcal{F}$ is *local* if for some k the value of $\hat{\xi}_\phi \in \Omega^0(M, \phi^*TX)$ at $m \in M$ depends only on the k-jet of ϕ at m. A vector field ξ on $\mathcal{F} \times M$ is said to be *decomposable and local* if it is the sum of a local vector field $\hat{\xi}$ on $\mathcal{F}$ and of a vector field η on M. Such vector fields preserve the bigrading on differential forms, and the Lie derivative by such vector fields commutes with both d and δ.

[13] See the problem sets for information about the Hodge $*$ operator.

Definition 2.33. (i) A local vector field $\hat{\xi}$ on $\mathcal{F}$ is a *generalized infinitesimal symmetry* of L if there exists

$$\alpha_{\hat{\xi}} \in \Omega^{0,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M) \tag{2.34}$$

such that

$$\mathrm{Lie}(\hat{\xi})L = d\alpha_{\hat{\xi}} \qquad \text{on } \mathcal{F} \times M \tag{2.35}$$

(ii) A decomposable and local vector field ξ on $\mathcal{F} \times M$ is a *manifest infinitesimal symmetry* if

$$\mathrm{Lie}(\xi)\mathcal{L} = 0 \qquad \text{on } \mathcal{F} \times M. \tag{2.36}$$

The definition of a manifest symmetry is the usual functorial notion—a symmetry of a mathematical structure is an automorphism which preserves all the data. A generalized symmetry is *nonmanifest* if we must choose $\alpha_{\hat{\xi}}$ in (2.3) nonzero; it only preserves the lagrangian up to an exact term. Note that there is an indeterminacy for nonmanifest symmetries: we can add any closed local form to $\alpha_{\hat{\xi}}$.

Definition 2.33 encodes the notion of an infinitesimal symmetry; there is also a definition of a symmetry such that the generator of a one-parameter group of symmetries is an infinitesimal symmetry. We leave the precise formulation to the reader.

We illustrate these definitions with some examples from mechanics.

Example 2.37 (time translation: manifest). Consider the point particle as defined in Example 2.12. Time translation is the action of[14] $\mathbb{R}$ on M^1: translation by $s \in \mathbb{R}$ is $T_s(t) = t + s$. This induces an action of $\mathbb{R}$ on $\mathrm{Map}(M^1, X) \times M^1$: $T_s(x,t) = \big(x \circ T_s^{-1}, T_s(t)\big)$. Differentiating with respect to s we obtain the desired vector field ξ on $\mathcal{F} \times M$. We specify it by giving its action on the "coordinate functions", which for function space is the evaluation map:

$$\begin{aligned} \iota(\xi)dt &= 1 \\ \iota(\xi)\delta x &= -\dot{x}. \end{aligned} \tag{2.38}$$

Note the minus sign here, which easily causes confusion. The vector field ξ is local, since (2.38) only depends on the 1-jet of x at t. It is clear that T_t is a preserves the total lagrangian, since T_t preserves the evaluation map and density $|dt|$, and the lagrangian and variational 1-form are expressed in terms of these. It follows by differentiation that (2.36) is satisfied.

Example 2.39 (time translation: nonmanifest). We take $\hat{\xi}$ to be the component of the manifest infinitesimal symmetry ξ along the space of fields $\mathcal{F}$. (Since ξ is decomposable, this makes sense.) In other words,

$$\iota(\hat{\xi})\delta x = -\dot{x}. \tag{2.40}$$

[14] As in previous computations we orient time.

Then $\hat{\xi}$ is a nonmanifest symmetry of the lagrangian (2.28) with

$$\alpha_{\hat{\xi}} = -\big(\frac{m}{2}|\dot{x}|^2 - V(x)\big) + C. \tag{2.41}$$

We put in a constant C to emphasize the indeterminacy which exists for nonmanifest symmetries. Equation (2.35) follows from the fact that ξ is a manifest symmetry. For writing $\xi = \hat{\xi} + \eta$, where $\eta = \partial_t$, we have $\operatorname{Lie}(\xi)L = 0$, from which

$$\operatorname{Lie}(\hat{\xi})L = -\operatorname{Lie}(\eta)L = -d\iota(\eta)L = -d\big(\frac{m}{2}|\dot{x}|^2 - V(x)\big). \tag{2.42}$$

A few remarks about the definition:

- As we mentioned above, there is ambiguity in the definition of $\alpha_{\hat{\xi}}$ for a nonmanifest symmetry—we can add any d-exact form.
- In the definition of a generalized infinitesimal symmetry we have said nothing about $\operatorname{Lie}(\hat{\xi})\gamma$. In fact, an argument based on Takens' theorem and its close cousins implies that for some $\beta_{\hat{\xi}} \in \Omega^{1,|-2|}_{\text{loc}}(\mathcal{F}\times M)$, the identity

$$\operatorname{Lie}(\hat{\xi})\gamma = \delta\alpha_{\hat{\xi}} + d\beta_{\hat{\xi}} \qquad \text{on } \mathcal{M}\times M \tag{2.43}$$

  holds on shell. We use this identity below to see that the Noether current is a local version of the conserved charge in symplectic geometry.
- In the manifest case it is awkward to have to compute that $\operatorname{Lie}(\xi)\gamma = 0$. In good cases this follows from $\operatorname{Lie}(\xi)L = 0$, which is much easier to check. Namely, set $E = M\times X$ and suppose each element in a one-parameter group of manifest symmetries is induced from a bundle automorphism of $E\to M$ which covers a diffeomorphism of M. (This is the case in Example 2.37.) If the lagrangian L depends only on the 1-jet of the fields, and if it is preserved by the diffeomorphism of $\mathcal{F}\times M$, then it follows that γ is also preserved and we have a manifest symmetry. We use this implicitly in what follows.

For a generalized infinitesimal symmetry, we summarize (2.35) and (2.43) in an off-shell diagram and an on-shell diagram:

$$\begin{array}{c|c}
 & 0 \quad \mathcal{F} \\ \hline
|0| & \operatorname{Lie}(\hat{\xi})L \\
 & \uparrow \\
|-1| & \alpha_{\hat{\xi}} \\
\multicolumn{2}{c}{M}
\end{array}
\qquad\qquad
\begin{array}{c|ccc}
 & 0 & & 1 \quad \mathcal{M} \\ \hline
|0| & \operatorname{Lie}(\hat{\xi})L & & \\
 & \uparrow & & \\
|-1| & \alpha_{\hat{\xi}} & \to & \operatorname{Lie}(\hat{\xi})\gamma \\
 & & & \uparrow \\
|-2| & & & \beta_{\hat{\xi}} \\
\multicolumn{4}{c}{M}
\end{array}$$

It is easy to check that for a generalized infinitesimal symmetry $\hat{\xi}$, we have

$$\operatorname{Lie}(\hat{\xi})\omega = d(-\delta\beta_{\hat{\xi}}) \qquad \text{in } \Omega^{2,|-1|}(\mathcal{M}\times M). \tag{2.44}$$

This is a local version of the condition to be a Hamiltonian vector field. (Note that in mechanics—dim $M = 1$—the right hand side vanishes.j) For a manifest infinitesimal symmetry ξ, the local symplectic form is preserved on the nose: $\mathrm{Lie}(\xi)\omega = 0$.

We are ready to define the Noether current associated to an infinitesimal symmetry. Consider first the manifest case. Suppose $\xi = \hat{\xi} + \eta$ is a decomposable and local vector field on $\mathcal{F} \times M$ which is a manifest infinitesimal symmetry: $\mathrm{Lie}(\xi)\mathcal{L} = 0$. Define the *Noether current*

$$j_\xi := \big[\iota(\xi)\mathcal{L}\big]^{0,|-1|} = \iota(\hat{\xi})\gamma + \iota(\eta)L \qquad \text{(manifest symmetry)}. \tag{2.45}$$

Then one can verify that on-shell we have

$$dj_\xi = 0 \qquad \text{on } \mathcal{M} \times M, \tag{2.46}$$

$$\delta j_\xi = -\iota(\hat{\xi})\omega - d\iota(\eta)\gamma \qquad \text{on } \mathcal{M} \times M, \tag{2.47}$$

Equation (2.46) is the assertion that the Noether current j_ξ is *conserved*. Equation (2.47) is the local version of the correspondence (1.19) in symplectic geometry between an infinitesimal symmetry and its associated charge. The exact term disappears upon integration over a slice at fixed time.

In the nonmanifest case there are somewhat different formulae. Thus suppose $\hat{\xi}$ is a vector field on $\mathcal{F}$ which is a generalized infinitesimal symmetry with associated $\alpha_{\hat{\xi}} \in \Omega^{0,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M)$. Then the Noether current is defined as

$$j_{\hat{\xi}} := \iota(\hat{\xi})\gamma - \alpha_{\hat{\xi}} \qquad \text{(nonmanifest symmetry)}. \tag{2.48}$$

It is easy to verify from (2.35), (2.43), and the Cartan formula that in this case (2.46) and (2.47) are replaced by

$$dj_{\hat{\xi}} = 0 \qquad \text{on } \mathcal{M} \times M, \tag{2.49}$$

$$\delta j_{\hat{\xi}} = -\iota(\hat{\xi})\omega + d\beta_{\hat{\xi}} \qquad \text{on } \mathcal{M} \times M, \tag{2.50}$$

Example 2.51 (time translation: manifest). Continuing Example 2.37, we compute the Noether current from (2.45) using (2.38):

$$\begin{aligned} j_\xi &= \iota(\xi)\mathcal{L} \\ &= \iota(\xi)\Big[\big\{\frac{m}{2}|\dot{x}|^2 - V(x)\big\}\,|dt| \;+\; m\langle \dot{x}, \delta x\rangle\Big] \\ &= \frac{m}{2}|\dot{x}|^2 - V(x) - m|\dot{x}|^2 \\ &= -\big(\frac{m}{2}|\dot{x}|^2 + V(x)\big). \end{aligned} \tag{2.52}$$

Of course, this is *minus* the total energy, or Hamiltonian, of the point particle. Note that the sign agrees with our convention in Lecture 1: the Hamiltonian is the conserved charge associated to the negative of infinitesimal time translation.

Example 2.53 (time translation: nonmanifest). Continuing Example 2.39 we define the Noether current $j_{\hat{\xi}}$ associated to (2.40) and (2.41) using (2.48). A short computation gives

$$j_{\hat{\xi}} = -\bigl(\frac{m}{2}|\dot{x}|^2 + V(x) + C\bigr). \tag{2.54}$$

The energy is defined only up to a constant C in this picture.

Example 2.55 (linear momentum). Continuing with the point particle, we consider the special case $X = \mathbb{E}^d$ and translation in the j^{th} coordinate direction. This is an isometry of $\mathbb{E}^d$, and if it preserves the potential V, then it is manifestly a symmetry of L, as L depends only on the target metric and V. The corresponding infinitesimal symmetry is a vector field ξ_j defined by:

$$\begin{aligned} \iota(\xi_j)dt &= 0 \\ \iota(\xi_j)\delta x^i &= \delta^i_j, \end{aligned} \tag{2.56}$$

where δ^i_j has its usual meaning. One can check directly that this is a manifest symmetry in case $\partial_j V = 0$. The associated Noether current is a component of the *linear momentum*:

$$j_{\xi_j} = m\dot{x}^j. \tag{2.57}$$

Finally, we give an example from field theory.

Example 2.58 (energy for a scalar field). We continue the notation of Example 2.28. Consider infinitesimal translation in the x^0 (time) direction. (We work in units where $c = 1$.) It defines a vector field ξ on $\mathcal{F} \times M$ by

$$\begin{aligned} \iota(\xi)\delta\phi &= -\partial_0\phi \\ \iota(\xi)dx^0 &= 1 \\ \iota(\xi)dx^1 &= 0. \end{aligned} \tag{2.59}$$

Then a routine computation shows that ξ is a *manifest* symmetry: $\operatorname{Lie}(\xi)L = 0$ on $\mathcal{F} \times M$. The associated Noether current is

$$j_\xi = -\frac{1}{2}\left\{(\partial_0\phi)^2 + (\partial_1\phi)^2\right\}\, dx^1 - \{\partial_0\phi\,\partial_1\phi\}\, dx^0. \tag{2.60}$$

The reader should check that this current is conserved, i.e., $dj_\xi = 0$ on-shell. The equations of motion must be used. The coefficient of dx^1 is minus the energy density of the field—the time derivative is the kinetic energy and the spatial derivative the potential energy. The global energy is the integral of $-j_\xi$ over a time-slice $x^0 =$ constant, and then the coefficient of dx^0 drops out. It is there, so to speak, to guarantee that $dj_\xi = 0$.

We can also regard infinitesimal time translation as a *nonmanifest* symmetry $\hat{\xi}$ by letting it operate only along $\mathcal{F}$:

$$\begin{aligned} \iota(\hat{\xi})\delta\phi &= -\partial_0\phi \\ \iota(\hat{\xi})dx^0 &= 0 \\ \iota(\hat{\xi})dx^1 &= 0. \end{aligned} \tag{2.61}$$

Then we compute

$$\mathrm{Lie}(\hat{\xi})L = d\alpha \qquad \text{on } \mathcal{F} \times M, \tag{2.62}$$

$$\mathrm{Lie}(\hat{\xi})\gamma = \delta\alpha + d\beta \qquad \text{on } \mathcal{M} \times M, \tag{2.63}$$

where

$$\begin{aligned} \alpha &= \frac{1}{2}\left[(\partial_0\phi)^2 - (\partial_1\phi)^2\right] dx^1 \in \Omega_{\mathrm{loc}}^{0,|-1|}, \\ \beta &= \partial_1\phi\,\delta\phi \qquad\qquad\qquad\quad \in \Omega_{\mathrm{loc}}^{1,|-2|}. \end{aligned} \tag{2.64}$$

We leave the reader to compute the Noether current from this point of view.

Hamiltonian structures

We can study lagrangian field theory on any spacetime M, and indeed this is common in field theory, string theory, and beyond. For example, we can consider electromagnetism in a nontrivial "gravitational background", that is, on a spacetime (other than Minkowski spacetime) which satisfies Einstein's equations. In quantum field theory one often "Wick rotates" the theory on Minkowski spacetime to a theory on Euclidean space, and then generalizes M to be any Riemannian manifold. Perturbative string theory is defined in terms of correlation functions on Riemann surfaces (with Riemannian metric). In these cases there is no interpretation in terms of classical physics; the quantities of interest are the correlation functions of the quantum theory. Nonetheless, many of the concepts we discussed carry over. Of course, in differential geometry we use the calculus of variations in settings which involve no physics. All this is to emphasize that we recover a classical system—symplectic manifold of states with a distinguished one-parameter group—only in the following case.

Definition 2.65. A *Hamiltonian structure* on a lagrangian field theory is an isometry $M \cong M^1 \times N$ of spacetime to time $\times$ space, where N is a Riemannian manifold and $M^1 \times N$ has the Lorentz metric $c^2 ds^2_{M^1} - ds^2_N$. Also, if fields are sections of a fiber bundle $E \to M$, then we require an isomorphism of E with a fiber bundle over $M^1 \times N$ which is pulled back from a fiber bundle $E_N \to N$.

This last condition means that fields take their values in a manifold which is independent of time, so it makes sense to compare fields at different times. In fact, we identify the space of fields $\mathcal{F}$ as the space of paths in a space $\mathcal{F}_N$ of fields on N. ($\mathcal{F}_N$ is the space of sections of $E_N \to N$, or in the case of a product $\mathcal{F}_N = \mathrm{Map}(N, X)$.) In that sense the Hamiltonian picture is the study of a particle moving in the infinite dimensional space $\mathcal{F}_N$.

The basic idea is to take local on-shell quantities of degree $(\bullet, |-1|)$ in the lagrangian theory and integrate over a time slice $\{t\} \times N$ to obtain global quantities on $\mathcal{M}$. Note that the integration is vacuous in mechanics, which is the case $N = \mathrm{pt}$.

Definition 2.66. In a lagrangian field theory with Hamiltonian structure, the *symplectic form* on the phase space $\mathcal{M}$ is

$$\Omega = \int_{\{t\}\times N} \omega \;\in\; \Omega^2(\mathcal{M}). \tag{2.67}$$

Typically N is noncompact and so to ensure convergence we only evaluate Ω on tangent vectors to $\mathcal{M}$ with compact support in spatial directions, or at least with sufficient decay at spatial infinity. The hyperbolicity of the classical equations of motion implies finite propagation speed of the classical solutions, and so the decay conditions are uniform in time. From (2.26) and Stokes' theorem, it follows that the right hand side of (2.58) is independent of $t \in M^1$ and also Ω is a closed 2-form on $\mathcal{M}$. In good cases this form is nondegenerate.

Similarly, if $j \in \Omega^{0,|-1|}_{\text{loc}}(\mathcal{F} \times M)$ is a conserved current—that is, $dj = 0$—then the associated *charge* Q_j is

$$Q_j = \int_{\{t\}\times N} j; \tag{2.68}$$

it is a function on the space of fields $\mathcal{F}$. Noether currents are conserved currents by (2.46) and (2.49); in that case the associated charge is called a *Noether charge*. If we restrict Q_j to $\mathcal{M}$, then since $dj = 0$ the right hand side is independent of t. This is a global conservation law. Local conservation laws are obtained by considering a domain $U \subset N$. For simplicity assume the closure of U is compact with smooth boundary ∂U. Let

$$q_t = \int_{\{t\}\times U} j \tag{2.69}$$

be the total charge contained in U at time t. Write

$$j = dt \wedge j_1 + j_2, \tag{2.70}$$

where j_1, j_2 do not involve dt. Stokes' theorem applied to integration over the fibers of the projection $M^1 \times U \to M^1$ implies

$$\frac{dq_t}{dt} + \int_{\{t\}\times \partial U} j_1 = 0. \tag{2.71}$$

This says that the rate of change of the total charge in U is minus the flux through the boundary.

Exercises

1. Recall the $*$ operator from the previous problem set. Now we define it for V is a finite-dimensional real vector space with a nondegenerate bilinear form, but no choice of orientation. First, define a real line $|\operatorname{Det} V^*|$ of densities on V. (Hint: A choice of orientation gives an isomorphism $|\operatorname{Det} V^*| \cong \operatorname{Det} V$.) Then define a $|-q|$-form on V to be an element of the vector space $\bigwedge^q V \otimes |\operatorname{Det} V^*|$. Finally, construct a $*$ operator

$$*\colon \textstyle\bigwedge^q V^* \longrightarrow \bigwedge^q V \otimes |\operatorname{Det} V^*|.$$

Compute formulas in a basis in some low-dimensional examples.

2. (a) (Stokes' theorem) Recall that for a single manifold X, assumed oriented and compact with boundary, that Stokes' theorem asserts that for any differential form $\alpha \in \Omega^\bullet(X)$ we have
$$\int_X d\alpha = \int_{\partial X} \alpha.$$
Now suppose we have a family of manifolds with boundary parametrized by a smooth manifold T, i.e., a fiber bundle $\pi\colon \mathcal{X} \to T$. Here $\mathcal{X}$ is a manifold with boundary, but T is a manifold without boundary. Assume first that the relative tangent bundle is oriented. Then integration along the fibers is defined as a map
$$\int_{\mathcal{X}/T} : \Omega^q(\mathcal{X}) \longrightarrow \Omega^{q-n}(T),$$
assuming the fibers to have dimension n. To fix the signs, we remark that on a product family $\mathcal{X} = T \times X$, for a form $\alpha = \alpha_T \wedge \alpha_X$ we set $\int_{\mathcal{X}/T} \alpha = \left\{\int_X \alpha_X\right\} \cdot \alpha_T$. If you are not familiar with this map, then construct it. Let $\partial\mathcal{X} \to T$ be the family of boundaries of the fibers. Then verify Stokes' theorem, at least in the case of a product family $\mathcal{X} = T \times X$: For $\alpha \in \Omega^q(\mathcal{X})$,
$$d\int_{\mathcal{X}/T} \alpha = \int_{\mathcal{X}/T} d\alpha + (-1)^{q-n} \int_{\partial\mathcal{X}/T} \alpha.$$
(b) Extend to the case when the tangent bundle along the fibers is not oriented. Then integration maps twisted forms on $\mathcal{X}$ to untwisted forms on T. Formulate Stokes' theorem. Be careful of the signs!

(c) Verify (2.71). You may choose not to use the generalities explained in the previous parts of this exercise.

3. A system of harmonic oscillators is described as a particle moving on a finite-dimensional real inner product space X with potential $V(x) = \frac{1}{2}|x|^2$. Let A be a skew-symmetric endomorphism of X and B a symmetric endomorphism. Consider the vector field on $\mathcal{F}$ defined by
$$\iota(\hat{\xi})\delta x = Ax + B\dot{x}$$
Verify that $\hat{\xi}$ is a nonmanifest infinitesimal symmetry. What is the corresponding Noether current? Can you identify these symmetries and currents physically? Are the $\hat{\xi}$ closed under Lie bracket on $\mathcal{F}$? What about on $\mathcal{M}$?

4. (a) Consider a *complex scalar field* on Minkowski spacetime. This is a map $\Phi\colon M^n \to \mathbb{C}$ with lagrangian
$$L = \left\{|d\Phi|^2 - m^2|\Phi|^2\right\} |d^n x|$$
Here m is a real parameter, and $|\cdot|$ is the usual norm of complex numbers. Compute the variational 1-form γ and the equations of motion. Verify that multiplication by unit complex numbers acts as a manifest symmetry of the theory. Write down the corresponding manifest infinitesimal symmetry $\hat{\xi}$ and the associated Noether current.

(b) More generally, consider a scalar field $\phi\colon M^n \to X$ with values in a Riemannian manifold X with potential function $V\colon X \to \mathbb{R}$. Suppose ζ is an infinitesimal isometry (Killing vector field) on X such that $\mathrm{Lie}(\zeta)V = 0$. Construct an induced manifest infinitesimal symmetry $\hat{\xi}$ and the corresponding Noether current.

(c) As a special case of the previous, consider a free particle on $\mathbb{E}^d$ ($n = 1$ and $V = 0$) and derive the formulas for linear and angular momentum, the conserved charges associated to the Lie algebra of the Euclidean group.

5. (a) For time translation as a nonmanifest symmetry of the real scalar field in 2 dimensions (see (2.61) in Example 2.58), compute the Lie derivative of the local symplectic form.

(b) Treat infinitesimal translation in the x^1 (space) direction both as a manifest and nonmanifest symmetry. Compute the associated Noether current and Noether charge. The latter is the momentum of the field.

6. (Energy-momentum tensor) In this problem we define the energy-momentum tensor. We caution, however, that there is another definition for fields coupled to a metric (defined more or less by differentiating with respect to the metric), and that the two do not always agree. The latter is always symmetric, whereas the one considered here is not.

(a) Consider a Poincaré-invariant lagrangian field theory $\mathcal{L} = L + \gamma$ on Minkowski spacetime M^n. We use the usual coordinates and the Lorentz metric with components $g_{\mu\nu}$ and inverse metric with components $g^{\mu\nu}$. As a consequence of Poincaré invariance we have that infinitesimal translation $\partial_\mu = \partial/\partial x^\mu$ induces a manifest infinitesimal symmetry of the theory. Let *minus* the associated Noether current be

$$\Theta_{\mu\nu} * dx^\nu = \Theta_{\mu\nu} g^{\nu\nu'}\ \iota(\partial_{\nu'})|d^n x|$$

for some functions

$$\Theta_{\mu\nu}\colon \mathcal{F} \times M \longrightarrow \mathbb{R}.$$

The tensor whose components are $\Theta = (\Theta_{\mu\nu})$ is called the energy-momentum tensor. Verify the conservation law

$$\sum_\nu \partial_\nu \Theta_{\mu\nu} = 0.$$

Prove that $\Theta_{\mu\nu} = \Theta_{\nu\mu}$ if and only if the current

$$\eta \cdot \Theta = \Theta_{\mu\nu}\eta^\mu * dx^\nu$$

is conserved for every infinitesimal Lorentz transformation η.

(b) Compute the energy-momentum tensor for the real scalar field (in two or more dimensions, as you prefer). Is it symmetric?

LECTURE 3
Classical Bosonic Theories on Minkowski Spacetime

Physical lagrangians; scalar field theories

In this lecture we consider theories defined on Minkowski spacetime M^n. As usual we let $\mathcal{F}$ denote the space of fields, L the lagrangian, γ the variational 1-form, $\mathcal{M}$ the space of classical solutions, and ω the local symplectic form. (In the theories we consider γ is determined canonically from L; see the first comment following (2.21).)

We first remark that our main interest in the lagrangians we write is for their use in *quantum* field theory, not classical field theory. The classical theory makes sense, certainly, and as explained in previous lectures the lagrangian encodes a classical Hamiltonian system, once a particular time is chosen (thus breaking Poincaré invariance). In the quantum theory, the exponential e^{iS} of the *action* $S = \int_{M^n} L$, is formally integrated over[15] $\mathcal{F}$ with respect to formal measure on $\mathcal{F}$. We will not discuss the quantum theory from this path integral point of view at all, but simply mention it again to remind the reader of the context for our discussion. Also, the path integrals are "Wick rotated" to integrals over fields on Euclidean space $\mathbb{E}^n$. There is a "Wick rotation" of lagrangians to Euclidean lagrangians, and we emphasize that they do not satisfy all of the requirements of physical lagrangians on Minkowski spacetime. We will not treat Wick rotation in these lectures either.

A word about units. We already used the universal constant c in relativistic theories; it has units of velocity, so converts times to lengths. In quantum theories there is a universal constant $\hbar$, called *Planck's constant*, which has units of action: mass $\times$ length^2/time. So using both c and $\hbar$ we can convert times and lengths to masses. This is typically done in relativistic quantum field theory. Physicists usually works in units where $c = \hbar = 1$, so the conversions are not evident.

The first requirement of a physical lagrangian is that it be *real*. We have already encoded that implicitly in our notation $L \in \Omega^{0,|0|}_{\text{loc}}(\mathcal{F} \times M)$, since we always use real (twisted) forms. But we could extend the formalism to complex (twisted) forms, and indeed we often must when writing Euclidean lagrangians. Also, we have assumed that $\mathcal{F}$ is a real manifold, but sometimes it presents itself more naturally as a complex manifold. Still, our point remains that the lagrangian is real when evaluated on fields (viewed as a real manifold).

[15]As remarked in a footnote in Lecture 1, the integral is taken over fields satisfying a certain finite action condition.

The second requirement is that the lagrangian be *local*. We have already built that into our formalism, and this property remains after Wick rotating to a Euclidean lagrangian. Locality holds for *fundamental* lagrangians, that is, lagrangians which are meant to describe nature at the smallest microscopic distance scales. *Effective* lagrangians describe nature at a larger distance scale, and these are often nonlocal, though usually only local approximations are written. In any case these lectures deal only with fundamental lagrangians. In fact, typically only first derivatives of the fields occur in the lagrangian. That constraint comes from the quantum theory and is beyond the scope of this course.

A final requirement is that the lagrangian be manifestly *Poincaré-invariant*. This ensures that the Poincaré group P^n acts on $\mathcal{M}$. The Poincaré group is a subgroup of the (global) symmetry group of the theory, and the entire symmetry group in a quantum theory is usually the product of P^n and a compact Lie group. (Under certain hypotheses this is guaranteed by the Coleman-Mandula theorem.) As explained in Lecture 2, symmetries give rise to conserved quantities. The conserved quantities associated to the Poincaré group include energy and momentum; conserved quantities associated to an external compact Lie group include electric charges and other "quantum numbers."

In these lectures we only write lagrangians which satisfy the conditions outlined here. In many cases they are the most general lagrangians which satisfy them, but we will not analyze the uniqueness question.

We begin with a real-valued scalar field $\phi\colon M^n \to \mathbb{R}$. Note that in this case $\mathcal{F}$ is an infinite-dimensional vector space. We work with standard coordinates $x^0, \dots, x^{n-1}$ as in (1.41), and set $\partial_\mu = \partial/\partial x^\mu$. The "kinetic energy" term for a real scalar field is

$$\begin{aligned} L_{\text{kin}} &= \frac{1}{2} d\phi \wedge *d\phi \\ &= \frac{1}{2}|d\phi|^2 \; |d^nx| \\ &= \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \; |d^nx| \\ &= \frac{1}{2}\big\{(\partial_0\phi)^2 - \sum_{i=1}^{n-1}(\partial_i\phi)^2\big\} \; |d^nx| \end{aligned} \tag{3.1}$$

The honest kinetic term is the first term in the last expression, the one involving the time derivative ∂_0; the terms with a minus sign are really potential energy terms. The signs are in accord with what we saw in mechanics: the lagrangian is kinetic minus potential. The various expressions make clear that L_{kin} is Poincaré-invariant, and in fact it is the Poincaré-invariant extension of the first term in (3.1), the usual kinetic energy. For this reason one often calls the entire L_{kin} the "kinetic term."

If we break Poincaré invariance and write $M^n = M^1 \times \mathbb{E}^{n-1}$ as time $\times$ space, then we can regard $\mathcal{F}$ as the space of paths in the vector space $\Omega^0(\mathbb{E}^{n-1})$. One can compute in this case that the symplectic structure on the space $\mathcal{M}$ of classical solutions is translation-invariant and the Hamiltonian is quadratic. Therefore, this is a free system: a *free real scalar field*. It is natural to ask what potential energy terms we can add to (3.1) to still have a free system. As we might expect, we can add a polynomial $V\colon \mathbb{R} \to \mathbb{R}$ of degree ≤ 2. We assume that V is bounded from below (see the discussion of energy below), so that the coefficient of the quadratic term is positive. We eliminate the linear and constant terms by assuming the minimum of V occurs at the origin and has value zero. Then the total lagrangian is

$$L = \left\{\frac{1}{2}|d\phi|^2 - \frac{m^2}{2}\phi^2\right\} |d^n x| \tag{3.2}$$

As written the constant m has units of inverse length. In a relativistic quantum theory it may be converted to a mass by replacing m with $mc/\hbar$, since $\hbar$ has units of *action*: mass times length squared divided by time. For this reason the potential term is called a *mass term*. The constant m is the mass of ϕ. This terminology is apparent from the quantization of the theory, as we discuss in Lecture 5. Finally, note that the lagrangian depends only on the 1-jet of ϕ, so as promised there is a canonical variational 1-form γ. We leave for the exercises a detailed computation of the variation of the lagrangian

$$\delta L = -\delta\phi \wedge \{d * d\phi + m^2\phi\,|d^n x|\} - d\{\delta\phi \wedge *d\phi\}, \tag{3.3}$$

the variational 1-form

$$\gamma = \delta\phi \wedge *d\phi, \tag{3.4}$$

and the local symplectic form

$$\omega = *d\delta\phi \wedge \delta\phi. \tag{3.5}$$

From this one derives the *classical field equation*, or Euler-Lagrange equation,

$$(\Box + m^2)\phi = 0. \tag{3.6}$$

Here

$$\begin{aligned} \Box &= (-1)^{n-1} * d * d \\ &= -d^* d \\ &= \partial_0^2 - \partial_1^2 - \cdots - \partial_{n-1}^2 \\ &= g^{\mu\nu}\partial_\mu\partial_\nu \end{aligned} \tag{3.7}$$

is the wave operator. We will analyze the solutions $\mathcal{M}$ in an exercise and again in Lecture 5. For now we can simply say that $\mathcal{M}$ is a real symplectic vector space which carries a representation of the Poincaré group P^n.

In nonfree theories, which of course are of greater interest than free theories, the potential is not quadratic. For example, later in the lecture we will consider a scalar field with a quartic potential. The most general model of this kind, called a *nonlinear σ-model*, starts with the data

$$\begin{array}{ll} X & \text{Riemannian manifold} \\ V\colon X \longrightarrow \mathbb{R} & \text{potential energy function} \end{array} \tag{3.8}$$

As usual, the space of fields is the mapping space $\mathcal{F} = \mathrm{Map}(M^n, X)$. The lagrangian of this model is

$$L = \left\{\frac{1}{2}|d\phi|^2 - \phi^* V\right\} |d^n x|. \tag{3.9}$$

Equations (3.4)–(3.6) generalize in a straightforward manner which incorporates the Riemannian structure of X. The special case $n = 1$ is the mechanical system we studied in Lecture 2—a particle moving on X—and we recover the equations we discussed there.

Hamiltonian field theory

To get a Hamiltonian interpretation of a Poincaré-invariant field theory, we break Poincaré invariance and choose an isomorphism $M^n \cong M^1 \times N$ for $N = \mathbb{E}^{n-1}$. (See Definition 2.66.) Of course, once we choose an affine coordinate system $x^0, x^1, \dots, x^{n-1}$, as we have been doing, then this splitting into time $\times$ space is determined; $x^0 = ct$ is a time coordinate and $x^1, \dots, x^{n-1}$ are coordinates on space. (We usually work in units where $c = 1$.) We consider a scalar field $\mathcal{F} = \operatorname{Map}(M^n, X)$ with values in a Riemannian manifold X, but what we say applies to other fields as well. In the Hamiltonian approach we view $\mathcal{F}$ as the space of paths in a space $\mathcal{F}_N = \operatorname{Map}(N, X)$ of fields on N:

$$\mathcal{F} \cong \operatorname{Map}(M^1, \mathcal{F}_N). \tag{3.10}$$

A *static* field $\phi \in \mathcal{F}$ is one which corresponds to a constant path under this isomorphism, and it is natural to identify $\mathcal{F}_N$ as the space of static fields.

Recall that in a mechanical system the *energy* is minus the Noether charge associated to infinitesimal time translation. More generally, in a lagrangian field theory $\mathcal{L} = L + \gamma$ we define the *energy density* to be minus the Noether current associated to infinitesimal time translation:

$$\Theta = -\iota(\xi_t)\mathcal{L}, \tag{3.11}$$

where ξ_t is infinitesimal time translation as a manifest infinitesimal symmetry. The energy at time t of a field ϕ is the integral of the energy density over the spatial slice:

$$E_\phi(t) = \int_{\{t\}\times N} \Theta(\phi). \tag{3.12}$$

For a static field the energy is constant in time. Just as typical fields in spacetime have infinite action, typical static fields have infinite energy. Define $\mathcal{F}\mathcal{E}_N \subset \mathcal{F}_N$ to be the space of static fields of finite energy. An important point is that whereas $\mathcal{F}_N$ may have fairly trivial topology, imposing finite energy often gives a space $\mathcal{F}\mathcal{E}_N$ of nontrivial topology.

For the scalar field the energy density is

$$\Theta(\phi) = \left\{ \frac{1}{2}|\partial_0\phi|^2 + \sum_{i=1}^{n-1} \frac{1}{2}|\partial_i\phi|^2 + V(\phi) \right\} |d^{n-1}x|, \tag{3.13}$$

where

$$|d^{n-1}x| = |dx^1 \cdots dx^{n-1}|. \tag{3.14}$$

The first term is the kinetic energy; the remaining terms are potential energy terms. Note that the kinetic term vanishes for a static field. We usually assume that the energy is bounded below, which for a scalar field means that the function V is bounded below. From (3.13) it is easy to see that fields of minimum energy are static and constant in space, i.e., constant in spacetime, and furthermore that constant must be a minimum of the potential V. Such static field configurations comprise a manifold

$$\mathcal{M}_{\mathrm{vac}} \subset \mathcal{F}\mathcal{E}_N \tag{3.15}$$

called the *moduli space of vacua*. For a real scalar field with potential V we usually assume that 0 is the minimum value of V, so

$$\mathcal{M}_{\mathrm{vac}} = V^{-1}(0). \tag{3.16}$$

For example, if $X = \mathbb{R}$ and $V = 0$ we have the theory of a massless real scalar field; in that case $\mathcal{M}_{\mathrm{vac}} \cong \mathbb{R}$. For a massive real scalar field we have $V(\phi) = \frac{m}{2}\phi^2$, so $\mathcal{M}_{\mathrm{vac}}$ is a single point $\phi = 0$. For a quartic potential

$$V(\phi) = \frac{\lambda}{8}(\phi^2 - a^2)^2 \tag{3.17}$$

(with λ, a positive real numbers), $\mathcal{M}_{\mathrm{vac}}$ consists of two points $\phi = \pm a$.

The notions of static fields, energy density, and vacuum solution extend to general field theories. Quite generally, on the space of static fields $\mathcal{F}_N$ we have the formula

$$\Theta = -\iota(\partial_t)L \qquad \text{on } \mathcal{F}_N. \tag{3.18}$$

Also, any critical point of energy on $\mathcal{F}_N$ is in fact a finite energy solution to the classical equations of motion. In particular, $\mathcal{M}_{\mathrm{vac}} \subset \mathcal{M}$. Finally, a vacuum solution is usually Poincaré-invariant. This is indeed true for the constant scalar fields. When we come to more complicated tensor fields, it is often true that Poincaré invariance already determines a unique field, which is then a unique vacuum.

A *soliton* is a static solution which is a critical point of energy, but not a minimum. Often the space $\mathcal{F}\mathcal{E}_N$ of finite energy static fields is not connected, and a soliton is a minimum energy configuration in a component where the minimum is not achieved. The two-dimensional real scalar field with quartic potential (3.17) provides an example. The space $\mathcal{F}\mathcal{E}_N$ has 4 components. The vacua are contained in two distinct components; in the other components we find solitons which minimize energy in that component.

Lagrangian formulation of Maxwell's equations

We first reformulate Maxwell's equations (1.39) in Minkowski spacetime M^4. Recall that in our previous formulation the electric field E is a time-varying 1-form on space N and the magnetic field B a time-varying 2-form on N. Now define the 2-form F on spacetime M^4 by

$$F = B - dt \wedge E. \tag{3.19}$$

Let $*$ be the $*$ operator on Minkowski spacetime and $*_N$ the $*$ operator on N; then

$$*F = \frac{1}{c} *_N E + c\,dt \wedge *_N B. \tag{3.20}$$

Maxwell's equations then simply reduce to the equations

$$\begin{aligned} dF &= 0 \\ d * F &= 0 \end{aligned} \tag{3.21}$$

These equations are Poincaré-invariant, as the $*$ operator on Minkowski spacetime is.

These equations are Maxwell's equations "in a vacuum"; the true Maxwell equations allow a current $j \in \Omega^3(M)$, which is constrained to have compact spatial support and satisfy $dj = 0$. Then the second Maxwell equation is $d * F = j$. For now, though, we concentrate on the case $j = 0$.

Note that (3.21) are first-order equations, whereas the Euler-Lagrange equations we have seen have been second-order. In fact, to construct a lagrangian formulation we introduce a 1-form $A \in \Omega^1(M^4)$ and define F as a function of A:

$$F_A = dA. \tag{3.22}$$

Note that $dF_A = 0$ for all A, so that the first Maxwell equation is immediately satisfied. Now the second Maxwell equation has second order in A,

$$d * F_A = d * dA = 0, \tag{3.23}$$

and we can expect to derive it from an action principle in a similar manner to our previous examples. (Note that there are action principles which lead to first-order Euler-Lagrange equations.) Some readers will recognize these equations as analogous to the equations of Hodge theory, and the lagrangian we introduce is also analogous to the one used in Hodge theory. Namely, introduce the following lagrangian, which is a function of A:

$$L = -\frac{1}{2} F_A \wedge *F_A. \tag{3.24}$$

Note that the lagrangian—and Maxwell's equations (3.22) and (3.23)—do not change if we change A by an exact 1-form: $A \to A + df$, $f \in \Omega^0(M^4)$. So for our space of fields we take the quotient of 1-forms by exact 1-forms:

$$\mathcal{F} = \Omega^1(M^4)/d\Omega^0(M^4). \tag{3.25}$$

Notice that the differential d identifies the space of fields with the space of exact 2-forms; it maps an equivalence class of gauge fields A to the exact 2-form F_A. We leave the reader to compute δL, γ, and to derive the Euler-Lagrange equation (3.23) from this lagrangian. Also, write the lagrangian in coordinates to see that it has the form kinetic energy (time derivatives) minus potential energy (spatial derivatives). We will write more general formulas later when we discuss gauge theories.

We remark that the lagrangian formulation we have just given works as well in n-dimensional Minkowski spacetime for arbitrary n.

Principal bundles and connections

To formulate gauge theory as used in lagrangians for quantum field theory, we quickly review some more differential geometry. The account we give here is very brief.

Let M be a manifold. Fix a Lie group G. A *principal G bundle* $P \to M$ is a manifold P on which G acts freely on the right with quotient $P/G \cong M$ such that there exist local sections. If P', P are principal G bundles over M, then an isomorphism of principal bundles $\varphi\colon P' \to P$ is a smooth diffeomorphism which commutes with G and induces the identity map on M. In case $P = P'$ such automorphisms are called *gauge transformations* of P. For each M there is a category of principal G bundles and isomorphisms.

A connection on a principal G bundle $\pi\colon P \to M$ is a G-invariant distribution in TP which is transverse to the vertical distribution $\ker d\pi$. In other words, at each $p \in P$ there is a subspace $V_p \subset T_pP$ of vectors tangent to the fiber at p. A connection gives, at each p, a complementary subspace $H_p \subset T_pP$ with the restriction that the distribution H be G-invariant. The infinitesimal version of the G action identifies each V_p with the Lie algebra $\mathfrak{g}$ of G. We can express a connection as the 1-form $A \in \Omega^1(P;\mathfrak{g})$ whose value at p is the projection $T_pP \to V_p \cong \mathfrak{g}$ with kernel H_p. The G-invariance translates into an equation on A which we do not write here. Connections, being differential forms on P, pull back under isomorphisms $\varphi\colon P' \to P$. Connections form a category, and there is a set of equivalence classes under isomorphisms.

A connection A has a curvature $F_A = dA + \frac{1}{2}[A \wedge A]$, which is a 2-form on P with values in $\mathfrak{g}$. Its transformation law under G, which again we omit, indicates that F_A is a 2-form on the base M with values in the *adjoint bundle*, the vector bundle of Lie algebras associated to P via the adjoint representation of G.

We can reformulate the lagrangian picture of Maxwell's equations in terms of connections. Namely, fix the Lie group $G = \mathbb{R}$ of translations. Then the space of equivalence classes of $\mathbb{R}$ connections on M may be identified with the space of fields (3.25). In fact, applying d we identify the quotient of 1-forms by exact 1-forms with the space of exact 2-forms. On the other hand, an $\mathbb{R}$ connection has a curvature which is an exact 2-form, any exact 2-form can occur, and equivalent connections have equal curvatures. From this point of view the space of fields is a category,[16] but the field theory is formulated on the set of equivalence classes. If we interpret A as an $\mathbb{R}$-connection, then F_A is its curvature. The lagrangian (3.24) depends only on F_A, so makes sense in this new formulation.

Gauge theory

The picture of Maxwell's equations given above is adequate for the classical theory. In quantum theory, however, we encounter a new ingredient: Dirac's charge quantization law. The charge in the previous story is the spatial integral of the current j (which we set to zero for simplicity). In that theory the charge can be any value, but in the quantum theory it must be an integer multiple of some fundamental value—that's what we mean by "quantization". In these lectures we will not have time to explain the hows and whys of charge quantization. Suffice it to say that there is an interesting geometric and topological story lurking behind. We simply use the formulation of Maxwell theory in terms of $\mathbb{R}$ connections and state that in the quantum theory charge quantization is achieved by replacing the group $\mathbb{R}$ by the *compact* group $\mathbb{R}/2\pi\mathbb{Z}$. (The '$2\pi$' is put in for convenience.)

[16]in fact, a *groupoid*

Once we have phrased Maxwell theory and charge quantization in these terms—this is certainly *not* how it was done historically!—it is easy to imagine a generalization in which the group $\mathbb{R}/2\pi\mathbb{Z}$ is replaced by any compact Lie group G. This bold step was taken by Yang and Mills in 1954 much before the connection to connections was established. Therefore, we are led to formulate a lagrangian field theory based on the following data:

$$\begin{array}{ll} G & \text{compact Lie group with Lie algebra } \mathfrak{g} \\ \langle\cdot,\cdot\rangle & \text{bi-invariant inner product on } \mathfrak{g} \end{array} \tag{3.26}$$

The theory we formulate is called *pure gauge theory*. We emphasize that in physics gauge theories with compact structure group are used in *quantum* field theory, not classical field theory. The choice of inner product incorporates *coupling constants* of the theory. For example, if G is a simple group then there is a 1-dimensional vector space of invariant inner products, any two of which are proportional. In the classical Maxwell theory, $G = \mathbb{R}$ and $\langle\cdot,\cdot\rangle$ is the standard inner product on the Lie algebra $\mathbb{R}$. We assume that $\langle\cdot,\cdot\rangle$ is nondegenerate; this amounts to a nondegeneracy assumption on the lagrangian (3.27) below. The space of fields is, as said above, the category of G connections over spacetime M^n, and everything we write is invariant under isomorphisms of connections. The lagrangian is

$$L = -\frac{1}{2}\langle F_A \wedge *F_A\rangle. \tag{3.27}$$

We leave the reader to compute δL and so derive the equation of motion—the *Yang-Mills equation*

$$d_A * F_A = 0, \tag{3.28}$$

the variational 1-form

$$\gamma = -\langle\delta A \wedge *F_A\rangle, \tag{3.29}$$

and the local symplectic form

$$\omega = \langle\delta A \wedge *d_A\delta A\rangle. \tag{3.30}$$

Here d_A is the differential in the extension of the de Rham complex to forms with values in the adjoint bundle (using the connection A); see (1.12).

If G is abelian, then the equations of motion are linear and the associated Hamiltonian system is free. If G is nonabelian, then the equations of motion are nonlinear. Geometers are familiar with the Yang-Mills equations (3.28) on Riemannian manifolds, where after dividing out by isomorphisms they are essentially elliptic. Here, on spacetime with a metric of Lorentz signature, the Yang-Mills equations are wave equations.

The energy density of the field A is

$$\Theta(A) = \Big\{\sum_{\mu<\nu} \frac{1}{2}|F_{\mu\nu}|^2\Big\}\ |d^{n-1}x|, \tag{3.31}$$

where $\mu, \nu = 0, \dots, n-1$ run over all spacetime indices and

$$F_A = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \tag{3.32}$$

is the curvature of A. So the minimum energy is achieved when $F_{\mu\nu} = 0$ for all μ, ν; i.e., when the curvature of A vanishes. On Minkowski spacetime all such connections are isomorphic, whence the moduli space of vacua consists of a single point:

$$\mathcal{M}_{\text{vac}} = \text{pt}\,. \tag{3.33}$$

As mentioned before, we work on the space of equivalence classes of connections.

Gauged σ-models

The most general bosonic field theory we consider on M^n combines the pure gauge theory of the previous section with the σ-models considered earlier.[17] Again, these are models used in quantum theory. The data we need to specify the model is the following:

$$\begin{array}{ll} G & \text{Lie group with Lie algebra } \mathfrak{g} \\ \langle\cdot,\cdot\rangle & \text{bi-invariant scalar product on } \mathfrak{g} \\ X & \text{Riemannian manifold on which } G \text{ acts by isometries} \\ V\colon X \longrightarrow \mathbb{R} & \text{potential function invariant under } G \end{array} \tag{3.34}$$

An important special case has X a real vector space with positive definite inner product and G acting by orthogonal transformations. The space $\mathcal{F}$ of fields in the theory consists of pairs

$$\begin{array}{ll} A & \text{connection on a principal } G\text{-bundle } P \longrightarrow M \\ \phi & \text{section of the associated bundle } P \times_G X \longrightarrow M \end{array} \tag{3.35}$$

In the linear case the associated bundle $P \times_G X$ is a vector bundle over M. In all cases it is often convenient to view ϕ as an equivariant map $\phi\colon P \to X$. The space of fields is again best seen as a category—an isomorphism $\varphi\colon P' \to P$ of principal bundles induces an isomorphism of fields $(A', \phi') \to (A, \phi)$. As usual, we consider fields up to isomorphism. The global symmetry group of the theory is the subgroup of isometries of X which commute with the G action and preserve the potential function V.

Physicists call models with this field content a *gauged (linear or nonlinear) σ-model.*

The lagrangian we consider combines (3.27) and (3.9):

$$L = \left\{ -\frac{1}{2}|F_A|^2 + \frac{1}{2}|d_A\phi|^2 - \phi^*V \right\} |d^n x|. \tag{3.36}$$

[17]For dimensions $n \leq 4$ this is all that is usually encountered in models without gravity. However, in higher dimensions—especially in supergravity—there may be other fields which locally are differential forms of degree > 1, just as connections with structure group $\mathbb{R}$ or $\mathbb{R}/2\pi\mathbb{Z}$ are locally 1-forms.

Note that the covariant derivative d_A replaces the ordinary derivative d encountered in the pure σ-model (3.9). This term "couples" the fields A and ϕ. Quite generally, if $L = L(\phi_1, \phi_2)$ is a lagrangian which depends on fields ϕ_1, ϕ_2, then we say that the fields are *uncoupled* if we can write $L(\phi_1, \phi_2) = L_1(\phi_1) + L_2(\phi_2)$. Again, we leave the reader to derive the equations of motion, variational 1-form, etc.

The energy density of the pair (A, ϕ) works out to be

$$\Theta(A,\phi) = \Big\{ \sum_{\mu<\nu} \frac{1}{2}|F_{\mu\nu}|^2 + \sum_{\mu} \frac{1}{2}|(\partial_A)_\mu \phi|^2 + \phi^* V \Big\} \; |d^{n-1}x|. \tag{3.37}$$

We seek vacuum solutions assuming that V has a minimum at 0, so that the energy is bounded below by 0. Thus we seek solutions of zero energy. The first term implies that for a zero-energy solution A is flat, so up to equivalence is the trivial connection with zero curvature. Then we can identify covariant derivatives with ordinary derivatives, and the second term implies that ϕ must be constant for a zero-energy solution. Finally, the last term implies that the constant is in the set $V^{-1}(0)$. Now recall that we consider pairs (A, ϕ) up to equivalence. A trivial connection A has a group of automorphisms isomorphic to G. This is the group of *global gauge transformations*. Now for a vacuum solution ϕ is a constant function into $V^{-1}(0)$, and the G-action of ϕ is simply the G-action on $V^{-1}(0)$. Thus the moduli space of vacua is then a subquotient space of X:

$$\mathcal{M}_{\text{vac}} = V^{-1}(0) \,/\, G. \tag{3.38}$$

In certain supersymmetric field theories the manifold X is restricted to be Kähler or hyperkähler and the potential is the norm square of an appropriate moment map. In those cases $\mathcal{M}_{\text{vac}}$ is the Kähler or hyperkähler quotient. (It was in this context that the latter was in fact invented.)

We mention that specific examples of these models have solitons, which recall are static solutions not of minimal energy. For example, in $n = 4$ dimensions the linear σ-model with $G = SO(3)$, X the standard real 3-dimensional representation, and V a quartic potential (3.17) has static *monopole* solutions.

Exercises

The lecture skipped many computations and verifications, so several of the problems ask you to fill in those gaps.

1. (a) Derive formula (3.3) for the variation of the free scalar field lagrangian.

   (b) What is the corresponding formula for the nonlinear σ-model (3.9)? What are the generalizations of (3.4) and (3.5)? Check your formulas against the formulas in Lecture 2 for the particle moving on X.

   (c) Verify formula (3.13) for the energy density of a scalar field.

2. Recall the (components of the) *energy-momentum* tensor $\Theta_{\mu\nu}$ from Problem Set 2. Note that the energy density may be expressed as $\Theta_{00} * dx^0$, and so the energy-momentum tensor is the Poincaré-invariant generalization of the energy density. Compute the energy-momentum tensor for the scalar field, and then recover (3.13).

3. In this problem you will find the solutions to the equations of motion of simple free systems, both for particles and for fields. The field in all cases is a map $\phi\colon M^n \to \mathbb{R}$ with potential $V(\phi) = \frac{1}{2}k\phi^2$.

   (a) First, consider $n = 1$. Write the Euler-Lagrange equations. What are the solutions for $k = 0$? For $k \neq 0$? (The latter is a harmonic oscillator, and is not usually called "free".)

   (b) If you didn't already do it, solve the previous problem using the 1-dimensional Fourier transform. My convention for the Fourier transform are as follows—you're welcome to use your own. Let V be an n-dimensional real vector space and $\phi\colon V \to \mathbb{C}$ a complex-valued function. Its Fourier transform $\hat{\phi}\colon V^* \to \mathbb{C}$ is a function on the dual space. The functions ϕ and $\hat{\phi}$ are related by the integrals

$$\hat{\phi}(k) = \frac{1}{(2\pi)^{n/2}} \int_V e^{-\sqrt{-1}\langle k,x\rangle} \phi(x)\,|d^n x|, \qquad k \in V^*$$
$$\phi(x) = \frac{1}{(2\pi)^{n/2}} \int_{V^*} e^{+\sqrt{-1}\langle k,x\rangle} \hat{\phi}(k)\,|d^n k|, \qquad x \in V.$$

   In analysis one analyzes carefully the class of functions for which these formulas make sense; we work formally here.

   (c) Now consider both $k = 0$ and $k \neq 0$ for the case of fields (arbitrary n). What is the support of the Fourier transform of the solution to the wave equation (3.6)? The reality of the field ϕ induces a reality condition on the Fourier transform $\hat{\phi}$; what is it? The equations are Poincaré-invariant, so the Poincaré group acts on the space of solutions. What can you say about the action?

4. We study the motion of a particle on an interval $[a, b] \in \mathbb{E}^1$ with no potential energy. As a classical system this is only a local system, since a particle moving at constant velocity—a motion which solves Newton's laws—runs off the end of space in finite time. Let's describe this system instead as particle motion on the line $\mathbb{E}^1$ with a potential function V which vanishes on $[a, b]$ and is positively infinite otherwise. Study this system as the $C \to \infty$ limit of a system whose potential outside $[a, b]$ is C. You may want to smooth out the potential at the points a, b. What happens to the particle as it gets close to the endpoints of the interval? What is the moduli space of vacua $\mathcal{M}_{\mathrm{vac}}$ in this model? I recommend that you study the quantum mechanics of this system as well. This includes the moduli space of vacua, correlation functions, their Wick rotation to Euclidean time, etc. Everything is computable and the computations illustrate features present in more complicated quantum field theories.

5. Consider a real scalar field in $n = 2$ dimensions with the quartic potential (3.17). A static field is a real-valued function on space $\mathbb{E}^1$. The space of all such is a vector space. Demonstrate that the space of *finite energy* static fields has 4 components. How are they distinguished? Find vacuum solutions. Find the soliton solutions mentioned in the lecture by writing the formula for energy and computing the critical point equation.

6. (a) Write Maxwell's equations (3.24) as a wave equation in A.

(b) Analyze the solutions using the Fourier transform. Remember that the space of fields is a quotient (3.25). This is not a particularly easy problem the first time around. Can you identify the (real) representation of the Poincaré group obtained?

7. Write the gauged σ-model for $G = \mathbb{R}$ and $X = \mathbb{E}^1$ with G acting by translations. What possible potentials V can we use? Write the equations of motion for this system. What modification to Maxwell's equations do you find?

8. (a) Write the lagrangians for gauge theory and for the gauged σ-model in coordinates so you can see what they really look like.

   (b) Fill in missing computations in the lecture: the equations of motion, local symplectic form, energy density etc. for pure gauge theory and for the gauged σ-model.

LECTURE 4
Fermions and the Supersymmetric Particle

In classical—that is, nonquantum—relativistic physics the objects which one considers are of two sorts: either particles, strings, and other extended objects moving in Minkowski spacetime M^n; or the electromagnetic field. Perhaps there are also sensible models with scalar fields. But as far as I know, there is not a good notion of a "classical fermionic field". Nonetheless, the remaining lectures incorporate fermionic fields, and in particular develop "classical supersymmetric field theory". As I have said repeatedly in these notes, although what we discuss uses the formalism of classical field theory, it is in the end used as the input into the (formal) path integral of the quantum field theory.

To write classical fermionic fields we need to bring in a new piece of differential geometry: supermanifolds. We refer the reader to John Morgan's lectures in this volume for an introduction, but recall a few key points here. The linear algebra underlying supermanifolds concerns $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces $V = V^0 \oplus V^1$. The homogeneous summand V^0 is called *even*; V^1 is termed *odd*. The parity-reversed vector space $\Pi V = V^1 \oplus V^0$ has the even and odd summands interchanged. The *sign rule* extends many notions of algebra to the $\mathbb{Z}/2\mathbb{Z}$-graded world by introducing a sign when odd elements are interchanged. For example, one of the axioms of a Lie bracket (on a $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra) is that $[a,b] = \mp[b,a]$, the plus sign occuring only if both a and b are odd. To understand fermionic fields, though, we have to come to grips with an odd vector space V^1 as a space, not just as an algebraic object, and for that we define it in terms of its functions. Namely, the ring of functions on V^1 is the $\mathbb{Z}/2\mathbb{Z}$-graded exterior algebra $\bigwedge^{\bullet}(V^1)^*$. (Compare: The algebraic functions on an even vector space V^0 is the symmetric algebra $\mathrm{Sym}^{\bullet}(V^0)^*$.) Since the exterior algebra contains nilpotents, we use intuition and techniques from algebraic geometry—specifically the *functor of points*—to understand the space V^1.

The supersymmetric particle

Fix a Riemannian manifold X. In the discussion of an ordinary, nonsupersymmetric particle moving on X we also have a potential energy function $V\colon X \to \mathbb{R}$. The supersymmetric version we consider forces $V = 0$. For the ordinary particle moving on X, as considered in Lecture 1, "spacetime" is simply time M^1, the field is a map $x\colon M^1 \to X$, and the lagrangian is

$$L_0 = \left\{\frac{1}{2}|\dot{x}|^2\right\} |dt|, \tag{4.1}$$

where t is a coordinate on M^1. We now want to add a second field to the theory, and it is a fermionic field. For simplicity we do this first in case $X = \mathbb{E}^1$, i.e., for a particle moving on a line. Then the fermionic field is a map $\psi\colon M^1 \to \Pi\mathbb{R}^1$ from time to a fixed one-dimensional odd vector space. So the space of fields is the product

$$\mathcal{F} = \operatorname{Map}(M^1, \mathbb{E}^1) \times \operatorname{Map}(M^1, \Pi\mathbb{R}^1). \tag{4.2}$$

The lagrangian for this theory is a function of the pair (x, ψ), and it is in fact the sum of the lagrangian L_0 for x and a lagrangian for ψ:

$$L = \Big\{\frac{1}{2}|\dot{x}|^2 + \frac{1}{2}\psi\dot{\psi}\Big\}\,|dt|. \tag{4.3}$$

The fields x and ψ are uncoupled in this model. Note that whereas the bosonic kinetic term is the square of the first derivative of the field, the fermionic kinetic term—the second term in (4.3)—is the field times its first derivative. This is typical of kinetic terms in higher dimensional field theories as well as in this particle example.

It is fruitful to consider the fermionic theory on its own. Then the space of fields is

$$\mathcal{F}_1 = \operatorname{Map}(M^1, \Pi\mathbb{R}^1) \tag{4.4}$$

and the lagrangian is

$$L_1 = \Big\{\frac{1}{2}\psi\dot{\psi}\Big\}\,|dt|. \tag{4.3}$$

To illustrate computations with fermionic fields, we compute carefully the variation of the lagrangian δL_1. As with our previous mechanics computations, we fix an orientation of time, so identify $|dt| = dt$. Then

$$\begin{aligned} \delta L_1 &= \big(\frac{1}{2}\delta\psi\,\dot{\psi} + \frac{1}{2}\psi\,\delta\dot{\psi}\big)\wedge dt \\ &= \frac{1}{2}\delta\psi\wedge d\psi + \frac{1}{2}\psi\,\delta d\psi \\ &= \frac{1}{2}\delta\psi\wedge d\psi - \frac{1}{2}\psi\, d\delta\psi \\ &= \frac{1}{2}\delta\psi\wedge d\psi - d\big(\frac{1}{2}\psi\,\delta\psi\big) + \frac{1}{2}d\psi\wedge\delta\psi \\ &= \delta\psi\wedge d\psi - d\big(\frac{1}{2}\psi\,\delta\psi\big). \end{aligned} \tag{4.5}$$

The first line is the Leibniz rule, but already one could raise an objection. Since δ is odd and ψ is odd, why don't we pick up a sign in the second term after commuting δ past ψ? In fact, it is perfectly consistent to employ sign rules based on parity + cohomological degree. But instead, we use sign rules based on the pair $(\text{parity}, \text{cohomological degree})$. In this notation δ has bidegree $(0,1)$ and ψ has bidegree $(1,0)$. In general when commuting elements of bidegrees (p,q) and (p',q'), we pick up a sign of $(-1)^{pp'+qq'}$. The second equation of (4.5) follows simply from

$\dot{\psi}\,dt = d\psi$. The third uses $\delta d = -d\delta$. In the fourth we commute d past ψ and again there is no sign. Finally, in the last equation we use $d\psi \wedge \delta\psi = \delta\psi \wedge d\psi$, which holds since the bidegrees of $d\psi$ and $\delta\psi$ are both equal to $(1,1)$. So define

$$\gamma_1 = \frac{1}{2}\psi\,\delta\psi, \tag{4.6}$$

and the equation of motion is the vanishing of $\delta L_1 + d\gamma_1 = \delta\psi \wedge d\psi = -\dot{\psi}\,\delta\psi \wedge dt$:

$$\dot{\psi} = 0. \tag{4.7}$$

So the classical space of solutions $\mathcal{M}$ is identified with constant values of ψ, i.e. with the odd vector space $\Pi\mathbb{R}^1$. The symplectic form is

$$\omega_1 = \delta\gamma_1 = \delta\psi \wedge \delta\psi. \tag{4.8}$$

Since $\delta\psi$ has bidegree $(1,1)$—that is, $\delta\psi$ has odd parity and cohomological degree 1)—there is no sign commuting $\delta\psi$ past itself. Indeed this symplectic form is a nondegenerate form on the odd vector space $\Pi\mathbb{R}^1$, and can be identified with the usual inner product on $\mathbb{R}^1$.

In the full model (4.3) the fields x and ψ are completely decoupled: $L = L_0 + L_1$. It follows that

$$\gamma = \gamma_0 + \gamma_1 = \dot{x}\,\delta x + \frac{1}{2}\psi\,\delta\psi, \tag{4.9}$$

and the symplectic form is similarly a sum from the free particle and (4.6). The equations of motion are the simultaneous equations

$$\ddot{x} = \dot{\psi} = 0. \tag{4.10}$$

The space of solutions is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space whose even part has dimension 2 and whose odd part has dimension 1. (We remark that odd vector spaces have a nice quantization only when the dimension is even, but as we are not quantizing now we will not let that worry us unduly.)

Life is much more interesting when we consider a particle moving in a Riemannian manifold X, and then introduce the correct fermionic "partner" field ψ. In fact,

$$\psi \in \Omega^0(M^1; x^*\Pi TX), \tag{4.11}$$

a section of the pullback *odd* tangent bundle. Note again the space of fermionic fields is an (infinite-dimensional) odd vector space. The entire space of fields $\mathcal{F}$ consists of pairs (x,ψ). It is not a product, but rather the projection $(x,\psi) \mapsto x$ is well-defined, so the space of fields $\mathcal{F}$ is an odd vector bundle over the space of bosonic fields $\mathcal{F}_0 = \mathrm{Map}(M^1, X)$. The important feature is that now the fields $x\colon M^1 \to X$ and ψ are coupled in the Lagrangian:

$$L = \left\{\frac{1}{2}|\dot{x}|^2 + \frac{1}{2}\langle\psi, (x^*\nabla)_{\partial_t}\psi\rangle\right\} dt. \tag{4.12}$$

The second term uses the covariant derivative on $x^*\Pi TX$ induced from the Levi-Civita connection. It is an instructive exercise, not at all trivial, to analyze this model in detail: compute the variational 1-form γ, the symplectic form ω, the equations of motion, and the Hamiltonian (energy). We simply state the answers here:

$$\begin{aligned}\gamma &= \langle \dot{x}, \delta x\rangle + \frac{1}{2}\langle \psi, \delta_\nabla \psi\rangle \\ \omega &= \langle \delta_\nabla \dot{x} \wedge \delta x\rangle + \frac{1}{2}\langle \delta_\nabla \psi \wedge \delta_\nabla \psi\rangle + \frac{1}{4}\langle \psi, R(\delta x \wedge \delta x)\psi, \rangle,\end{aligned} \tag{4.13}$$

where R is the curvature of X (pulled back to M^1 using $x\colon M^1 \to X$). The equations of motion are

$$\begin{aligned}\nabla_{\dot{x}}\dot{x} &= \frac{1}{2}R(\psi,\psi)\dot{x} \\ \nabla_{\dot{x}}\psi &= 0.\end{aligned} \tag{4.14}$$

Finally, the Hamiltonian is the same as for the ordinary particle:

$$H = \frac{1}{2}|\dot{x}|^2. \tag{4.15}$$

The new feature of this model is a nonmanifest symmetry which exchanges x and ψ. In fact, there is a one-parameter family of such, and its infinitesimal generator is therefore an *odd* vector field on $\mathcal{F}$. We describe it by introducing an *auxiliary odd parameter* η, which is best understood in the "functors of points" paradigm, and then writing formulas for the *even* vector field $\hat{\zeta}$ which is the product of η and the odd vector field just mentioned:

$$\begin{aligned}\iota(\hat{\zeta})\delta x &= -\eta\psi \\ \iota(\hat{\zeta})\delta_\nabla \psi &= +\eta\dot{x}.\end{aligned} \tag{4.16}$$

Both sides of the second equation are odd sections of the pullback tangent bundle x^*TX. This infinitesimal symmetry is not manifest; rather

$$\mathrm{Lie}(\hat{\zeta})L = d\alpha_{\hat{\xi}} \tag{4.17}$$

for

$$\alpha_{\hat{\xi}} = d\big(\iota(\hat{\zeta})\gamma + \eta\langle \psi, \dot{x}\rangle\big). \tag{4.18}$$

The corresponding Noether charge is then

$$j_{\hat{\xi}} = \iota(\hat{\zeta})\gamma - \alpha_{\hat{\xi}} = -\eta\langle \psi, \dot{x}\rangle. \tag{4.19}$$

Now define

$$Q = \langle \psi, \dot{x}\rangle; \tag{4.20}$$

we use the opposite sign as in (4.19) just as the Hamiltonian is the opposite of the conserved charge associated to infinitesimal time translation. On the space of solutions to (4.14), or state space $\mathcal{M}$, the odd function Q is conserved under time evolution, just as the Hamiltonian H is. In fact, $\mathcal{M}$ is a symplectic supermanifold, and we have the Poisson bracket

$$\{Q, Q\} = -2H. \tag{4.21}$$

A brief word about supersymmetric quantum mechanics

For the ordinary point particle moving on X, say with no potential, then by fixing a time we identify the state space $\mathcal{M}_0$ with the tangent bundle TX (see (1.6)). This is the statement that a solution to Newton's law, a second order ordinary differential equation, is determined by an initial position and velocity. Now the equation of motion of the fermionic field ψ—the second equation in (4.14)—is first-order, so the solution is determined by an initial value. It follows, if X is complete, that by fixing a time we may identify the state space $\mathcal{M}$ of solutions to the simultaneous equations (4.14) as the total space of the bundle

$$\pi^*\Pi TX \longrightarrow TX, \tag{4.22}$$

where $\pi\colon TX \to X$ is the ordinary tangent bundle. This is a symplectic supermanifold. The symplectic form on each (odd) fiber of (4.22) is really the Riemannian metric on the corresponding ordinary tangent space.

Quantization—often said to be an art rather than a functor—is meant to convert symplectic (super)manifolds into (graded) Hilbert spaces and map the Poisson algebra of functions—classical observables—into the Lie algebra of self-adjoint operators—quantum observables. For the particle moving on X, whose state space is $\mathcal{M}_0 \cong TX$, there is a standard answer: The Hilbert space $\mathcal{H}^0$ is the space of L^2 functions on X and the quadratic Hamiltonian maps[18] to the second-order Laplace operator Δ. We approach the quantization of (4.22) in two steps: first quantize the fibers and then the base. Now each fiber is an odd symplectic vector space, which can be viewed as an ordinary vector space with a nondegenerate symmetric bilinear form. In fact, the bilinear form is positive definite. The Poisson algebra of functions may be identified with the Clifford algebra of that inner product. So what we seek is a representation of the Clifford algebra—a Clifford module. As we work in the super world, we take the Clifford module to be $\mathbb{Z}/2\mathbb{Z}$-graded. This works more naturally for even dimensional odd vector spaces, in which case the unique irreducible Clifford module[19] is the graded spinor representation $S^+ \oplus S^-$. To do that fiberwise over all of TX requires that X be a spin manifold. The classical system makes sense for any Riemannian manifold X, whereas for the quantization we need a spin structure on X. This is a typical situation in quantization: there is often an obstruction—or *anomaly*—which prevents the quantization. In this case the anomaly is the obstruction to putting a spin structure on X. If there is a spin structure, then combining the quantization of TX discussed earlier with the quantization of the fibers, we see that the appropriate graded Hilbert space for the supersymmetric particle is

$$\mathcal{H} = L^2(X; S^+) \oplus L^2(X; S^-), \tag{4.23}$$

the graded Hilbert space of L^2 spinor fields on X.

What are the operators corresponding to the classical observables Q and H? As before, H is the Laplace operator, but now acting on spinor fields. In the

[18] In fact, there is an indeterminacy; we can map the Hamiltonian to $\Delta + C$ for any constant C.

[19] Recall that in ordinary quantization we also take irreducible representations: the quantization of the symplectic (p, q)-plane is, for example, the set of L^2 functions of q, not L^2 functions of both q and p.

quantization of the odd fibers, ψ becomes Clifford multiplication, and so it is not unreasonable to believe from (4.20) that Q becomes the Dirac operator. Note that the Dirac operator is odd: it exchanges even and odd elements of $\mathcal{H}$.

A study of the path integral in this model leads to a physicists' proof of the Atiyah-Singer index theorem for Dirac operators, which was discovered in the early '80s.

Superspacetime approach

Roughly speaking, the supersymmetry (4.16) is a square root of infinitesimal time translation. Namely, if we let $\hat{Q}$ be the odd vector field such that $\hat{\zeta} = \eta\hat{Q}$, and let $\hat{\xi}$ be infinitesimal time translation, then

$$[\hat{Q}, \hat{Q}] = 2\hat{\xi}. \tag{4.24}$$

we are led to wonder if there is a square root of infinitesimal time translation whose action on fields induces $\hat{Q}$, just as infinitesimal time translation on M^1 induces $\hat{\xi}$. Of course, this cannot happen on M^1, nor on any ordinary manifold. The "square" of a vector field—one-half its bracket with itself—vanishes for even vector fields, since the Lie bracket is skew-symmetric. So we replace M^1 by a supermanifold and seek an odd vector field as the square root. Then we write the fields in our theory as functions on that supermanifold so that such an odd vector field induces an action on fields.

Such a construction exists and it leads to a formulation of the superparticle in which the supersymmetry is manifest. The supermanifold we seek goes under the name *superspacetime*. (Of course, 'superspacetime' is usually rendered 'superspace'. In this example, 'supertime' would be even more appropriate.) The superspacetime formulation of supersymmetric theories goes back to the mid '70s when it was introduced by Salam and Strathdee. Be warned that not all supersymmetric theories have a superspacetime formulation. We give a general construction of superspacetime in Lecture 6. Here we simply introduce the superspacetime relevant to the superparticle.

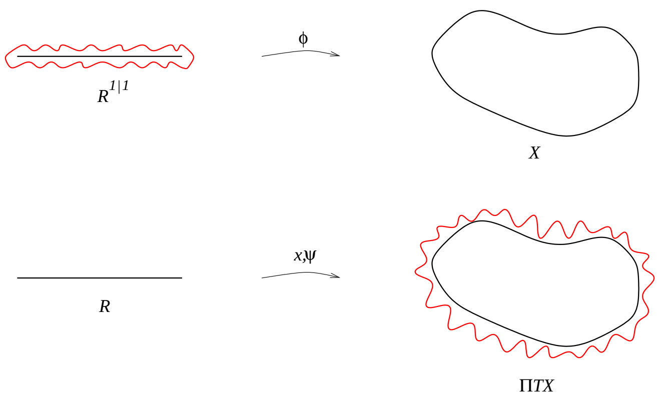
Let $M^{1|1}$ denote the affine space whose ring of functions is

$$C^\infty(M^{1|1}) = C^\infty(M^1)[\theta]$$

for an odd variable θ. Note:

- θ is *not* an *auxiliary* odd variable—it is a *bona fide* odd function on $M^{1|1}$.
- In our *component* formulation of the superparticle we had fields x, ψ which map an ordinary manifold into a supermanifold. Now we consider a *superfield* formulation in which the field Φ maps a supermanifold into an ordinary manifold, as sketched in the nonartist's rendering below.

It is instructive to work out the space of maps $M^{1|1} \to X$ into a manifold X by considering the induced algebra homomorphism $C^\infty(X) \to C^\infty(M^{1|1})$. A single map reduces simply to a path $x = x(t)$ in X, but things become more interesting if we introduce an auxiliary odd parameter η. In other words, we consider a one-parameter family of maps with parameter η. This is equivalent to an algebra homomorphism $C^\infty(X) \to \operatorname{Spec}\mathbb{R}[\eta] \times C^\infty(M^{1|1})$. Then the pullback of a



function $f \in C^\infty(X)$ may be written as $a_t(f) + \theta\eta b_t(f)$. Using the fact that the pullback is an algebra homomorphism, one can deduce that $a_t(f) = f\big(x(t)\big)$ for some path $x(t) \in X$, and $b_t(f) = V_t f$ for some path of tangent vectors $V_t \in T_{x(t)}X$. The fermionic component field $\psi(t) = \eta V_t$.

Let t be a standard affine coordinate on M^1, so that t, θ are global coordinates on $M^{1|1}$. Let

$$i\colon M^1 \hookrightarrow M^{1|1} \tag{4.25}$$

be the inclusion defined by

$$\begin{aligned} i^* t &= t \\ i^*\theta &= 0. \end{aligned} \tag{4.26}$$

We introduce a global framing of $M^{1|1}$ by the vector fields

$$\begin{aligned} \partial_t &= \frac{\partial}{\partial t} \\ D &= \partial_\theta - \theta\partial_t = \frac{\partial}{\partial\theta} - \theta\partial_t. \end{aligned} \tag{4.27}$$

Here ∂_t is even and D is odd. It is unfortunate that the vector field 'D' has the same symbol as the total differential considered in earlier parts of the lecture. *Tant pis*, there just aren't enough dees in the world! We also introduce the odd vector field

$$\tau_Q = \partial_\theta + \theta\partial_t. \tag{4.28}$$

In fact, $M^{1|1}$ is the supermanifold underlying a Lie group on which $\{\partial_t, D\}$ is a basis of left invariant vector fields and $\{\partial_t, \tau_Q\}$ a basis of right invariant vector fields. They satisfy the bracket relations

$$\begin{aligned} [D, D] &= -2\partial_t \\ [\tau_Q, \tau_Q] &= +2\partial_t \\ [D, \tau_Q] &= \quad 0. \end{aligned} \tag{4.29}$$

The vector field ∂_t commutes with both D and τ_Q.

We now formulate a field theory with spacetime $M^{1|1}$ and space of fields

$$\mathcal{F} = \{\Phi\colon M^{1|1} \longrightarrow X\}, \tag{4.30}$$

where X is our fixed Riemannian manifold. As we saw above, Φ leads to component fields x, ψ which are a path in X and an odd tangent vector field along the path We recover these *component fields* from Φ by restricting D-derivatives of the superfield to the underlying Minkowski spacetime:

$$\begin{aligned} x &:= i^*\Phi \\ \psi &:= i^*D\Phi. \end{aligned} \tag{4.31}$$

The vector fields ∂_t, τ_Q act on the field Φ directly by differentiation, and it is not hard to work out the action on component fields. For ∂_t it is simply differentiation in t, and for τ_Q we use the one parameter group $\varphi_u = \exp(u\eta\tau_Q)$ generated by $\hat{\zeta} = \eta\tau_Q$ for an odd parameter η. Recall that the action of a diffeomorphism of (super)spacetime on fields uses pullback by the *inverse* (see Example 2.37, for example):

$$\begin{aligned} \iota(\hat{\zeta})\delta x = \hat{\zeta}\cdot x &= \frac{d}{du}\Big|_{u=0} (\varphi_u^{-1})^* i^*\Phi \\ &= \frac{d}{du}\Big|_{u=0} i^*(\varphi_u^{-1})^*\Phi \\ &= -\eta i^*\tau_Q\Phi \\ &= -\eta i^*D\Phi \\ &= -\eta\psi \end{aligned} \tag{4.32}$$

Similarly, we compute

$$\begin{aligned} \iota(\hat{\zeta})\delta\psi &= i^*D\psi\tau_Q\Phi \\ &= -\eta i^*\tau_Q D\Phi \\ &= -\eta i^*D^2\Phi \\ &= \eta i^*\partial_t\Phi \\ &= \eta\dot{x} \end{aligned} \tag{4.33}$$

The lagrangian density in superspacetime is

$$\mathcal{L} = |dt|\,d\theta\,\Big\{-\frac{1}{2}\langle D\Phi, \partial_t\Phi\rangle\Big\} = |dt|\,d\theta\;\ell. \tag{4.34}$$

Here $|dt|\,d\theta$ is a bi-invariant density on $M^{1|1}$—it is invariant under ∂_t, D, τ_Q.

Before integrating we pause to point out that we have now made the supersymmetry manifest. Namely, the lagrangian $\mathcal{L}$ is invariant under the vector fields $\{\zeta, \xi\}$ on $\mathcal{F}\times M^{1|1}$ induced by $\{\eta\tau_Q, \partial_t\}$. Explicitly, we have

$$\begin{aligned} \iota(\zeta)dt &= \eta\theta & \iota(\xi)dt &= 1 \\ \iota(\zeta)d\theta &= \eta & \iota(\xi)d\theta &= 0 \\ \iota(\zeta)\delta\Phi &= -\eta\tau_Q\Phi & \iota(\xi)\delta\Phi &= -\partial_t\Phi. \end{aligned} \tag{4.35}$$

The invariance of $\mathcal{L}$ follows *a priori* from the fact that τ_Q, ∂_t commute with D, ∂_t and the fact that the density $|dt|\, d\theta$ is invariant.

Now to the integration. We define the *component lagrangian* L from the superspacetime lagrangian $\mathcal{L}$ by "integrating out" the odd variable θ. This is the *Berezin integral* which in this case amounts to

$$L = (i^* D\ell)\, dt. \tag{4.36}$$

(This is a definite finesse: We simply introduce this definition without more explanation!) So we compute

$$\begin{aligned} i^* D\ell &= -\frac{1}{2} i^* D\langle D\Phi, \partial_t \Phi\rangle \\ &= -\frac{1}{2} i^* \Big\{ \langle \nabla_D D\Phi, \partial_t\Phi\rangle - \langle D\Phi, \nabla_D \partial_t \Phi\rangle \Big\} \\ &= -\frac{1}{2} i^* \Big\{ -|\partial_t \Phi|^2 - \langle D\Phi, \nabla_{\partial_t} D\Phi\rangle \Big\} \\ &= \frac{1}{2}|\dot{x}|^2 + \frac{1}{2}\langle \psi, \nabla_{\dot{x}} \psi\rangle. \end{aligned} \tag{4.37}$$

Hence we recover the superparticle component lagrangian (4.12).

The reader would do well to compute that other features of the superspacetime model match the component formulation. For example, compute the action of τ_Q on the component fields using the definition (4.31) of the component fields to recover (4.16). This computation is a bit tricky, but can be viewed as a problem in ordinary differential geometry—the odd variables cause no additional difficulties. Also, you can do the classical mechanics directly on $M^{1|1}$: Compute γ, ω, the equations of motion, the supercharge, etc. This is a nontrivial exercise in calculus on supermanifolds; the component formulas in the text may be used as a check.

Exercises

1. The idea here is to analyze the lagrangian (4.12) for the supersymmetric particle. You can do it from several points of view.

   (a) First, start out working in local coordinates. That is, choose local coordinates $x^1, x^2, \dots, x^d$ on X, and write the Riemannian metric in these coordinates as $g_{ij}dx^i dx^j$. The ordinary particle lagrangian is

   $$L_0 = \Big\{ \frac{1}{2} g_{ij}\dot{x}^i \dot{x}^j \Big\}\, |dt|.$$

   Write the supersymmetric particle lagrangian L (4.12) in this notation. Note that the Christoffel symbols depend on the map x.

   (b) Now compute δL. You can explicitly take a 1-parameter family of fields x_u, ψ_u and differentiate with respect to u. Deduce the classical equations (4.14).

   (c) Do the same computation without introducing coordinates. Be careful to use δ_∇ when covariant derivatives are needed and be mindful of (1.13).

   (d) In your computations the variational 1-form γ (4.13) should be staring at you. Compute the symplectic form ω.

2. (a) Do the computations of the same quantities in superspacetime, as suggested at the end of the lecture.
   (b) Recover the formulas in components from these computations.

3. Check that $\hat{\zeta}$, as defined in (4.16), is a nonmanifest symmetry. In other words, compute $\mathrm{Lie}(\hat{\zeta})L$.

4. (a) Verify (4.24).
   (b) Verify (4.21).

5. Show that there is no way to add a potential term $-x^*V$ to the lagrangian L (4.3) in such a way that supersymmetry is maintained. You are allowed to add terms to (4.16)—change the supersymmetry transformation laws of the fields—but even allowing this it is not possible.

LECTURE 5
Free Theories, Quantization, and Approximation

Quantization of free theories: general theory

Recall that a Hamiltonian system consists of three ingredients: states, observables, and a one-parameter family of motions. In general the observables have a Lie-type bracket on them. In Lecture 1 we saw that for a classical Hamiltonian system the state space is a symplectic manifold $(\mathcal{M}, \Omega)$, the space of observables is $\Omega^0(\mathcal{M})$ with its Poisson bracket, and the motion is by a one-parameter family of symplectic diffeomorphisms generated by a Hamiltonian function $H\colon \mathcal{M} \to \mathbb{R}$. We saw in Lecture 4 that to accommodate fermions, we should allow $\mathcal{M}$ to be a supermanifold with an appropriate symplectic structure; observables may then be even or odd, but the Hamiltonian is necessarily even. Also, we saw in Lecture 3 that for field theories on Minkowski spacetime the one-parameter family of time-translations is embedded in an action of the Poincaré group on $\mathcal{M}$. In Lecture 4 we saw the first glimpses of an extension of the Poincaré group which acts in supersymmetric systems.

In a quantum mechanical system from this Hamiltonian point of view, the state space is a complex (separable) Hilbert space $\mathcal{H}$, the observables are self-adjoint operators on $\mathcal{H}$ with bracket the usual commutator of operators, and there is a Hamiltonian operator $\hat{H}$ which generates a one-parameter group of unitary transformations which represent time translation. The extension which accommodates fermions is algebraic: the state space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ is a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space and there is a $\mathbb{Z}/2\mathbb{Z}$-grading on the operators—even operators preserve the grading and odd operators exchange $\mathcal{H}^0$ and $\mathcal{H}^1$. The unitary time-translations are required to be even. In a *relativistic* quantum mechanical system the unitary representation of time-translations is extended to a unitary representation of the Poincaré group. (We review the structure of the physically relevant representations later in this lecture.) In supersymmetric systems there is a larger *super Poincaré group* which acts.

The last paragraph is not quite right: The state space is the complex projective space $\mathbb{P}\mathcal{H}$ formed from a Hilbert space $\mathcal{H}$. For the free theories discussed below, we will explicitly see that appropriate groups of symmetries are only represented projectively.

In general there is no canonical way to pass back and forth between a classical system and a quantum system. Rather, there are usually parameters in a theory—whether or not it be classical or quantum—and only for certain limits of the parameters is there a reasonable correspondence. The basis for this is a *precise* correspondence for *free* theories. Recall that a classical system $(\mathcal{M}, \Omega, H)$ is free if $(\mathcal{M}, \Omega)$ is a symplectic affine space and H is quadratic (so generates a one-parameter family of symplectic affine transformations.)[20] We allow a "super" version of this: the affine space is a supermanifold whose underlying vector space of translations is $\mathbb{Z}/2\mathbb{Z}$-graded. For free classical systems there is a canonically associated quantum system. This has been studied in great detail mathematically, both in case $\mathcal{M}$ is finite dimensional and $\mathcal{M}$ is infinite dimensional. (Quantization in the latter case requires an additional choice.) We give a brief overview here.

Consider first the bosonic case where $(\mathcal{M}^0, \Omega^0)$ is an "ordinary" (even) symplectic affine space. The subspace of affine observables is closed under Poisson bracket; it forms a Heisenberg Lie algebra, a nontrivial central extension of the commutative algebra of translations. As well, the subspace of quadratic observables is closed under Poisson bracket; it is a central extension of the Lie algebra of affine symplectic transformations. Now what we might hope for in (free) quantization is a map

$$(\text{classical observables}), \{\cdot,\cdot\} \longrightarrow (\text{quantum observables}, [\cdot,\cdot]) \tag{5.1}$$

which is a homomorphisms of Lie algebras. That turns out to be impossible (unless the quantization is trivial). In fact, we only demand (5.1) be a homomorphism on affine functions; it follows that it is a homomorphism on quadratic functions as well. In other words, we would like a representation of the Heisenberg algebra by self-adjoint operators, or, by exponentiation, a unitary representation of the Heisenberg group. Physicists call this a representation of the *canonical commutation relations*. It is a basic theorem that a unitary *irreducible* representation is unique, up to isomorphism.[21] In fact, the infinitesimal representation extends to a representation of quadratic observables, and on the group level we obtain a representation of a cover of the affine symplectic group. This describes the quantization in general terms.

We now give an algebraic description. It depends from the beginning on a choice: We choose both an origin for the affine space $\mathcal{M}^0$ and a *polarization* of the vector space U^0 of translations of $\mathcal{M}^0$, i.e., a decomposition

$$U^0 \cong L \oplus L' \tag{5.2}$$

of the symplectic vector space U^0 as a sum of complementary *lagrangian* subspaces. It is important that we allow a complex polarization, that is, $L, L' \subset U^0 \otimes \mathbb{C}$. Using the choice of origin we identify $\mathcal{M}^0 \cong U^0$, and then polynomial functions on $\mathcal{M}^0$ as elements of $\operatorname{Sym}^\bullet\bigl((U^0)^*\bigr)$. With these choices we take

$$\mathcal{H} = \overline{\operatorname{Sym}^\bullet(L^*) \otimes \mathbb{C}}, \tag{5.3}$$

the Hilbert space completion of the polynomial functions on one of the lagrangian subspaces. The representation on linear observables—that is, elements of $(U^0)^*$—into operators on $\operatorname{Sym}^\bullet(L^*)$ is defined by

$$\begin{aligned} \ell^* &\longmapsto \text{multiplication by } \ell^* \\ \ell'^* &\longmapsto \text{contraction with } \ell'^*, \end{aligned} \tag{5.4}$$

[20]This definition—which I allow may not be standard—has the strange consequence that a harmonic oscillator is a free system.

[21]In the infinite dimensional case we need to specify a class of polarizations to fix the representation. We will do so in our examples by requiring that energy be nonnegative.

where $\ell^* \in L^*$, $\ell'^* \in L'^*$, and the contraction uses the nondegenerate pairing of L^* and L'^* induced by the symplectic form. In the lingo of physics, ℓ^* acts as a *creation operator* and ℓ'^* acts as an *annihilation operator*. The *vacuum* is the element $1 \in \mathrm{Sym}^0(L^*)$. In field theory the 1-particle Hilbert space is $\mathrm{Sym}^1(\mathcal{H})$, the 2-particle Hilbert space $\mathrm{Sym}^2(\mathcal{H})$, etc. One can easily check that the operator bracket matches the Poisson bracket of linear functions, so we have a representation of the Heisenberg algebra. This is a purely algebraic description of the representation on a dense subspace of Hilbert space.

The odd case is exactly parallel if we work in the language of supermanifolds. So suppose (U^1, Ω^1) is an odd symplectic vector space; that is, an odd vector space equipped with a nondegenerate symmetric bilinear form, which we assume is positive definite. In short, we have a Euclidean space, viewed as being odd. Recall that the algebra of functions on U^1 is $\mathrm{Sym}\big((U^1)^*\big)$; i.e., it is the exterior algebra on the dual to the U^1 viewed as an ordinary vector space. Again it is a Poisson algebra, and the affine (linear + constant) and functions of degree ≤ 2 are each closed under Poisson brackets. The affine functions form a nontrivial central extensions of the odd vector space of linear functions with trivial bracket, by analogy with the Heisenberg algebra in the even case. The purely quadratic functions are even and under Poisson bracket form a Lie algebra isomorphic to the algebra of skew-symmetric endomorphisms of the ordinary vector space underlying U^1. As before, we ask to represent the affine functions by a homomorphism into self-adjoint operators on some $\mathbb{Z}/2\mathbb{Z}$*-graded* Hilbert space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$; it follows that the representation extends to a homomorphism of quadratic functions as well.

It is useful to describe this situation as follows. If f and g are linear functions, and $O(f), O(g)$ the corresponding odd linear operators, then[22]

$$[O(f), O(g)] = i\langle f, g\rangle, \tag{5.6}$$

where $\langle\cdot,\cdot\rangle$ is the symplectic form, and $i = \sqrt{-1}$. The factor of i is there since $O(f)$ is self-adjoint, in the appropriate graded sense.[23] Better to take skew-adjoint operators $c(f) = i^{1/2}O(f)$, in which case we obtain the usual Clifford algebra relation

$$[c(f), c(g)] = -\langle f, g\rangle. \tag{5.7}$$

Then $\mathcal{H}$ is a graded module for the Clifford algebra generated by the linear functions. Note that since $\mathcal{H}$ is complex, we may as well take the complex Clifford algebra. Again: The associative algebra generated by the affine functions with the relation that the commutator be the Poisson bracket is a Clifford algebra, and the quantum Hilbert space is an irreducible graded Clifford module for this Clifford algebra.

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[22] I follow the sign rule, so that the commutator of two homogeneous elements a, b in a $\mathbb{Z}/2\mathbb{Z}$-graded algebra is

$$[a, b] = ab - (-1)^{|a|\,|b|}ba. \tag{5.5}$$

[23] The usual equation $\langle Tv, v'\rangle = \langle v, T^*v'\rangle$ which defines the adjoint picks up a sign if both T and v are odd.

If $\dim U^1$ is even (or infinite), we can describe this Clifford module, the quantum Hilbert space, in terms of polarizations as before. But here we necessarily use complex polarizations, since there are no real isotropic subspaces. Thus we write

$$U^1 \otimes \mathbb{C} \cong L_{\mathbb{C}} \oplus \overline{L}_{\mathbb{C}} \tag{5.8}$$

for a totally isotropic L. We take

$$\mathcal{H} = \overline{\mathrm{Sym}^{\bullet}(L_{\mathbb{C}}^*)} \tag{5.9}$$

as in the even case (5.3), but since $L_{\mathbb{C}}$ is odd this is the exterior algebra on the ordinary vector space underlying $L_{\mathbb{C}}^*$. If U^1 is finite dimensional, for example, then $\mathcal{H}$ is finite dimensional and there is no Hilbert space completion necessary. In any case it is a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space. The 1-particle subspace $\mathrm{Sym}^1(L_{\mathbb{C}}^*)$ is odd, the 2-particle subspace $\mathrm{Sym}^2(L_{\mathbb{C}}^*)$ is even, etc. The formulas for the action of linear operators are exactly the same as in (5.4). In the finite dimensional case, this is a standard construction of the Clifford module in even dimensions.

As stated above, the representation of the Poisson algebra of linear and constant functions extends to a representation of the Poisson algebra of purely quadratic functions, which recall is isomorphic to the Lie algebra of the orthogonal group. Exponentiating we obtain a representation of the double cover of the orthogonal group, called the Pin group, which should be viewed as $\mathbb{Z}/2\mathbb{Z}$-graded. The identity component is the Spin group, which operates by even unitary transformations. So the Hilbert space in this case is the Hilbert space underlying the spin representation.

Quantization of free theories: free fields

We now apply these general remarks to the case of a free scalar field, as discussed in Lecture 3. We work on Minkowski spacetime M^n, the field is a real-valued function $\phi\colon M^n \to \mathbb{R}$, and the classical field equation (3.6) is the linear wave equation (3.7):

$$(\partial_0^2 - \partial_1^2 - \cdots - \partial_{n-1}^2 + m^2)\phi = 0, \tag{5.10}$$

where $m \geq 0$ is the mass of the field. (We work in units where $\hbar = c = 1$.) The space of solutions to this wave equation is an infinite dimensional symplectic vector space $\mathcal{M}$; the symplectic form is the integral of (3.5) over a spacelike hypersurface in M^n. We analyze (5.10) using the Fourier transform. Fix an origin in M^n, so identify ϕ as a function $\phi\colon V \to \mathbb{R}$. Its Fourier transform $\hat{\phi}\colon V^* \to \mathbb{R}$ is the coefficient function in an expansion of ϕ as a linear combination of plane waves. Namely,

$$\phi(x) = \frac{1}{(2\pi)^{n/2}} \int_{V^*} \hat{\phi}(\alpha) e^{i\langle x,\alpha\rangle}\, |d^n\alpha|, \tag{5.11}$$

where $\langle\cdot,\cdot\rangle$ is the pairing between V and V^* and $|d^n\alpha|$ is the density associated to the inverse Lorentz metric on V^*. The Fourier transform $\hat{\phi}$ is a *complex*-valued function, but because ϕ is real $\hat{\phi}$ satisfies the reality condition

$$\hat{\phi}(-\alpha) = \overline{\hat{\phi}(\alpha)}. \tag{5.12}$$

Under Fourier transform derivatives become multiplication operators, so the second-order differential operator (5.10) becomes the quadratic equation

$$(|\alpha|^2 - m^2)\hat{\phi}(\alpha) = 0. \tag{5.13}$$

In other words, the support of the Fourier transform of a solution ϕ lies on the *mass shell*

$$\mathcal{O}_m = \{\alpha \in V^* : |\alpha|^2 = m^2\} \subset V^* \tag{5.14}$$

In case $m > 0$ this is a hyperbola; if $m = 0$ it is the dual lightcone. We have not specified what class of functions ϕ and $\hat{\phi}$ lie in, but at least the Fourier transform must exist. Since equation (5.10) is Poincaré-invariant, the vector space of solutions $\mathcal{M}$ is a *real* representation of the Poincaré group P^n.

According to the general discussion above, the quantization is determined by a polarization of the symplectic vector space $\mathcal{M}$. Here we use a complex polarization, a decomposition of $\mathcal{M} \otimes \mathbb{C}$ into a sum of lagrangians. Now $\mathcal{M} \otimes \mathbb{C}$ is the space of complex-valued functions on $\mathcal{O}_m$ with no reality condition. For $m > 0$ there is a decomposition

$$\mathcal{O}_m = \mathcal{O}_m^+ \cup \mathcal{O}_m^- \qquad \text{(disjoint)}, \tag{5.15}$$

where $\mathcal{O}_m^+ = \mathcal{O}_m \cap \{\alpha_0 > 0\}$ is the subset of covectors of *positive energy* and $\mathcal{O}_m^- = \mathcal{O}_m \cap \{\alpha_0 < 0\}$ the subset of covectors of negative energy. (We use linear coordinates $x^0, x^1, \dots, x^{n-1}$ on V and dual linear coordinates $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ on V^*. In appropriate units the first coordinate α_0 is energy and the remaining coordinates α_i are momenta. The norm square of α is the mass square, which is energy square minus momentum square.) The subspace L_m of $\mathcal{M}_{\mathbb{C}}$ consisting of $\hat{\phi}$ supported on $\mathcal{O}_m^+$ is lagrangian, as is the subspace $\overline{L_m}$ of $\mathcal{M}_{\mathbb{C}}$ consisting of $\hat{\phi}$ supported on $\mathcal{O}_m^-$. Thus we take the Hilbert space of the massive particle to be

$$\mathcal{H} = \overline{\mathrm{Sym}^\bullet(L_m^*)}. \tag{5.16}$$

The subspace $\mathrm{Sym}^\bullet(L_m^*)$ is called the *Fock space*; the Hilbert space is a completion. The 1-particle Hilbert space $\mathrm{Sym}^1(L_m^*)$ is an irreducible (complex) unitary representation of the Poincaré group. In the massless case the decomposition (5.15) has an additional piece:

$$\mathcal{O}_0 = \mathcal{O}_0^+ \cup \mathcal{O}_0^- \cup \{0\} \qquad \text{(disjoint)}. \tag{5.15}$$

We would like to take the lagrangian decomposition as before, so that L_0 consists of Fourier transforms supported on $\mathcal{O}_0^+$ and $\overline{L_0}$ of Fourier transforms supported on $\mathcal{O}_0^-$, but we have the bothersome 0 to worry about. In fact, it is not a problem in dimensions $n \geq 3$, but does manifest itself in two-dimensional field theory (and in quantum mechanics).

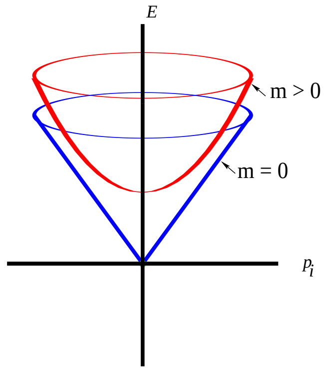
There is a similar story for other free fields. In general the Fourier transform of a solution to the linear classical equation of motion has support on a mass hyperbola or lightcone, but rather than simply being a complex-valued function it is a section of a complex vector bundle (which satisfies a reality condition). There is again a complex polarization, determined by the positive energy condition, and a similar picture of the quantum Hilbert space.

Representations of the Poincaré group

A relativistic quantum particle is defined to be an irreducible unitary positive energy representation $\mathcal{H}$ of the Poincaré group P^n. (Recall the definition (1.45) at the end of Lecture 1.) Such representations were classified by Wigner long ago, and we quickly review the construction. First, we restrict the representation to the translation subgroup V. Since V is abelian, the representation decomposes as a direct sum (really direct integral) of one-dimensional representations on which V acts by a character

$$v \longmapsto \text{multiplication by } e^{i\alpha(v)/\hbar}, \qquad \alpha \in V^*.$$

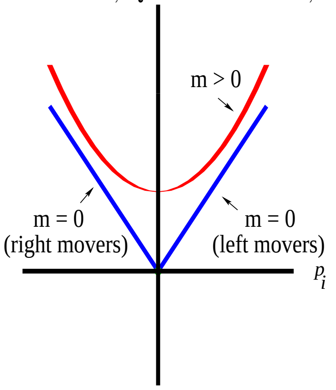
The set of infinitesimal characters α which occur are permuted by the action of the identity component of $O(V)$ on V^*. Since the representation $\mathcal{H}$ is irreducible, these infinitesimal characters form an orbit of the action. There are two types of orbits which have positive energy. These are indicated in the figure below. Notice that the axes are labeled E for energy and p_i for momentum. The mass square $m^2 = E^2/c^4 - \sum p_i^2/c^2$ is constant on an orbit and so is an invariant of an irreducible representation. The two orbit types correspond to *massless* ($m = 0$) and *massive* ($m > 0$) representations.



In the two-dimensional case ($n = 2$) the massless orbit breaks up into two distinct orbits along the two rays of the positive lightcone, as indicated in the next figure. We call them *right movers* and *left movers* since the corresponding characters are functions of $ct - x$ and $ct + x$ respectively. So there is a more refined classification of massless particles in two dimensions.

Now the representation $\mathcal{H}$ of P^n is obtained by constructing a homogeneous complex hermitian vector bundle over the orbit. More precisely, the total space of the bundle carries an action of $\mathrm{Spin}(V)$ covering the action (of its quotient by $\{\pm 1\}$) on the orbit. Such bundles may be constructed by fixing a point on the orbit and constructing a finite dimensional unitary representation of the stabilizer subgroup of that point, whose reductive part is called the *little group*. (Any finite dimensional representation factors through the reductive part.)

Consider first the massive case and fix the mass to be m. For convenience we set $c = 1$ and take as basepoint $(m, 0, \dots, 0)$. Then the stabilizer subgroup, or



little group, is easily seen to be isomorphic to $\mathrm{Spin}(n-1)$. Thus a massive particle corresponds to a representation of $\mathrm{Spin}(n-1)$.

In the massless case we consider the basepoint $(1, 1, 0, \ldots, 0)$. The stabilizer subgroup in this case is a double cover of the Euclidean group of orientation-preserving isometries of an $(n-2)$-dimensional Euclidean space. We can see this as follows. The group $O(V)$ is the group of conformal transformations of the $(n-2)$-dimensional sphere. We can view the sphere as the set of rays in the forward lightcone. An element in the identity component of $O(V)$ acts on the forward lightcone, and the subgroup H of transformations which fix a ray R is isomorphic to the identity component of the Euclidean group of the complement plus dilations. The action on points of R is given by the dilation factor at the corresponding point of the sphere, whence the claimed stabilizer subgroup. Massless particles correspond to finite dimensional unitary representations of the corresponding subgroup $\tilde{H}$ of $\mathrm{Spin}(V)$. Such representations are necessarily trivial on the lift to $\tilde{H}$ of the subgroup of H consisting of translations. In other words, they factor through representations of a group isomorphic to $\mathrm{Spin}(n-2)$.

Representations of the little group are classified by their *spin*, (which is called *helicity* in the massless case). What do we mean by the "spin" of a representation of $\mathrm{Spin}(m)$? Suppose W is such a representation. Fix a 2-plane in $\mathbb{R}^m$ and consider the subgroup $SO(2) \subset SO(m)$ of rotations in that plane which fix the perpendicular plane. We work with the double cover $\mathrm{Spin}(2) \subset \mathrm{Spin}(m)$. Restricted to $\mathrm{Spin}(2)$ the representation W decomposes as a sum of one-dimensional complex representations, on each of which $\mathrm{Spin}(2)$ acts by $\lambda \mapsto \lambda^{2j}$, where $\lambda \in \mathrm{Spin}(2)$ and j is a half-integer. (We use half-integers so that the two-dimensional tautological representation of the $SO(2)$ subgroup of $SO(m)$ is the sum of the representations $j = 1$ and $j = -1$.) The spin of the representation W is the largest $|j|$ which occurs in the decomposition. For example, the trivial representation has spin 0. It represents a *scalar particle*. The m-dimensional defining representation of $SO(m)$ has spin 1; the corresponding particle is sometimes called the *vector particle*. The reader can check that all exterior powers of this representation (except Λ^m) also have spin 1. The spin representations of $\mathrm{Spin}(m)$ have spin 1/2. One obtains higher spin by looking at the symmetric powers of the defining representation.

There are physical reasons why in interacting local quantum field theories one only sees massless particles of low spin. More precisely, in theories without gravity only massless particles of spin 0, spin 1/2, and spin 1 occur. Massless spin 1 particles only occur in *gauge theories*; a theory whose only massless particles have spin 0 and spin 1/2 is a *σ-model*. The *graviton*—the particle which mediates the gravitational force—is a massless particle of spin 2 and in theories of supergravity there are also massless particles of spin 3/2. That's it! There are no massless particles of higher spin in realistic theories.

Given a homogeneous vector bundle, we take $\mathcal{H}$ to be the space of L^2 sections of the bundle over the orbit. (There is an invariant measure on the orbit.)

- Recall that the Poincaré group P^n projects onto the identity component of $O(V)$, which consists of transformations which preserve both orientation and the splitting of the lightcone into forwards and backwards. In a local quantum field theory the *CPT theorem* states that the representation of P^n extends to a (projective) representation of the larger group which allows for orientation reversal (but still preserves the splitting of the lightcone). Elements in the new component are represented by antilinear maps. The condition that the representation extend can be stated in terms of the little group; the precise statement depends on the parity of n. For n even it says that the representation of the little group is self-conjugate, i.e., either real or quaternionic. The statement for n odd is more complicated and we omit it.
- To incorporate fermions we consider representations of the little group on $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces. The even part corresponds to bosons, the odd to fermions.
- In unitary local quantum field theories there is a connection between the spin of a particle and its *statistics*—whether it is a boson or a fermion. A particle of integral spin is a boson and a particle of half-integral spin is a fermion. This spin-statistics connection is often violated in nonunitary theories, and in particular in the topological field theories which have mathematical applications.

Free fermionic fields

Given a particle representation of P^n we can ask for a free field theory whose 1-particle Hilbert space is the given representation. We saw earlier that a real scalar field gives a spin 0 particle. The spin 1 particle is the 1-particle Hilbert space associated to the lagrangian (3.24) on a quotient (3.25) of 1-forms. (This is a nice exercise.) Thus it remains for us to construct the spin 1/2 representation of Poincaré as the quantization of a free field. The spin-statistics theorem implies that it should be a fermionic field. We will not carry out the quantization in these notes—that is left for the exercises (or references)—but we will describe the theory.

Let S be any *real* spin representation of $\mathrm{Spin}(V)$. We build a theory whose space of fields is

$$\mathcal{F} = \mathrm{Map}(M^n, \Pi S), \tag{5.17}$$

a space of spinor fields on M^n. The fermionic field is odd—it is a map into the odd vector space ΠS—and so the space $\mathcal{F}$ is an infinite dimensional odd vector space. One remarkable fact about the spin group in Lorentz signature, not true in other signatures, is that the symmetric square of any real spinor representation contains a copy of the vector representation, except for $n = 2$. In fact, there always exists a symmetric $\mathrm{Spin}(V)$-equivariant pairing

$$\tilde{\Gamma}\colon S \otimes S \longrightarrow V. \tag{5.18}$$

We will not prove these facts about the spin group, but rather illustrate them in a few cases.

For $n = 2$ the identity component $SO^+(V)$ of $O(V)$ is isomorphic to the multiplicative group $\mathbb{R}^{>0}$, so $\mathrm{Spin}(V) \cong \mathbb{R}^{>0} \times \mathbb{Z}/2\mathbb{Z}$. In any spinor representation the nontrivial element in $\mathbb{Z}/2\mathbb{Z}$ acts by -1. There are two inequivalent irreducible real spinor representations $S^{\pm}$ on which $\lambda \in \mathbb{R}^{>0}$ acts as $\lambda^{\pm 1}$. The vector representation $V \cong (S^+)^{\otimes 2} \oplus (S^-)^{\otimes 2}$.

For $n = 3$ we have $\mathrm{Spin}(V) \cong SL(2;\mathbb{R})$. There is a unique irreducible spinor representation S of dimension 2, the standard representation of $SL(2;R)$, and $V = \mathrm{Sym}^2(S)$.

For $n = 4$ we have $\mathrm{Spin}(V) \cong SL(2;\mathbb{C})$. There is again a unique real spinor representation S, the real 4-dimensional representation underlying the standard 2-dimensional representation S' of $SL(2;\mathbb{C})$. Let S'' be the conjugate to S'; then $V \otimes \mathbb{C} \cong S' \otimes S''$.

The exceptional isomorphisms for orthogonal groups continue up to dimension $n = 6$; in that case $\mathrm{Spin}(V) \cong SL(2;\mathbb{H})$. It is significant that dimensions 3,4,6 have Lorentz spin groups isomorphic to $SL(2;\mathbb{F})$ over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. There is even a sense in which the Lorentz spin group in $n = 10$ dimensions is $SL(2)$ over the octonions!

In any dimension either there is a unique irreducible real spinor representation S and any real spinor representation has the form $S^{\oplus N}$ for some N, or there are two distinct irreducible real spinor representations $S, \tilde{S}$ and any spinor representation has the form $S^{\oplus N} \oplus \tilde{S}^{\oplus \tilde{N}}$ for some $N, \tilde{N}$. The latter occurs in dimensions $n \equiv 2, 6 \pmod 8$.

In general, given S and the pairing (5.18) we write a kinetic lagrangian

$$L_{\text{kinetic}} = \left\{ \frac{1}{2}\tilde{\Gamma}(\psi, \partial\psi) \right\} |d^n x|. \tag{5.19}$$

Choose a basis $\{e_\mu\}$ for V and a basis $\{f^a\}$ of S. Then we can expand a spinor field ψ as

$$\psi(x) = \psi_a(x) f^a. \tag{5.20}$$

Write

$$\tilde{\Gamma}(f^a, f^b) = \tilde{\Gamma}^{\mu ab} e_\mu. \tag{5.21}$$

Then the kinetic term may be written

$$\tilde{\Gamma}(\psi, \partial\psi) = \tilde{\Gamma}^{\mu ab}\psi_a \partial_\mu \psi_b = \psi \not{D}\psi, \tag{5.22}$$

the last expression being the most common. Note that it has the same general form as the fermionic term we used in (4.3); it is a product of the fermionic field with its first derivative.

For a given spinor representation S there may or may not be a mass term possible. A mass term is specified by a quadratic function $\Pi S \to \mathbb{R}$, that is, by a skew-symmetric pairing

$$M\colon {\textstyle\bigwedge}^2 S \longrightarrow \mathbb{R}. \tag{5.23}$$

A nonzero pairing may or may not exist. Then the full lagrangian is

$$L = \Big\{ \frac{1}{2}\,\psi \not{D}\psi - \frac{1}{2}\,\psi M\psi \Big\}\,|d^nx|. \tag{5.24}$$

The 1-particle Hilbert space of this free fermionic field is a spin $1/2$ representation of P^n which is the space of sections of a vector bundle of rank $\dim S/2$ over the appropriate mass shell. There is an exception to this last description in $n=2$, where it is possible to have the Fourier transform of a massless spinor field supported on half of the lightcone; then it is a section of a vector bundle of rank $\dim S$.

The general free theory

To specify a free theory of scalar, spinor, and 1-form fields we need to give the precise field content and the masses of the fields. Since we have not developed the theory of Lorentz spinors in detail, we will be somewhat schematic about the spin $1/2$ fields.

Fix a dimension n. Let $S_0, \tilde S_0$ be the irreducible real spinor representations, with fixed pairings (5.18); if there is only one real representation, then set $\tilde S = 0$. The kinetic data for the free theory is a set of even real[24] vector spaces each equipped with positive definite inner product:

$$W_0, W_{1/2}, \tilde W_{1/2}, W_1 \qquad \text{positive definite inner product spaces} \tag{5.25}$$

The spinor fields take values in the odd vector space

$$\mathcal{W}^{\mathrm{odd}}_{1/2} = \Pi S_0 \otimes W_{1/2} \;\oplus\; \Pi\tilde S_0 \otimes \tilde W_{1/2}. \tag{5.26}$$

In a free theory we can have quadratic potential functions, which are mass terms and are specified by *mass matrices*. These are nonnegative quadratic forms

$$\begin{aligned} M_0 &\in \operatorname{Sym}^2(W_0^*)\\ M_{1/2} &\in \operatorname{Sym}^2\big((\mathcal{W}^{\mathrm{odd}}_{1/2})^*\big)\\ M_1 &\in \operatorname{Sym}^2(W_1^*). \end{aligned} \tag{5.27}$$

The fields in the theory all live in linear spaces, as expected for a free theory. A field is a triple $\Phi = (\phi,\psi,\alpha)$ where

$$\begin{aligned} &\phi\colon M^n \longrightarrow W_0\\ &\psi\colon M^n \longrightarrow \mathcal{W}^{\mathrm{odd}}_{1/2}\\ &\alpha \in \Omega^1(M^n;W_1)/d\Omega^0(M^n;W_1) \end{aligned} \tag{5.28}$$

[24]In some dimensions we take $S_0, \tilde S_0$ and $W_{1/2}, \tilde W_{1/2}$ to be complex conjugate vector spaces.

Recall the gauge equivalence we have on 1-forms, which is the reason to divide by exact 1-forms. The free field lagrangian is quadratic. The fields are completely decoupled. For each field there is a kinetic term and a mass term:

$$L = \Big\{ \frac{1}{2}|d\phi|^2 + \frac{1}{2}\langle\psi, \not{D}\psi\rangle - \frac{1}{2}|\alpha|^2 - \frac{1}{2}M_0(\phi) - \frac{1}{2}M_{1/2}(\psi) - \frac{1}{2}M_1(\alpha) \Big\} \, |d^n x|. \tag{5.29}$$

There is an associated quantum field theory, which may be specified by its 1-particle ($\mathbb{Z}/2\mathbb{Z}$-graded) Hilbert space. The even part is a sum of spin 0 and spin 1 representations; the odd part a sum of spin 1/2 representations. To describe it we need to compute the (nonnegative real number) masses which occur and describe a vector space of particles with that mass. For the bosons ϕ, α this is straightforward: using the inner products on W_0, W_1 we may express the mass forms M_0, M_1 as nonnegative symmetric matrices and then decompose W_0, W_1 according to their eigenvalues and eigenspaces. The eigenvalues are the masses and the eigenspaces encode the multiplicity of particles with a particular mass. For the spinor field the computation of masses involves a bit more algebra of spinors.

General theory

The theories of interest are not free, of course, and there is a wide variety of physically interesting lagrangians which satisfy the basic criteria outlined at the beginning of Lecture 3. We extract a general class of lagrangians which covers many examples, just as we described a general class of bosonic theories (gauged σ-models) at the end of Lecture 3. These are theories with fields of spins 0, 1/2, and 1.

The data we need to write the fields and kinetic terms is:

(5.30)

G	Lie group with Lie algebra $\mathfrak{g}$
$\langle\cdot,\cdot\rangle$	bi-invariant scalar product on $\mathfrak{g}$
X	Riemannian manifold on which G acts by isometries
$W, \tilde{W} \longrightarrow X$	real vector bundles with metrics, connections, and orthogonal G action

Consider $\Pi S_0, \Pi\tilde{S}_0$ as constant vector bundles over X, and define

$$\mathcal{W}^{\mathrm{odd}} = \Pi S_0 \otimes W \;\oplus\; \Pi\tilde{S}_0 \otimes \tilde{W} \longrightarrow X. \tag{5.31}$$

A field is then a triple $\Phi = (\phi, \psi, A)$, where

(5.32)

A	connection on a principal G-bundle $P \to M$
ϕ	G-equivariant map $P \to X$
ψ	G-equivariant lift of ϕ to $\mathcal{W}^{\mathrm{odd}}$

The collection of fields is a category $\mathcal{F}$, in fact, a groupoid—all the morphisms in the category are equivalences (invertible). If we divide by these equivalences (think of them as gauge transformations), we obtain the space $\overline{\mathcal{F}}$ of equivalence classes of fields. The kinetic lagrangian is:

$$L_{\text{kinetic}} = \left\{ \frac{1}{2}|d_A\phi|^2 + \frac{1}{2}\langle \psi, \not{D}_{A,\phi}\psi\rangle - \frac{1}{2}|F_A|^2 \right\} |d^nx|. \tag{5.33}$$

In this formula d_A is computed using the connection A and the Dirac operator $\not{D}_{A,\phi}$ on M uses the connection A as well as the pullback of the connections on $W, \tilde{W}$ via the map ϕ. Note that for fixed ϕ the field ψ is a spinor field on M coupled to a vector bundle over M whose connection depends on both A and ϕ.

While the kinetic terms always look this way, there is a variety of possible potential terms. Certainly the basic one is a potential for ϕ and ψ:

$$V = V^{(0)} + V^{(2)} + \cdots \qquad G\text{-invariant section of } \operatorname{Sym}^{\text{even}}\bigl((\mathcal{W}^{\text{odd}})^*\bigr) \tag{5.34}$$

Then the lagrangian of the theory is

$$L = L_{\text{kinetic}} - V(\phi, \psi)\ |d^nx|. \tag{5.35}$$

The scalar potential $V^{(0)}\colon X \to \mathbb{R}$ is the component of V whose value lies in $\operatorname{Sym}^0\bigl((\mathcal{W}^{\text{odd}})^*\bigr)$. Mass terms for spinor fields are included in $V^{(2)}$, as are *Yukawa couplings*. The "4-fermi term" $V^{(4)}$ occurs in supersymmetric σ-models (as in (7.31)) and also in supergravity theories. One can view the total potential V as an even function on the supermanifold determined by the vector bundle $\mathcal{W}^{\text{odd}} \to X$.

There are additional possible potential terms. For example, in dimension $n = 2$ the data

$$\langle\cdot\rangle \qquad \text{Ad-invariant linear form on } \mathfrak{g} \tag{5.36}$$

determines a term

$$\langle F_A\rangle \tag{5.37}$$

in the lagrangian. This term is a 2-form, not a density, so a theory with this term in it is not invariant under isometries of M^2 which reverse the orientation; we need to fix an orientation to integrate (5.37) so to define the action.

The theory we have defined is Poincaré-invariant. There is also a commuting compact Lie group of global manifest symmetries—it is the group which preserves all of the given data. It acts by isometries on X and $\mathcal{W}^{\text{odd}}$, commutes with the G action, and preserves the potentials.

A vacuum solution of the general theory has A trivial, $\psi = 0$, and ϕ constant. (Compare with the discussion of bosonic models in Lecture 3.) Note that any vacuum solution is Poincaré-invariant. So the moduli space of vacua is

$$\mathcal{M}_{\text{vac}} = V^{-1}(0)/G. \tag{5.38}$$

Perturbation theory

In a general lagrangian field theory we can consider "small fluctuations" of the fields around any fixed $\Phi_0 \in \mathcal{F}$. The space of these small fluctuations is simply the tangent space $T_{\Phi_0}\mathcal{F}$. The idea is to construct a new lagrangian field theory whose space of fields is $T_{\Phi_0}\mathcal{F}$ and whose lagrangian is an approximation to the lagrangian L of the original theory. More precisely, the approximate lagrangian is the N^{th} order Taylor series of L at Φ_0. The quadratic approximation gives a free field theory. It is typical to consider such an approximation at a vacuum solution Φ_0, rather than at an arbitrary point of field space. The free field theory approximation may be quantized, as discussed earlier in this lecture, and that is the first step in understanding the perturbative quantum theory at the particular vacuum in question. The information in the free quantum field theory—the vector spaces (5.25) and the mass matrices (5.27)—may be read off from the geometric data. This is an exercise in differential geometry; Feynman diagrams (and more subtle quantum reasoning) enter only when we keep higher order terms in the approximate lagrangian.

We work with a general theory, as described in the previous section. Fix $\Phi_0 = (A_0, \phi_0, \psi_0)$, which we assume to be a vacuum solution. So A_0 is a trivial connection, $\psi_0 = 0$, and fixing a trivialization ϕ_0 is a constant map to X. Recall that the fields $\mathcal{F}$ form a category, and we work essentially on the space $\overline{\mathcal{F}}$ of equivalence classes. The tangent space at the equivalence class $[\Phi_0]$ of Φ_0 may be described by expressed by specifying the vector spaces (5.25). To that end, note that the infinitesimal G action on X induces a linear map

$$\rho_{\phi_0}\colon \mathfrak{g} \longrightarrow T_{\phi_0}X. \tag{5.39}$$

Then the vector spaces are

$$\begin{aligned} W_0 &= \operatorname{coker}\rho_{\phi_0} \\ W_{1/2}, \tilde{W}_{1/2} &= \text{fibers of } W, \tilde{W} \text{ at } \phi_0 \\ W_1 &= \mathfrak{g} \end{aligned} \tag{5.40}$$

We leave it as an exercise for reader to write the lagrangian to second order at $\Phi_0 = (A_0, \phi_0, 0)$ and so to derive the mass matrices

$$\begin{aligned} M_0 &= \operatorname{Hess}_{\phi_0} V^{(0)} \\ M_{1/2} &= V^{(2)}(\phi_0) \\ M_1 &= \text{pullback of metric on } T_{\phi_0}X \text{ under } \textit{action} \end{aligned} \tag{5.41}$$

This completely specifies the free field approximation at the vacuum Φ_0.

There is some terminology associated to the free field approximation:

1. Let $G_{\phi_0} \subset G$ be the stabilizer group of the G action at $\phi_0 \in X$. Physicists say, "The gauge group G is broken to G_{ϕ_0}." The 1-forms with values in its Lie algebra $\mathfrak{g}_{\phi_0} \subset \mathfrak{g}$, which is the kernel of ρ_{ϕ_0}, are the the massless 1-form fields in the free field approximation. The remaining 1-form fields are massive. The fact that these fields have a mass is called the *Higgs mechanism*.

2. Let $[\phi_0]$ denote the equivalence class of ϕ_0 in $\mathcal{M}_{\text{vac}} = V^{-1}(0)$. Assume that $[\phi_0]$ is a smooth point. Then the massless scalar fields in the free field approximation are maps into $T_{[\phi_0]}\mathcal{M}_{\text{vac}}$.
3. The global symmetry group H is broken to the subgroup H_{ϕ_0} which fixes ϕ_0. The quotient of the Lie algebras $\mathfrak{h}/\mathfrak{h}_{\phi_0}$ is a subspace of $T_{[\phi_0]}\mathcal{M}_{\text{vac}}$. Massless scalar fields in this subspace are called *Goldstone bosons*.

Supersymmetric theories put restrictions on the data which defines a theory. The restrictions depend on the dimension n and on the particular supersymmetry. In various cases it constrains X to be Kähler, hyperkähler, etc. It is useful to view theories in terms of this geometric data to keep track of the zoo of examples.

Exercises

1. In this problem consider a two-dimensional symplectic affine space, which we take to have affine coordinates p, q and symplectic form $dp \wedge dq$.
   (a) Compute the Lie algebra of functions at most quadratic in p, q. What are the symplectic gradients? Do the same for affine functions.
   (b) Quantize this space: Choose a lagrangian decomposition, build the symmetric algebra, etc.
   (c) Quantize the free particle on $\mathbb{E}^1$ and the harmonic oscillator, described as a particle moving on $\mathbb{E}^1$ subject to the potential $V(x) = (k/2)\,x^2$ for some $k > 0$.

2. (a) Quantize a $2n$-dimensional symplectic affine space explicitly. Choose coordinate functions p_i, q^j, $1 \le i, j \le n$, so that the symplectic form is $dp_i \wedge dq^i$. Choose L to be the span of the $\partial/\partial q^j$, etc. You should see creation and annihilation operators explicitly.
   (b) Repeat for the odd case.

3. Verify that the subspaces in the lagrangian decomposition we used to quantize scalar fields are indeed isotropic.

4. (a) Quantize a free 1-form field. This is a bit tricky because of the gauge invariance. You need to Fourier transform solutions to the wave equation and identify them as sections of a bundle over some mass shell.
   (b) Quantize a free fermion field. The story in general requires more algebra of spinors than I have given here. So you should try some examples in 2 and 3 dimensions.

5. (a) Write the lagrangian for the general theory (5.35) as explicitly as you can. Surely you'll want to consider special cases. For example, start with no fermions. You may take X to be a vector space, and perhaps start with an abelian gauge group, or no gauge group at all. Then take the vector bundle W to be a constant vector space.
   (b) Recover all lagrangians considered in these lectures as special cases of the general theory.

6. Write the quadratic approximation to the general theory at a vacuum. Verify that you get the vector spaces and mass matrices given in (5.40) and (5.41).

7. Translate the following description of theories to the geometric data (5.30) and (5.34). What are the global symmetry groups of these models?

   (a) "A theory with gauge group $SU(2)$, a massless adjoint scalar, and a massive spinor in the vector representation."

   (b) "A nonlinear σ-model with target S^n and a circle subgroup of $SO(n+1)$ gauged."

   (c) "A theory with gauge group $SU(3) \times SU(2) \times U(1)$ with fermions in the (3,2,1) representation, fermions in the (1,2,1) representation, and scalar fields in the (1,2,1) representation."

8. Compute the moduli space of vacua and the particle content (and masses) at the vacua for each of the following theories.

   (a) Gauge group $U(1)$ and with charged complex scalar fields, i.e., $X = \mathbb{C}$ with the standard action of $U(1)$.

   (b) The same theory with potential

   $$V(\phi) = \|\phi\|^2(1 - \|\phi\|^2)^2.$$

   (c) Let $\mathbb{T}$ denote the circle group of unit norm complex numbers. Then $G = \mathbb{T} \times \mathbb{T}$ with standard inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}\sqrt{-1}\mathbb{R} \times \sqrt{-1}\mathbb{R}$. Take $X = \mathbb{C}$ with the G-action $(\lambda_1, \lambda_2) \cdot z = \lambda_1 z \lambda_2^{-1}$ and the G-invariant potential as above.

   (d) Add spinor fields to these theories. You may work in low dimensions if necessary.

LECTURE 6
Supersymmetric Field Theories

Introductory remarks and overview

We continue to work with a Poincaré-invariant quantum field theory defined on Minkowski spacetime[25] M^n, and restrict to theories of the general type outlined in the last lecture. In particular, they are theories of scalar fields, spinor fields, and gauge fields. Earlier we remarked that under certain hypotheses there is a theorem of Coleman-Mandula which asserts that the group of global symmetries of the theory has the form $P^n \times H$, where P^n is the Poincaré group and H a commuting compact Lie group. Given a theory in terms of geometric data we can read off the group H. Turning this around, we can specify H in advance and then ask for a theory with symmetry group H. For example, a free theory is determined by a set of vector spaces (5.25) and mass matrices (5.26), and imposing a symmetry group H means that the vector spaces carry a representation of H which fix the mass matrices. A similar constraint holds on the nonlinear data (5.30) for nonfree theories.

The Coleman-Mandula theorem seemed to rule out other possible symmetry groups, but in the 1970s another possibility was discovered. Namely, in each dimension n there are extensions of the Poincaré group P^n to *super Poincaré groups* $P^{n|s}$, and quantum field theories may admit these super Lie groups as symmetry groups. Furthermore, a generalization of the Coleman-Mandula theorem, due to Haag-Lopuszański–Sohnius, states that the only allowed super Lie groups are the product of a super Poincaré group $P^{n|s}$ and a compact Lie group H. In this lecture we introduce these super Lie groups.

A super Lie group is, naturally, the marriage of a supermanifold and a Lie group, or, in street lingo, "a group object in the category of supermanifolds." We do not treat supermanifolds systematically, though, and in any case will only really use the infinitesimal version. A *super Lie algebra* is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1 \tag{6.1}$$

equipped with a bracket operation

$$[\cdot,\cdot]\colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g} \tag{6.2}$$

[25]As in previous lectures, we take M^n to be an affine space with underlying vector space of translations V.

which is skew-symmetric and satisfies the Jacobi identity. The skew-symmetry and Jacobi must be written using the sign rule, and it is understood that the bracket has degree 0. Thus $[\mathfrak{g}^1, \mathfrak{g}^1] \subset \mathfrak{g}^0$ and this operation is symmetric as a map of ungraded vector spaces.

We use the notation $\mathfrak{p}^{n|s}$ for a supersymmetric extension of the Poincaré algebra $\mathfrak{p}^n = \mathrm{Lie}(P^n)$ whose odd part has dimension s. The corresponding super Poincaré group is denoted $P^{n|s}$. A theory with symmetry group $P^{n|s}$ is said to have s supersymmetries. As we see below, the odd part of $\mathfrak{p}^{n|s}$ is a real spinor representation of the Lorentz group $\mathrm{Spin}(V)$ in dimension n, so its dimension s is bounded below by a number which increases exponentially with n, roughly $2^{n/2}$.

We saw the simplest super Poincaré algebra $\mathfrak{p}^{1|1}$ in Lecture 4, e.g., equation (4.24).

Suppose now we specify a super Poincaré group $P^{n|s}$ and ask for a theory with symmetry group $P^{n|s}$. For free theories this determines a restriction on the possible vector spaces (5.25) (particle content) and masses (5.26) which can occur. Free classical theories give rise to quantum theories and in particular to the unitary representation of the Poincaré group P^n on the 1-particle Hilbert space. The restrictions on particle content and masses may be seen by requiring that this representation extend to a representation of $P^{n|s}$. It is not too difficult to catalog representations of a given $P^{n|s}$ which correspond only to free scalar fields, spinor fields, and 1-form fields. Recall that an irreducible representation of P^n may be realized on a Hilbert space of sections of a homogeneous vector bundle over a mass shell. The rank of that vector bundle is the number of *physical degrees of freedom*. An irreducible representation of $P^{n|s}$ may be realized as the space of sections of a homogeneous $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle over a mass shell. One important feature is that the number of bosonic degrees of freedom equals the number of fermionic degrees of freedom, except in special two-dimensional cases (which we will not meet in these lectures).

Possible nonfree classical field theories with super Poincaré symmetry are similarly constrained. In fact, the free approximation at any vacuum gives rise to a representation of $P^{n|s}$ as above, and so the constraints on free theories give rise to constraints on nonfree theories. Furthermore, the existence of a given free theory suggests the existence of a corresponding nonfree theory. This logic was used by Nahm in the late '70s for theories with gravity to predict the existence of certain supergravity theories, which were then constructed quite rapidly.

# SUSY	maximal dimension	σ-model	gauge theory
1	2	S	S
2	3	S	S
4	4	S (Kähler)	S
8	6	✓(hyperkähler)	S
16	10		✓

Supersymmetric theories

The chart gives an overview of supersymmetric field theories. The first column shows the number of supersymmetries, which is a power of 2. The second column shows the maximum dimension of spacetime in which that number of supersymmetries may occur. Given a theory in n spacetime dimensions with s supersymmetries, there are corresponding theories in all dimensions $n' \leq n$ by a process called *dimensional reduction*. Namely, if we have a theory on M^n, and we pick a subspace $U \subset V$ of *spatial* translations—the induced metric on U is negative definite—then given a theory with space of fields $\mathcal{F}$ and lagrangian density L, we can restrict the theory to the subspace of fields invariant under translations in U. That subspace of fields can be identified with a space of fields on a lower dimensional Minkowski spacetime.

A few remarks on this table:

- In Lecture 3 we discussed two formulations of the supersymmetric particle. First, we described the theory in *components*, that is, in terms of ordinary fields. Then we gave a super(space)time formulation in terms of *superfields*. All theories with a superspacetime formulation (marked 'S' in the table) may be described in terms of either superfields or in terms of component fields. Theories with no superspacetime formulation only have a description in terms of "component fields", though these fields are not the components of anything—they are just fields on ordinary Minkowski spacetime. The component formulation of theories which have a superspacetime formulation often includes *auxiliary fields* which enter the lagrangian algebraically.
- An 'S' in the table indicates that there is a superspacetime formulation; a '✓' indicates that the theory exists, but there is no adequate superspacetime formulation (off-shell); and a blank spot in the table means that there is no theory. The blank spot can be predicted from the representation theory of the supersymmetry group.
- In a σ-model with 4 supersymmetries the target manifold X is constrained to be Kähler; in a model with 8 supersymmetries it must be hyperkähler.
- There are special theories not obtained by dimensional reduction, and even in theories which are dimensional reductions there are sometimes terms one can add to the lagrangian which do not come from the higher dimensional theory.
- Theories with a superspacetime formulation have manifest supersymmetry in the superspacetime formulation. The supersymmetry algebra closes off-shell. In the component formulation of such theories the supersymmetry is not manifest, but the algebra still closes off-shell. (For that we need to include the auxiliary fields if there are any.) If there is no superspacetime formulation, then the supersymmetry algebra does not close off-shell. (The coupled harmonic oscillators in the exercises for Lecture 2 provide an analog of these phenomena in ordinary classical mechanics.)
- The superspacetime formulations are most useful for understanding the supersymmetry, since in this way it is manifest. However, to see the physics of the theory it is often best to work in components. Some computations are easier in superspacetime, some easier in components. If there is a superspacetime formulation, then it is useful in the quantum theory to give *a priori* constraints on the possible quantum corrections. Such corrections must respect the supersymmetry, and the constraints imposed are more easily seen in superspacetime.

- The most commonly used superspacetimes are those in dimensions $n \le 4$ with $s \le 4$ supersymmetries. For theories with more supersymmetry in dimensions $n \le 4$, we can still use the $s = 4$ superspacetime to keep part of the supersymmetry manifest. Nevertheless, superspacetimes with $s > 4$ have striking applications.
- We indicated that there is a superspacetime formulation of supersymmetric gauge theories with 8 supersymmetries, but this only applies to *pure* gauge theories. The superspacetime formulation in 6 dimensions is a bit deficient in some ways; the reduction to 4 dimensions is better.
- There are superspacetime formulations of σ-models with 8 supersymmetries for special kinds of hyperkähler manifolds, but as far as I know none which works in general.

Super Minkowski spacetime and the super Poincaré group

Recall that Minkowski spacetime M^n is an affine space whose underlying vector space V of translations has a Lorentz metric, and that the Poincaré group P^n is a cover of the component of affine symmetries of M^n which preserve the metric and contains the identity. The constructions in this section provide a generalization to supermanifolds with nontrivial odd part.

Fix a dimension n. We give a general description for any n, and then describe things more explicitly for small n. To define a superspacetime we need to fix a real spin representation S, which we assume has dimension s. Recall that this is the data which we used in Lecture 5 to define the fermion field. The important extra ingredient is the symmetric pairing (5.18)

$$\tilde{\Gamma}\colon S \otimes S \longrightarrow V. \tag{6.3}$$

It was used in (5.19) to write the kinetic lagrangian for the spinor field. The supersymmetry algebra depends on a related pairing

$$\Gamma\colon S^* \otimes S^* \longrightarrow V, \tag{6.4}$$

which is used, as we will see, to write "square roots" of translations. Both Γ and $\tilde{\Gamma}$ are positive definite in the sense that once we choose a positive cone $C \subset V$ of timelike vectors, then

$$\Gamma(s^*, s^*), \tilde{\Gamma}(s, s) \in \overline{C} \tag{6.5}$$

for all $s \in S$, $s^* \in S^*$, and these quantities vanish only when the input vanishes. Furthermore, there is a Clifford relation between Γ and $\tilde{\Gamma}$ which is specified most easily in terms of bases. Let $\{P_\mu\}$ be a basis of V and $\{Q^a\}$ a basis of S, with dual basis $\{Q_a\}$ of S^*. Then we write

$$\begin{aligned} \Gamma(Q_a, Q_b) &= \Gamma^\mu_{ab} P_\mu \\ \tilde{\Gamma}(Q^a, Q^b) &= \tilde{\Gamma}^{\mu ab} P_\mu . \end{aligned} \tag{6.6}$$

Let $g_{\mu\nu}$ be the coefficients of the Lorentz metric with respect to the basis $\{P_\mu\}$, and $g^{\mu\nu}$ the coefficients of the inverse metric on V^*. Then the Clifford relation is

$$\Gamma^\mu_{ab}\,\tilde{\Gamma}^{\nu bc} + \Gamma^\nu_{ab}\,\tilde{\Gamma}^{\mu bc} = 2g^{\mu\nu}\delta^c_a, \tag{6.7}$$

where δ is the usual Kronecker δ-function. The theory of spin representations in Lorentz signature guarantees the existence of $\Gamma, \tilde{\Gamma}$ with these properties; in fact, Γ determines $\tilde{\Gamma}$ uniquely.

As mentioned we construct a Lie algebra in which elements of S^* act as square roots of infinitesimal translations, which are elements of V. Thus introduce the $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra

$$\mathcal{L} = V \oplus S^* \tag{6.8}$$

with V central and the nontrivial odd bracket

$$[Q_a, Q_b] = -2\Gamma^{\mu}_{ab} P_{\mu}. \tag{6.9}$$

There is a corresponding super Lie group whose underlying supermanifold is *super Minkowski spacetime*

$$M^{n|s} = M^n \times \Pi S^*. \tag{6.10}$$

Corresponding to the given bases on V and S^* are linear coordinates x^μ on V and θ^a on the odd supermanifold ΠS^*. So altogether x^μ, θ^a are global coordinates on $M^{n|s}$. The coordinate vector fields are $\partial_\mu, \partial/\partial\theta^a$. Now the action of the Lie algebra $\mathcal{L}$ on $M^{n|s}$ gives rise to a basis $\{\partial_\mu, D_a\}$ of left-invariant vector fields and a commuting basis $\{\partial_\mu, \tau_{Q_a}\}$ of *right* invariant vector fields. Note that ∂_μ is both left and right invariant since V is central. Also, as usual right invariant vector fields give rise to left actions, so that the τ_{Q_a} are part of the infinitesimal left action of $P^{n|s}$. The vector fields D_a and τ_{Q_a} are given by the formulas

$$\begin{aligned} D_a &= \frac{\partial}{\partial\theta^a} - \Gamma^{\mu}_{ab}\theta^b\partial_\mu \\ \tau_{Q_a} &= \frac{\partial}{\partial\theta^a} + \Gamma^{\mu}_{ab}\theta^b\partial_\mu. \end{aligned} \tag{6.11}$$

as the reader may check. The nontrivial brackets are

$$\begin{aligned} [D_a, D_b] &= -2\Gamma^{\mu}_{ab}\partial_\mu \\ [\tau_{Q_a}, \tau_{Q_b}] &= +2\Gamma^{\mu}_{ab}\partial_\mu. \end{aligned} \tag{6.12}$$

The brackets of the left invariant D_a are as in the Lie algebra $\mathcal{L}$; brackets of the right invariant τ_{Q_a} are opposite. (This is a general feature of right and left actions, as explained in the text preceding (1.17).) Also

$$[D_a, \tau_{Q_b}] = 0 \tag{6.13}$$

since left invariant vector fields commute with right invariant vector fields.

The super Poincaré algebra is the graded Lie algebra

$$\mathfrak{p}^{n|s} = \big(V \oplus \mathfrak{so}(V)\big) \;\oplus\; S^*. \tag{6.14}$$

Its even part is the usual Poincaré algebra. It also contains the super Lie algebra (6.8) of translations as a subalgebra. The bracket of elements of $\mathfrak{so}(V)$ and S^*

is by the infinitesimal spin representation. The super Poincaré group is the semidirect product $\mathrm{Spin}(V) \ltimes \exp(\mathcal{L})$, expressed by the split exact sequence

$$1 \longrightarrow \exp(\mathcal{L}) \longrightarrow P^{n|s} \longrightarrow \mathrm{Spin}(V) \longrightarrow 1. \tag{6.15}$$

We mention briefly two more concepts connected with the super Poincaré algebra and illustrate them below. First, it may happen that $\mathrm{Sym}^2 S^*$ contains some copies of the trivial representation, i.e., that there is a symmetric pairing

$$S^* \otimes S^* \longrightarrow \mathbb{R}^c. \tag{6.16}$$

Then we can form new super Lie algebras by adding $\mathbb{R}^{c'}$ to the even part of (6.14) for any $c' \leq c$:

$$\tilde{\mathfrak{p}}^{n|s} = \bigl(V \oplus \mathbb{R}^{c'} \oplus \mathfrak{so}(V)\bigr) \ \oplus \ S^*. \tag{6.17}$$

The subspace $\mathbb{R}^{c'}$ is central; its elements are called *central charges*. Second, there may be outer automorphisms of $\mathfrak{p}^{n|s}$ which fix the Poincaré algebra. These are called infinitesimal *R-symmetries*; the connected group we obtain by exponentiation is the *R-symmetry group*. The R-symmetry group is compact, since the pairing Γ is positive definite.

The central charges already arise in classical field theories. This is a general feature of symplectic geometry, which is encoded in the exact sequence (1.18). Namely, if $\mathfrak{p}^{n|s}$ is a Lie algebra of symmetries of some theory, then the corresponding Lie algebra of observables is in general a central extension. In the quantum theory it is the Lie algebra of classical observables which gives rise to a Lie algebra of quantum observables—self-adjoint operators.

The infinitesimal R-symmetries, which act in a quantum theory as automorphisms of the symmetry algebra, are represented (projectively) on the Hilbert space of the theory.

Examples of super Poincaré groups

We already met $\mathfrak{p}^{1|1}$ when we discussed the supersymmetric particle. There is a single even infinitesimal translation P, which is infinitesimal time translation, and a single odd infinitesimal translation Q; the nontrivial bracket is

$$[Q,Q] = -2P \tag{6.18}$$

as above. The Lie algebra of the Lorentz group is trivial, but the Lorentz group is cyclic of order 2; its action on Q is nontrivial. (The nonidentity element of the Lorentz group maps Q to $-Q$.) There are no nontrivial R-symmetries. More conceptually, the basic real spin representation of the Lorentz group $\mathrm{Spin}(V) \cong \mathbb{Z}/2\mathbb{Z}$ is a one-dimensional space S, and we identify[26] $V \cong (S)^{\otimes(-2)}$.

There is a simple extension of this example to $\mathfrak{p}^{1|s}$ for any $s \geq 0$. Namely, let Q_a be a basis for an s-dimensional vector space of odd infinitesimal translations, and then $\{P, Q_a\}$ is a basis of $\mathfrak{p}^{1|s}$. The nontrivial brackets are

$$[Q_a, Q_a] = -2P \tag{6.19}$$

[26] The dual of a one-dimensional vector space S is often denoted $S^{\otimes(-1)}$.

for all a. More abstractly, the Q_a span a vector space S^* and the bracket is defined from a positive definite symmetric pairing

$$\Gamma\colon S^* \otimes S^* \longrightarrow \mathbb{R} \cdot P. \tag{6.20}$$

The R-symmetry algebra is the orthogonal algebra of the pairing, and the corresponding R-symmetry group is isomorphic to $Spin(s)$. For some values of s this algebra arises by dimensional reduction, as we see below.

Jump now to $n = 3$ spacetime dimensions. The Lorentz group is isomorphic to $SL(2;\mathbb{R})$ and so the basic real spin representation S has dimension 2. We identify $V = \mathrm{Sym}(S^*)$. In other words, the pairing $\Gamma\colon S^* x S^* \to V$ induces an isomorphism $\mathrm{Sym}(S^*) \cong V$. (This was also the case in our first example $\mathfrak{p}^{1|1}$, but the induced map is not usually an isomorphism.)

It is natural to label basis vectors of V as P_{ab} where the subscript is symmetric in the indices and all indices run from 1 to 2. The bracketing relations in the supersymmetry algebra are

$$[Q_a, Q_b] = -2P_{ab}. \tag{6.21}$$

Corresponding to the basis elements P_{ab} of V are the coordinate vector fields ∂_{ab}, which are related to the previous $\partial_\mu = \partial/\partial x^\mu$ by

$$\begin{aligned} \partial_{11} &= \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}, \\ \partial_{22} &= \frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}, \\ \partial_{12} &= \frac{\partial}{\partial x^2}. \end{aligned} \tag{6.22}$$

Notice that ∂_{11} and ∂_{22} are lightlike vectors, whereas ∂_{12} is spacelike. In this case there are no possible central charges and the R-symmetry group is trivial. The super Poincaré algebra is denoted $\mathfrak{p}^{3|2}$ and the corresponding super Poincaré group is $P^{3|2}$.

Now we consider the dimensional reduction to $n = 2$ spacetime dimensions. If we imagine a classical field theory with an action of this algebra, then we want to restrict it to the subspace of fields on which a one-dimensional subspace T of spatial translations acts trivially. For a quantum field theory dimensional reduction means restriction to the Hilbert subspace on which the operators corresponding to elements of T act trivially. Either way, the abstract supersymmetry algebra is obtained by setting a single infinitesimal translation to zero. For $\mathfrak{p}^{3|2}$ it is convenient to pick the spatial translation ∂_{12} and set it to zero. This gives a supersymmetry algebra in 2 dimensions which is denoted $\mathfrak{p}^{2|(1,1)}$. We explain the notation now.

The Lorentz group in $n = 2$ is $\mathbb{R}^{>0} \times \mathbb{Z}/2\mathbb{Z}$. In any spin representation $\mathbb{Z}/2\mathbb{Z}$ acts nontrivially, and as pointed out in Lecture 5 there are two inequivalent one-dimensional real spin representations S^+, S^-. The group of infinitesimal translations is identified as

$$\begin{aligned} V &= (S^+)^{\otimes(-2)} \oplus (S^-)^{\otimes(-2)} \\ &= \quad V^+ \quad \oplus \quad V^-. \end{aligned} \tag{6.23}$$

This makes clear that the superspacetime corresponding the spinor representation $S^+ \oplus S^-$ is globally a product:

$$M^{2|(1,1)} = M^{1|1} \times M^{1|1}. \tag{6.24}$$

This splitting corresponds to the splitting of the lightcone in two dimensions. For any $s^+, s^- > 0$ there is a superspacetime $M^{2|(s^+,s^-)}$ and a corresponding super Poincaré group; they are constructed starting with the spin representation $S = (S^+)^{s^+} \oplus (S^-)^{s^-}$. The existence of two distinct real spin representations explains the notation $s = (s^+, s^-)$. Similar notation is used for all $n \equiv 2, 6 \pmod 8$.

In terms of the bases written above it is customary to use '+' for the index '1' and '−' for the index '2'. Set $\partial_+ = \partial_{11}$ and $\partial_- = \partial_{22}$. Waves which are functions of x^+ are *left-moving*, and waves which are functions of x^- are right-moving. The entire picture splits as the Cartesian product of left- and right-movers.

There is a new feature of $P^{2|(1,1)}$ not encountered in $P^{3|2}$: the possibility of central charges. Namely, the symmetric square of S^* in this case is

$$\mathrm{Sym}^2\big((S^+)^* \oplus (S^-)^*\big) \cong V^+ \oplus V^- \oplus \mathbb{R}. \tag{6.25}$$

The centrally extended algebra $\tilde{\mathfrak{p}}^{2|(1,1)}$ has a single central charge Z, and the nontrivial brackets are

$$\begin{aligned} [Q_+, Q_+] &= -2\partial_+ \\ [Q_-, Q_-] &= -2\partial_- \\ [Q_+, Q_-] &= 2Z. \end{aligned} \tag{6.26}$$

Note that $\mathfrak{p}^{2|(1,1)}$ is obtained by setting $Z = 0$.

The further dimensional reduction to $n = 1$ is obtained by setting the spatial translation $\partial_+ - \partial_-$ to zero. This recovers the algebra $\mathfrak{p}^{1|2}$ constructed above. In this case there is an R-symmetry group Spin(2) which we may understand from the original 3 dimensional picture. Namely, we have now set a two-dimensional space of infinitesimal translations to zero, so broken the original Lorentz group in 3 dimensions down to a Lorentz group in 1 dimension. The infinitesimal rotations of that 2-dimensional plane, which are part of the Lorentz group in 3 dimensions, are the R-symmetries of the dimensionally reduced algebra.

Representations of the super Poincaré group

Recall that in any relativistic quantum mechanical system particles are irreducible representations of the Poincaré group P^n. In a supersymmetric theory this representation extends to a unitary representation of $P^{n|s}$ for some s depending on the amount of supersymmetry. The irreducible representations of $P^{n|s}$ are generally not irreducible when restricted to the subgroup P^n, but rather break up as a finite sum. Such collections of particles are called supersymmetric *multiplets*. The particle content of a supersymmetric theory is organized into such multiplets. There is a general story, but to understand supersymmetric quantum field theories one needs to learn the taxonomy of multiplets in various dimensions with various amounts of supersymmetry. Here we outline the general theory and then illustrate with a few low dimensional examples.

Consider the supersymmetry group $P^{n|s}$. (We sue the notation $s = (s^+, s^-)$ if $n \equiv 2, 6 \pmod 8$.) As for the Poincaré group we isolate the subgroup of ordinary translations:

$$1 \to V \to P^{n|s} \to \mathrm{Spin}(V) \ltimes \Pi S^* \to 1 \tag{6.27}$$

Suppose we are given an irreducible nonnegative energy representation of $P^{n|s}$. Restrict to V to obtain an irreducible orbit of infinitesimal characters of some mass $m \geq 0$ in V^*. The supersymmetric little groups

$$\begin{aligned} m > 0: &\qquad \mathrm{Spin}(n-1) \ltimes \Pi S^* \\ m = 0: &\qquad \mathrm{Spin}(n-2) \ltimes \Pi S^* \end{aligned} \tag{6.28}$$

are super groups; the underlying ordinary group is the little group of the Poincaré group, but there are odd parts as well. (Recall that the *little group* is defined to be the reductive part of the stabilizer; a finite dimensional unitary representation of the stabilizer factors through the little group.) The representation of $P^{n|s}$ is induced from a projective graded finite dimensional unitary representation of the little group—as before, we form a homogeneous graded hermitian vector bundle over the orbit and take the Hilbert space of sections. The fact that the representation is projective is due to the nontrivial bracket on S^*. More precisely, at $\lambda \in V^*$ we seek a representation of the central extension constructed using the quadratic form q_λ on S^* given by

$$q_\lambda(Q_1, Q_2) = \lambda([Q_1, Q_2]).$$

Recall the positivity condition on the pairing (6.4). Since λ is in the closure of the positive dual lightcone, we conclude that q_λ is negative semidefinite. In the massive case the form is negative definite, whereas in the massless case it has a kernel. Except for the exceptional case $s = 1$ this kernel has dimension $s/2$. (This follows by splitting V into the sum of a two-dimensional Lorentz space and a negative definite complement, thus reducing to the $n = 2$-dimensional case and an explicit computation.) In the massless case we work with the quotient of S^* by the kernel, and then in both cases we obtain a negative definite space. The projective representations of (3.15) we want are then graded Clifford modules

$$W = W^0 \oplus W^1 \tag{6.29}$$

for S^* (modulo the kernel in the massless case) together with an intertwining action of the appropriate Spin group.

Some remarks:

- In this construction we end up considering spinors of spinors—a Clifford module for the vector space of spinors. Also, keep in mind that S is a *real* space, whereas the representation of the little group we construct—the spinors of spinors—is *complex*. The space S is a representation of $\mathrm{Spin}(n-1)$ or $\mathrm{Spin}(n-2)$ by restriction of the $\mathrm{Spin}(V)$ action.
- Usually $\dim S$ is even and we construct the unique irreducible graded Clifford module by complexifying S and choosing a lagrangian splitting, as described in (5.8).

- The irreducible massless representations tend to be smaller than the irreducible massive representations because of the drop in dimension. This does not hold in low dimensions, however.
- The representation theory of the supersymmetry groups with central charges is similar. The important concept of a *BPS particle* is introduced there. We will not give details here, but remark that we will meet an analog for fields in the next lecture.
- Allowable representations in a local quantum field theory are constrained to be CPT invariant, as mentioned in Lecture 5.
- We choose the grading of W—which piece is even and which piece odd—according to the spin-statistics principle.
- The even and odd pieces of a Clifford module have equal dimensions. This means that *a supersymmetry multiplet has an equal number of bosonic degrees of freedom and fermionic degrees of freedom.* (The number of degrees of freedom is the dimension of the representation of the little group.) There is an exception for chiral theories in $n = 2$, however. For example, there exists a theory with symmetry group $P^{2|(1,0)}$ with one (chiral) fermion and no bosonic degrees of freedom.

Now some examples.

Example 1 ($P^{3|2}$). We first review the representations of the Poincaré group P^3. For massive representations the little group is $\mathrm{Spin}(2)$, and its irreducible complex representations are one-dimensional and labeled by the spin j. All of the representations are CPT invariant. The spin $j = 0$ representation is called a *massive scalar particle*, the spin $j = 1/2$, $j = -1/2$ representations *massive spinor particles*, and the spin $j = 1$, $j = -1$ representations *massive half-vector particles*. (The *massive vector particle* is the sum of the $j = 1$ and $j = -1$ representations.) In the massless case the little group is $\mathrm{Spin}(1) \cong \mathbb{Z}/2$. There are two irreducible complex representations, each of dimension one. The trivial one is the *massless scalar particle* and the nontrivial one is the *massless spinor particle*.

Now we turn to the supersymmetry group $P^{3|2}$. In the massive case we have in addition to the little group $\mathrm{Spin}(2)$ a negative definite Clifford algebra generated by S^*, which here has dimension 2. After complexification we choose a basis $\{Q_+, Q_-\}$ where $\mathrm{Spin}(2)$ acts on $Q_\pm$ by $j = \pm 1/2$. A physicist describes the Clifford module in the following notation. Fix a *vacuum* vector $|0\rangle$ and postulate

$$Q_-|0\rangle = 0.$$

Then the Clifford module is

$$W = \mathbb{C} \cdot |0\rangle \;\oplus\; \mathbb{C} \cdot Q_+|0\rangle.$$

To define the action of $\mathrm{Spin}(2)$ we need to specify a spin j for $|0\rangle$; then the spin of $Q_+|0\rangle$ is $j + 1/2$.

To summarize: A massive multiplet for $P^{3|2}$ is a pair of particles of adjacent spins j and $j + 1/2$.

We show some possibilities in the table above, which indicates the number of representations with a given spin j which occur in each multiplet. We only list

	-1	$-1/2$	0	$1/2$	1
massive scalar multiplet		1	1		
massive scalar multiplet			1	1	
massive vector multiplet	1	1		1	1

Massive multiplets for $P^{3|2}$

multiplets with spin at most 1; they are the relevant multiplets for supersymmetric theories without gravity. The multiplets are usually named after the highest spin boson. There are two *massive scalar multiplets* because of the two different possibilities for a massive spinor. In addition to the *massive vector multiplet* shown, there are also *massive half-vector multiplets*.

Recall that free classical theories can be quantized, and the 1-particle Hilbert space is a sum of particle representations. It is instructive to see how the two different spinor particles arise from free fields—the difference comes from the sign in the mass term in the lagrangian. We will not consider massive vectors in these lectures, but it is amusing to see how the massive half-vectors can be written in terms of fields—the 3-dimensional Chern-Simons term enters.

Now we consider the massless multiplets. In this case the quadratic form on S^* is degenerate, and the quotient by the kernel is one-dimensional. The irreducible complex Clifford module $W = W^0 \oplus W^1$ is still two-dimensional. The little group $\mathrm{Spin}(1) \cong \mathbb{Z}/2$ acts trivially on W^0 and nontrivially on W^1. So there is a unique massless multiplet, the *massless scalar multiplet,* which contains one scalar particle and one spinor particle.

Example 2 ($\tilde{P}^{2|(1,1)}$). This example illustrates the representation theory for a centrally extended supersymmetry group. We refer to (6.26) for the bracketing relations in the central extension $\tilde{\mathfrak{p}}^{2|(1,1)}$ of the symmetry algebra $\mathfrak{p}^{2|(1,1)}$. From these brackets we easily compute

$$\frac{1}{4}[Q_+ \pm Q_- \, , \; Q_+ \pm Q_-] = -\partial_t \pm Z, \tag{6.30}$$

where

$$2\partial_t = \partial_+ + \partial_- \tag{6.31}$$

is infinitesimal time translation. Note that the left hand side of (6.30) is the square of an odd element. Thus the quantum operators $\hat{H} \pm \hat{Z}$ corresponding to the right hand side of (6.30) are nonnegative; in other words,

$$\hat{H} \geq |\hat{Z}|. \tag{6.32}$$

Here $\hat{H}$ is the quantum hamiltonian. In an irreducible supersymmetry representation the operator $\hat{Z}$ is a constant since Z is central. For $Z = 0$ the inequality (6.32) is a special case of the general argument that the quantum hamiltonian in a supersymmetric theory is nonnegative. But with the central charge we have a stronger bound. The Poincaré invariant statement is the *BPS bound*: the *mass* is bounded below by one-half the absolute value of the central charge. Furthermore, if we have equality—the mass equal to one-half the absolute value of the central charge—then the quadratic form on the odd part of the little group has a half-dimensional kernel, just as for massless representations. (In this case that is evident in (6.30).) So we obtain special irreducible massive multiplets—*BPS multiplets*—which typically have fewer degrees of freedom than the usual massive multiplets. If this is the case, then they are stable under perturbations. Hence these BPS representations are an important source of stable particles in supersymmetric theories.

The states in a BPS representation are annihilated by $1/2$ of the supersymmetry, in this example by $Q_+ + Q_-$ or $Q_+ - Q_-$. There are more complicated situations where a different fraction of the supersymmetry annihilates a representation, which is also termed BPS.

In Lecture 7 we will see a classical field configuration which satisfies a classical version of the BPS condition for the supersymmetry group $\tilde{P}^{2|(1,1)}$: it solves the equations of motion and is annihilated by half of the classical supersymmetry.

Exercises

1. (Clifford algebras) Let U be a real vector space with a positive definite inner product $\langle\cdot,\cdot\rangle$. The Clifford algebra $\mathrm{Cliff}(U)$ is the quotient of the tensor algebra by the ideal generated by
$$u \otimes u' + 2\langle u, u'\rangle \cdot 1$$
for all $u, u' \in U$.
   (a) What algebra do you get for $\dim U = 1, 2, 3$? (Hint: Choose an orthonormal basis for U and write everything explicitly in terms of that basis.)
   (b) Prove that $\mathrm{Cliff}(U)$ is finite dimensional if U is finite dimensional.
   (c) So far we have viewed $\mathrm{Cliff}(U)$ as an ordinary algebra. Show how it may be viewed as a $\mathbb{Z}/2\mathbb{Z}$-graded superalgebra. Is it (super)commutative?
   (d) Find irreducible $\mathbb{Z}/2\mathbb{Z}$-graded modules for $\mathrm{Cliff}(U)$ in low dimensions.

2. (super Lie algebras) Let
$$\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$$
be a super Lie algebra.
   (a) Show that the ($\mathbb{Z}/2\mathbb{Z}$-graded) skew-symmetric bracket on $\mathfrak{g}$
$$[\cdot,\cdot]\colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$$
is equivalent to a bracket on $\mathfrak{g}^0$, an map $\mathfrak{g}^0 \otimes \mathfrak{g}^1 \to \mathfrak{g}^1$, and a symmetric pairing $\mathfrak{g}^1 \otimes \mathfrak{g}^1 \to \mathfrak{g}^0$.
   (b) Show that the Jacobi identity for $\mathfrak{g}$ is equivalent to verifying that $\mathfrak{g}^0$ is a Lie algebra, the map $\mathfrak{g}^0 \otimes \mathfrak{g}^1 \to \mathfrak{g}^1$ is an action, and $[Q,[Q,Q]] = 0$ for all $Q \in \mathfrak{g}^1$.
   (c) Show that $\mathfrak{p}^{n|s}$ as defined in the lecture is a super Lie algebra.

3. (a) Using (6.11), verify (6.12) and (6.13).

   (b) Use the "functor of points" point of view explained in John Morgan's lecture to construct $\exp \mathcal{L}$ as a super Lie group. In other words, for every supercommutative ring (think of $R = \mathbb{R}[\eta_1, \eta_2, \dots, \eta_N]$ with η_i odd and mutually commuting) construct an ordinary Lie group over $\operatorname{Spec} R$. (Hint: Use the Campbell-Hausdorff formula, which terminates quickly in this case.)

   (c) Show that (6.11) are indeed left- and right-invariant vector fields on $\exp \mathcal{L}$.

4. Construct a basis for the entire super Lie algebra $\mathfrak{p}^{3|2}$ and compute all brackets.

5. In this problem you'll learn about dimensional reduction. Begin on Minkowski M^n with coordinates $x^0, x^1, \dots, x^{n-1}$ as usual. We dimensionally reduce fields by demanding that they be translation-invariant in the x^{n-1} direction. The idea is to identify translation-invariant fields with fields on M^{n-1}.

   (a) Just to be sure you understand what we're talking about, show that a scalar field in n dimensions dimensionally reduces to a scalar field in $n-1$ dimensions.

   (b) A 1-form field in n dimensions is given by n functions of n variables. Demanding that the 1-form be translation-invariant gives n functions of $n-1$ variables. What fields do we obtain in M^{n-1}? In other words, what fields do we get by dimensionally reducing a 1-form field on M^n? (The n functions of $n-1$ variables organize into fields according to their transformation law under the Poincaré group P^{n-1}. What are these fields?)

   (c) What fields do we obtain if we dimensionally reduce a connection A on M^n for some compact gauge group A?

   (d) In both of the previous examples we have gauge symmetries. Translations act on the space $\overline{\mathcal{F}}$ of equivalence classes of all fields $\mathcal{F}$. Think through dimensional reduction in this way: identify the subspace of $\overline{\mathcal{F}}$ of equivalence classes invariant under translation in the x^{n-1} direction with the space of equivalence classes of some fields on M^{n-1}.

   (e) What happens when you dimensionally reduce a spinor field? Try some low dimensional examples, e.g. starting in $n = 3$.

6. Let $\mathcal{H}$ be the Hilbert space underlying some nonnegative energy unitary representation of the Poincaré group P^n. Let $\mathcal{H}' \subset \mathcal{H}$ be the subspace annihilated by the infinitesimal translation operator $\hat{P}_{n-1}$. Show that the Poincaré group P^n acts on $\mathcal{H}'$. What representation do you get? The answer depends, of course, on the representation you begin with. Recall that representations are characterized by a mass and finite-dimensional unitary representation of the little group. So the dimensional reduction should be expressed as a map (mass, rep of little group in n dimensions) to (mass, rep of little group in $n-1$ dimensions).

7. Obtain $M^{1|1}$ as a dimensional reduction of $M^{2|(1,0)}$.

8. The lectures on mirror symmetry concern $P^{2|(2,2)}$. This problem is meant to familiarize you with this super Poincaré group and the underlying superspacetime $M^{2|(2,2)}$.

(a) What is the real spin representation out of which $M^{2|(2,2)}$ is constructed?

(b) It is usual to take a basis of *complex* left-invariant vector fields. The even elements are the real vector fields ∂_+, ∂_- as in the lecture. The odd vector fields are $D_+, \overline{D}_+, D_-, \overline{D}_-$ with nontrivial brackets

$$\begin{aligned} [D_+, \overline{D}_+] &= -\partial_+ \\ [D_-, \overline{D}_-] &= -\partial_- \end{aligned}$$

Verify that this *is* the super Poincaré algebra as defined in the text. What is the pairing Γ?

(c) What are the possible central extensions of this algebra by adjoining central charges?

(d) What is the Lie algebra of R-symmetries? Write the action on $P^{2|(2,2)}$.

9. Discuss the representation theory of $P^{2|(2,2)}$, without and with the possible central charges.

LECTURE 7
Supersymmetric σ-Models

In this lecture we consider one of the simplest supersymmetric field theories. Initially we consider it in dimension $n = 3$, which is the maximal dimension in which it is defined. Its dimensional reductions to $n = 1$ and $n = 1$ dimensions relate to interesting topics in geometry: Morse theory and index theory. We will see some indications of this in the classical theory, but the real power comes after quantization (which we do not discuss in these lectures).

3-dimensional theory

The supersymmetric σ-model has fields of spin 0 and spin 1/2, but no spin 1 fields. Recall from (5.30) that in general a spin 0 field is specified by a Riemannian manifold X and a spin 1/2 field by a real vector bundle $W \to X$ with metric and connection. We may also add potential terms (5.34). Theories constructed from this data are invariant under the Poincaré group P^3. Now we ask for a theory which is supersymmetric with 2 supersymmetries.[27] In other words, we ask that the larger supergroup $P^{3|2}$ act by symmetries on the theory. We expect that this leads to constraints on this basic data. In fact, we will find that $W = TX$ (where we do not include any auxiliary fields in what we mean by 'X') and that the potential (5.34) is constructed from a single real function $h\colon X \to \mathbb{R}$. Rather than begin with the general theory and attempt to derive these constraints, we simply construct the supersymmetric theory directly. There is a superspacetime formulation on $M^{3|2}$ which makes the supersymmetry manifest, and we use it to derive the theory in terms of *component fields* on M^3

Recall that in 3 dimensions $\mathrm{Spin}(V) \cong SL(2;\mathbb{R})$; we take the spin representation S to be the standard 2-dimensional representation. In Lecture 7 we defined the framing ∂_{ab}, D_a of $M^{3|2}$ by left-invariant vector fields. The indices $a, b, \dots$ run from 1 to 2, and the index ab is symmetric in a and b. We now additionally make note of the skew-symmetric form

$$\epsilon\colon S \otimes S \longrightarrow \mathbb{R}, \tag{7.1}$$

[27] There is a supersymmetric σ-model with a single supersymmetry in one dimension, which we constructed in Lecture 4 as the supersymmetric particle. There are also supersymmetric σ-models with 4 supersymmetries, where the target X is Kähler, and σ-models with 8 supersymmetries, where the target X is hyperkähler. These models exist classically in dimensions 4 and 6 respectively, so by dimensional reduction in smaller dimensions as well.

which in terms of $SL(2;\mathbb{R})$ is simply the volume form, and its dual form[28] on S^*. We choose the bases so that $\epsilon^{12} = \epsilon_{12} = 1$.

Fix a Riemannian manifold X. The field Φ in the superspacetime formulation is a scalar field on superspacetime with values in X:

$$\Phi\colon M^{3|2} \longrightarrow X.$$

Fields on superspacetime are usually called *superfields*. The left action of the super Poincaré group $P^{3|2}$ on $M^{3|2}$ induces an action on superfields by pullback. The corresponding infinitesimal action is by *right*-invariant vector fields, which we call τ_{Q_a}.

Just as we did for the supersymmetric particle (see (4.31)) we define *component fields* on Minkowski spacetime M^3 by restricting normal derivatives of the superfield to $M^3 \subset M^{3|2}$. Since the θ^a are nilpotent, a small number of normal derivatives determines the superfield completely. In the case of the supersymmetric particle on $M^{1|1}$ there is a single odd direction, so we only needed one first derivative. In $M^{3|2}$ we have two odd directions, so have two first derivatives and one (mixed) second derivative; all others vanish because of the oddness of the derivatives. The complete list of component fields is:

$$\begin{aligned} \phi &= i^*\Phi \\ \psi_a &= i^* D_a\Phi \\ F &= -\frac{1}{2}\epsilon^{ab} i^* D_a D_b \Phi \end{aligned} \tag{7.2}$$

Several remarks are in order:

- The leading bosonic component ϕ is a scalar field on M^3 with values in X, namely

$$\phi\colon M^3 \to X. \tag{7.3}$$

- The field ψ_a is odd, since the vector field D_a is odd. It is an odd section of the pullback tangent bundle ϕ^*TX. Together the two fields ψ_1, ψ_2 transform as a spinor field on Minkowski spacetime, since D_a correspond to basis elements of S^*, so we can combine them into

$$\psi \in \Omega^0\bigl(M^3; \Pi S \otimes \phi^*TX\bigr). \tag{7.4}$$

- The field F is even and is also a section of the pullback tangent bundle:

$$F \in \Omega^0(M^3; \phi^*TX). \tag{7.5}$$

- The outer (leftmost) derivative in the definition of F is a covariant derivative using the Levi-Civita connection, and in detailed computations we meet

[28] Any nondegenerate symmetric or skew-symmetric bilinear form on a vector space S induces a nondegenerate bilinear form on the dual space, called the dual form. Note that in the skew case the form determines an isomorphism $S \to S^*$ only up to a sign, but that isomorphism enters twice in transferring the bilinear form to the dual, so the sign ambiguity disappears. Be warned that physicists in this circumstance often use *minus* the dual form.

multiple covariant derivatives of Φ. The fact that Levi-Civita is torsion-free means the first two derivatives acting on Φ commute, but when commuting further derivatives we pick up curvature terms. This explains why curvature terms enter the component lagrangian below.

- In case $X = \mathbb{R}$, the fields ϕ, ψ comprise precisely the field content of a massless or massive scalar multiplet in $n = 3$ dimensions. (Recall the discussion of representations of $P^{3|2}$ in Lecture 6.) That is, if we write the free massless or massive lagrangian for these fields, then upon quantization the one-particle Hilbert space is that irreducible supersymmetry multiplet. So we are led to inquire: What is F doing here? We will see that F is an *auxiliary field* in the sense that it does not contribute physical degrees of freedom in the quantization. In fact, F enters the lagrangian only algebraically—no derivatives—so the equations of motion determine F algebraically in terms of the other fields. We will eliminate F using this algebraic equation.
- Still, the presence of F allows us to write down supersymmetry transformation laws (see (7.7) below) for the component fields which close *off-shell*. In other words, we have an action of $\mathfrak{p}^{3|2}$ on the entire space of fields $\mathcal{F}$ (off-shell), not just on the solutions $\mathcal{M} \subset \mathcal{F}$ to the equation of motion. If we use the (algebraic) equation of motion to substitute for F in the transformation laws, then the algebra no longer closes off-shell.
- The component fields $\{\phi, \psi, F\}$ are often referred to as a *multiplet*, in this case a *scalar multiplet*.
- The statement that we have a complete list of component fields is the statement that the supermanifold of superfields $\mathcal{F}_\Phi$ is isomorphic to the supermanifold of component fields $\mathcal{F}_{\{\phi,\psi,F\}}$. In each case we must use the "functor of points" point of view to make sense of the statement. Also, the statement implicitly assumes that the space of maps between supermanifolds is itself an infinite dimensional supermanifold.
- We can do the classical field theory in superspacetime $M^{3|2}$ with the superfield Φ or in ordinary spacetime M^3 with the component fields ϕ, ψ, F. In the first case we do calculus on $\mathcal{F}_\Phi \times M^{3|2}$; in the second calculus on $\mathcal{F}_{\{\phi,\psi,F\}} \times M^3$. The supersymmetry is *manifest* in the superfield formulation but not in the component formulation.

Next, we write down the supersymmetry transformation law for the component fields. In superspacetime the infinitesimal supersymmetry is the induced action on Φ of $-\tau_{Q_a}$, where we use a minus sign to obtain a left action. Introduce odd parameters η^a and consider the action of the even vector field $-\eta^a\tau_{Q_a}$. This induces a vector field $\hat{\zeta}$ on the space of component fields $\mathcal{F}_{\{a,\psi,F\}}$. Quite generally, a component field f (see (7.2)) is defined by a formula

$$f = i^* D^r \Phi, \tag{7.6}$$

where 'D^r' denotes a sum of products of D_a. Then a short argument shows that under the infinitesimal supersymmetry transformation $\hat{\zeta}$, the field f transforms as

$$\hat{\zeta} f = -\eta^a i^* D_a D^r \Phi. \tag{7.7}$$

The right hand side can be rewritten in terms of component fields. This formula works in any model with a superfield formulation, not just in the $3|2$ supersymmetric

σ-model. In the case at hand we find

$$\begin{aligned} \hat{\zeta}\phi &= -\eta^a\psi_a \\ \nabla_{\hat{\zeta}}\psi_a &= \eta^b(\partial_{ab}\phi - \epsilon_{ab}F) \\ \nabla_{\hat{\zeta}}F &= \eta^a\big[(\not{D}\psi)_a + \frac{1}{3}\epsilon^{bc}R(\psi_a,\psi_b)\psi_c\big], \end{aligned} \tag{7.8}$$

where R is the Riemann curvature tensor of X and the Dirac operator is

$$(\not{D}\psi)_a = -\epsilon^{bc}\partial_{ab}\psi_c. \tag{7.9}$$

Note the appearance of the covariant derivatives in the action of the vector field $\hat{\zeta}$ on sections of the (pulled back) tangent bundle. For this reason when checking commutators of the supersymmetry transformations (7.8) for vector fields $\hat{\zeta}, \hat{\zeta}'$ we encounter curvature terms.

This computation shows off the advantages of the superspacetime formulation—the supersymmetry transformation laws of the component fields are determined *a priori*.

The lagrangian density for the supersymmetric σ-model is

$$\mathcal{L} = |d^3x|\, d^2\theta\, \frac{1}{4}\epsilon^{ab}\langle D_a\Phi, D_b\Phi\rangle. \tag{7.10}$$

We have not given a detailed discussion of densities and integration on supermanifolds, and as before we finesse that point. Suffice it to say that $|d^3x|\, d^2\theta$ is a $P^{3|2}$-invariant density on $M^{3|2}$. The lagrangian function ℓ which appears after it in the lagrangian density (7.10) is obviously $P^{3|2}$-invariant, since $P^{3|2}$ acts on the left and the D_a are left-invariant. So $\mathcal{L}$ is manifestly supersymmetric—invariant under $P^{3|2}$. It is also worth mentioning that the lagrangian is constrained by asking that the kinetic term for the bosonic field be quadratic in first derivatives, as in (3.1). From this point of view each $d\theta$ counts as half of a bosonic derivative (see (7.13) below), as does each D_a. Thus (7.10) has a total of two bosonic derivatives, as it should.

We need to "integrate out the θs" to define a component lagrangian L. Quite generally, for a superspacetime model on $M^{n|s}$ the superspacetime lagrangian density is[29]

$$\mathcal{L} = |d^nx|\, d^s\theta\ \ell. \tag{7.11}$$

There is a standard notion of (Berezin) integration to integrate out the odd variables. Let

$$\pi\colon M^{n|s} \longrightarrow M^n \tag{7.12}$$

be the projection given by our construction of superspacetime and π_* the corresponding integration. Then the integral over the odd variables amounts to differentiation:

$$\pi_* = i^*\frac{\partial}{\partial\theta^s}\cdots\frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\theta^1}. \tag{7.13}$$

[29] To write this we need to choose an orientation on S to order the basis elements θ. That choice appears again in (7.13) below, so cancels out.

Rather than integrate, our approach is to find in each case a certain sum of products of D_a, which we denote 'D^s', such that

$$\pi_*\mathcal{L} = (i^* D^s \ell + \Delta i^* \ell)\, |d^n x| \tag{7.14}$$

for some Poincaré invariant differential operator Δ on M^n. Then instead of using $\pi_*\mathcal{L}$ as the component lagrangian, we define the component lagrangian to be

$$L = (i^* D^s \ell)\, |d^n x|. \tag{7.15}$$

A few brief remarks:

- In general this differs from the straight integration of the odd variables, as indicated by the presence of Δ. The particular D^s that we use is chosen so that the component lagrangian density L involves only first derivatives of the fields.
- In this lecture we have $\Delta = 0$. Examples with $\Delta \neq 0$ occur in superspacetimes with $s = 4$ odd directions, as we indicate in the exercises.
- Formula (7.15) is effective in that we find strings of D_a acting on superfields which we then express in terms of component fields using the commutation relations.
- The component lagrangian is supersymmetric, but the supersymmetry is not manifest. There is an explicit formula to determine the exact term by which it changes under a supersymmetry transformation. Recall from (2.48) that this term appears in the Noether current for the supersymmetry transformation, often called the *supercurrent*.

For $M^{3|2}$ we define the component lagrangian to be

$$L = i^*\bigl(-\frac{1}{2}\epsilon^{ab} D_a D_b\bigr)\ell\, |d^3 x|. \tag{7.16}$$

As remarked above, this agrees with the definition using the Berezinian integral $\int d^2\theta$. For the σ-model lagrangian (7.10) we have

$$L = -\frac{1}{8}\epsilon^{ab}\epsilon^{cd} i^* D_a D_b \langle D_c\Phi, D_d\Phi\rangle. \tag{7.17}$$

To illustrate the manipulations, consider a typical term in (7.17), omitting the numerical factor:

$$i^*\langle D_1 D_2 D_1 \Phi, D_2\Phi\rangle. \tag{7.18}$$

We emphasize that the outer two derivatives are *covariant* derivatives. When we commute them we pick up a curvature term:

$$\begin{aligned}\langle D_1 D_2 D_1 \Phi, D_2\Phi\rangle = &-\langle D_2 D_1 D_1 \Phi, D_2\Phi\rangle \\ &- 2\langle \partial_{12} D_1\Phi, D_2\Phi\rangle + \langle R(D_1\Phi, D_2\Phi) D_1\Phi, D_2\Phi\rangle.\end{aligned} \tag{7.19}$$

Since $D_1^2 = -\partial_{11}$ and $[\partial_{11}, D_2] = 0$, we have

$$D_2 D_1 D_1 \Phi = -D_2 \partial_{11}\Phi = -\partial_{11} D_2 \Phi. \tag{7.20}$$

Restricting to M^3 we obtain

$$i^*\langle D_1D_2D_1\Phi, D_2\Phi\rangle = \langle\partial_{11}\psi_2,\psi_2\rangle - 2\langle\partial_{12}\psi_1,\psi_2\rangle + \langle R(\psi_1,\psi_2)\psi_1,\psi_2\rangle. \tag{7.21}$$

Of course, there are systematic algebraic shortcuts one can develop for these manipulations. In any case, at the end of the day one obtains the component lagrangian for the supersymmetric σ-model:

$$L = \Big\{\frac{1}{2}|d\phi|^2 + \frac{1}{2}\langle\psi \not{D}_\phi\psi\rangle + \frac{1}{12}\epsilon^{ab}\epsilon^{cd}\langle\psi_a, R(\psi_b,\psi_c)\psi_d\rangle + \frac{1}{2}|F|^2\Big\}\ |d^3x|. \tag{7.22}$$

Some remarks:

- As promised, F appears only algebraically in the lagrangian; it is an auxiliary field. Its equation of motion is simply

$$F = 0. \tag{7.23}$$

- The spinor field ψ depends on ϕ (see (7.4)), as does the Dirac form. Also, the Dirac form is defined using a covariant derivative.
- There is a quartic potential $V^{(4)} = \frac{1}{12}\epsilon^{ab}\epsilon^{cd}\langle\psi_a, R(\psi_b,\psi_c)\psi_d\rangle$ for the fermionic field ψ.
- The *bosonic lagrangian* is obtained by setting the fermion ψ to zero:

$$L_{\text{bos}} = \frac{1}{2}|d\phi|^2\ |d^3x| \tag{7.24}$$

  after eliminating the auxiliary field F. This is the nonlinear σ-model lagrangian (3.9) with zero potential. So we have indeed obtained a supersymmetric extension of the nonlinear σ-model with two supersymmetries in three dimensions.
- There is not much to say about this classical field theory: the moduli space of vacua is X and the global symmetry group is the group of isometries of X.

We need the formula for the *supercurrent* j_a, which is *minus* the Noether current corresponding to the supersymmetry τ_{Q_a}. Here we write the full $|-1|$-form on M^3:

$$j_a = \iota(\partial_{cd})\,|d^3x|\,\Big\{\epsilon^{cb}\epsilon^{de}\langle\partial_{ae}\phi,\psi_b\rangle - \epsilon^{bd}\delta^c_a\langle F,\psi_b\rangle\Big\}. \tag{7.25}$$

We are only interested in this on-shell, in which case we could set $F = 0$, but we need the formula later in cases where $F \neq 0$. The *supercharges* are obtained by integrating j_a over a fixed time slice:

$$\begin{aligned} Q_1 &= \int_{x^0=\text{const}} \Big\{\langle\partial_{11}\phi,\psi_1\rangle + \langle\partial_{12}\phi,\psi_2\rangle\Big\}\ |dx^1dx^2| \\ Q_2 &= \int_{x^0=\text{const}} \Big\{\langle\partial_{22}\phi,\psi_2\rangle + \langle\partial_{12}\phi,\psi_1\rangle\Big\}\ |dx^1dx^2| \end{aligned} \tag{7.26}$$

In these last formulas we did set $F = 0$.

A supersymmetric potential

The theory is more fun when we add a potential term, which of course we must do in a supersymmetric way. This is easy in superspacetime. Let

$$h\colon X \longrightarrow \mathbb{R} \tag{7.27}$$

be a real-valued function on X. It is *not* the potential function V of the model; we compute V in terms of h below. Simply add the pullback of h by the superfield Φ to the superspacetime lagrangian (7.10):

$$\mathcal{L}' = |d^3x|\, d^2\theta\, \Big\{\frac{1}{4}\epsilon^{ab}\langle D_a\Phi, D_b\Phi\rangle + \Phi^*(h)\Big\}. \tag{7.28}$$

It is easy to work out the contribution of the new term to the component lagrangian:

$$\Big\{\langle F, \phi^* \operatorname{grad} h\rangle + \frac{1}{2}\epsilon^{ab}\phi^*(\operatorname{Hess} h)(\psi_a, \psi_b)\Big\}\; |d^3x|. \tag{7.29}$$

So the new component lagrangian is the sum of (7.22) and (7.29). Now the equation of motion of F is more interesting than before:

$$F = -\phi^* \operatorname{grad} h. \tag{7.30}$$

So upon eliminating F from the component lagrangian we obtain

$$\begin{aligned} L' = \Big\{\frac{1}{2}|d\phi|^2 + \frac{1}{2}\langle\psi \not{D}_\phi \psi\rangle - \frac{1}{2}\phi^*|\operatorname{grad} h|^2 + \frac{1}{2}\epsilon^{ab}\phi^*(\operatorname{Hess} h)(\psi_a, \psi_b) \\ + \frac{1}{12}\epsilon^{ab}\epsilon^{cd}\langle\psi_a, R(\psi_b, \psi_c)\psi_d\rangle\Big\}\; |d^3x|. \end{aligned} \tag{7.31}$$

The presence of h leads to two new terms:

- A potential energy

$$V^{(0)} = \frac{1}{2}|\operatorname{grad} h|^2 \tag{7.32}$$

  for the scalar field ϕ. Note that the potential is nonnegative, as it must be in a supersymmetric theory.
- A mass term

$$V^{(2)} = \frac{1}{2}\epsilon^{ab}\phi^*(\operatorname{Hess} h)(\psi_a, \psi_b) \tag{7.33}$$

  for the fermions.

The bosonic lagrangian in this case is

$$L_{\text{bos}} = \Big\{\frac{1}{2}|d\phi|^2 - \frac{1}{2}\phi^*|\operatorname{grad} h|^2\Big\}\; |d^3x|, \tag{7.34}$$

a nonlinear σ-model with nonzero potential.

We also record the *on-shell* supersymmetry transformation laws for the physical fields ϕ, ψ from (7.8), using the equation of motion for F:

$$\begin{aligned} \hat{\zeta}\phi &= -\eta^a \psi_a \\ \nabla_{\hat{\zeta}} \psi_a &= \eta^b (\partial_{ab}\phi + \epsilon_{ab}\phi^* \operatorname{grad} h). \end{aligned} \tag{7.35}$$

The supercharges are given by the formulas

$$\begin{aligned} Q_1 &= \int_{x^0=\text{const}} \Big\{ \langle \partial_{11}\phi, \psi_1 \rangle + \langle \partial_{12}\phi, \psi_2 \rangle - \langle \phi^* \operatorname{grad} h, \psi_2 \rangle \Big\} \, |dx^1 dx^2| \\ Q_2 &= \int_{x^0=\text{const}} \Big\{ \langle \partial_{22}\phi, \psi_2 \rangle + \langle \partial_{12}\phi, \psi_1 \rangle + \langle \phi^* \operatorname{grad} h, \psi_1 \rangle \Big\} \, |dx^1 dx^2| \end{aligned} \tag{7.36}$$

Now we investigate the classical vacuum states—the minimal energy field configurations. As we discussed in Lecture 3 these occur when $\psi = 0$ and $\phi \equiv \phi_0$ is constant. If ϕ_0 is at a minimum of the potential V—in other words, $V(\phi_0) = 0$—then ϕ_0 is a critical point of h. So the moduli space of vacua is the critical point set of h. A vacuum is invariant under supersymmetry, i.e., the vector field $\hat{\zeta}$ vanishes at a vacuum, as is obvious from (7.35). Thus we say that *supersymmetry is unbroken* in these vacua. It is natural to assume that h is a Morse function, or perhaps a generalized Morse function in the sense of Bott. Then we might have critical manifolds, and at vacua which lie on such manifolds there are massless scalars and massless fermions. The fact that supersymmetry is unbroken means that these come in supersymmetric pairs, i.e., in multiplets. More generally, the masses of the scalar fields and of the fermions agree at such a vacuum. This is easily verified by computing the masses directly from the lagrangian (7.31). Or, recall our formulas (5.41). For the scalar field ϕ, the squares of the masses are the eigenvalues of the Hessian of the potential V, which at a critical point of h are the squares of the eigenvalues of $\operatorname{Hess} h$. On the other hand, the mass matrix M for the fermions is $\operatorname{Hess} h$, and so the fermion mass squares are again the squares of the eigenvalues of the Hessian.

We can also consider a vacuum which is a local minimum of V, but not a global minimum. (A critical point of V which is not a local minimum is unstable.) For such a local minimum ϕ_0 we need to shift the energy by $-V(\phi_0)$ in order to have zero energy for the vacuum. Since

$$dV = \langle \operatorname{grad} h, \nabla \operatorname{grad} h \rangle,$$

such critical points of V occur when $\operatorname{grad} h$ is nonzero but in the kernel of $\operatorname{Hess} h$. Such a point could be isolated, and at such a vacuum we might have no massless scalar fields. But there is a massless fermion since the fermion mass matrix $\operatorname{Hess} h$ has a kernel. In fact, this massless fermion can be predicted from the supersymmetry. Notice that this vacuum is *not* supersymmetric—the vector field $\hat{\zeta}$ in (7.35) does not vanish there. So we say that *supersymmetry is broken* in this vacuum. Notice that the Poincaré symmetry is always assumed to be unbroken in a vacuum, but as we see this is not true for the *super* Poincaré symmetry. Now just as broken (even) global symmetries lead to massless bosons—Goldstone bosons—so too do broken odd symmetries lead to massless fermions: *Goldstone fermions*.

Dimensional reduction to $n = 2$ dimensions

We dimensionally reduce the model in components, though we could dimensionally reduce the superspacetime formulation to obtain a superspacetime formulation in $n = 2$ dimensions.

Let's begin by collecting the formulas we need. Restrict to fields f which satisfy $\partial_{12} f = 0$. (Recall from (6.22) that ∂_{12} is infinitesimal translation along a spatial direction.) As in Lecture 6 we use a basis of *lightlike* vector fields ∂_+, ∂_-. Also, for spinors recall that we use '+' for the index '1' and '−' for the index '2'. Then from (7.35) we find the supersymmetry transformations

$$\begin{aligned} \hat{\zeta}\phi &= -(\eta^+\psi_+ + \eta^-\psi_-) \\ \hat{\zeta}\psi_+ &= \eta^+\partial_+\phi + \eta^-\phi^* \operatorname{grad} h \\ \hat{\zeta}\psi_- &= \eta^-\partial_-\phi - \eta^+\phi^* \operatorname{grad} h \end{aligned} \tag{7.37}$$

and from (7.36) the supercharges

$$\begin{aligned} Q_+ &= \int_{x^0=\text{const}} \Big\{ \langle \partial_+\phi, \psi_+\rangle - \langle \phi^* \operatorname{grad} h, \psi_-\rangle \Big\}\, |dx^1| \\ Q_- &= \int_{x^0=\text{const}} \Big\{ \langle \partial_-\phi, \psi_-\rangle + \langle \phi^* \operatorname{grad} h, \psi_+\rangle \Big\}\, |dx^1|. \end{aligned} \tag{7.38}$$

Field theory in $n = 2$ dimensions is special for many reasons, among them the fact that spatial infinity is disconnected. This has the following consequence. Let $\mathcal{FE}_{\text{bos}}$ denote the space of static bosonic field configurations of finite energy. Now the energy density of a field $\phi(x^1)$ on space is (see (3.13))

$$\begin{aligned} \Theta &= \Big\{ \frac{1}{2}\,|\partial_1\phi|^2 + \frac{1}{2}\,\phi^*|\operatorname{grad} h|^2 \Big\}\, |dx^1| \\ &= \frac{1}{2}\,|\partial_1\phi \pm \phi^* \operatorname{grad} h|^2\, |dx^1| \;\mp\; d\phi^*(h). \end{aligned} \tag{7.39}$$

If a field $\phi(x^1)$ on space has finite energy, then the integral of the first term is finite, in which case $\phi(x^1)$ has limits as $x^1 \to \pm\infty$ which are critical points of h. The integral over space of the topological term is plus or minus the difference

$$Z = h\big(\phi(\infty)\big) - h\big(\phi(-\infty)\big) \tag{7.40}$$

of the limiting values. So the space $\mathcal{FE}_{\text{bos}}$ of finite energy field configurations splits into a disjoint union according to the limiting values at plus or minus infinity. The parameter space for the components is the Cartesian square $\pi_0 \operatorname{Crit}(h)^{\times 2}$ of the set of components of the critical point set. The *central charge* Z is constant on each component.

Example. A typical example is $X = \mathbb{R}$ with

$$h(\phi) = \frac{\phi^3}{3} - a^2\phi, \qquad a \in \mathbb{R}. \tag{7.41}$$

Then we have a quartic potential

$$V^{(0)}(\phi) = \frac{1}{2}(\phi^2 - a^2)^2 \tag{7.42}$$

with nondegenerate zeros at $\phi = \pm a^2$. So in the $n = 2$-dimensional theory $\mathcal{FE}_{\text{bos}}$ has four components.

In a "diagonal" component of $\mathcal{FE}_{\text{bos}}$, where $h(\infty) = h(-\infty)$, there are field configurations of zero energy: constant fields with values in the critical point set. (For an isolated critical point there is a unique such field.) These are the classical vacua. But in components where $h(\infty) \neq h(-\infty)$ there are no vacua. In any component the energy E, which is the spatial integral of the energy density Θ (7.39), is bounded below by the absolute value of the central charge:

$$E \geq |Z|. \tag{7.43}$$

From (7.39) we see that a field configuration of minimal energy satisfies the *first order* differential equation for a flow line:

$$\partial_1 \phi \pm \phi^* \operatorname{grad} h = 0. \tag{7.44}$$

It is easy to verify that a solution to (7.44) necessarily satisfies the *second order* Euler-Lagrange equation of motion.

In general, recall that nonvacuum field configurations of locally minimal energy are called solitons. Physically a classical soliton represents a stable localized lump of energy sitting still in space. Upon quantization we might expect that it gives a particle, and because of the inequality (4.34) we can expect it to be massive. In more detail: When we quantize the theory we need to quantize the space $\mathcal{M}$ of all finite energy classical solutions to the field equations. This space divides into components as indicated above. The lowest energy state in a diagonal component is a vacuum of zero energy. The free approximation around the vacuum give massless and massive particles according to the Hessian of V, at least in perturbation theory. Nondiagonal components have no state of zero mass. Rather, the states of smallest mass saturate the inequality (7.43) (where we replace energy by mass) and form a symplectic manifold on which the Poincaré group acts. The free approximation now involves quantization of this symplectic manifold as well as quantization of the quadratic approximation in its infinite dimensional normal bundle. What we expect, then, is a collection of *quantum solitons*. They are massive, and in perturbation theory their mass is very large compared to the masses of the particles constructed from small fluctuations. Of course, this story obtains quantum corrections as we move away from the free approximation.

The discussion so far has ignored the fermion and the supersymmetry. In fact, the central charge Z appears in the supersymmetry algebra as was anticipated in (6.26), where we discussed a central extension of the supersymmetry algebra $\mathfrak{p}^{2|(1,1)}$. Now we realize the central extension classically by computing the Poisson bracket $\{Q_+, Q_-\}$ of the supersymmetry charges. The Poisson bracket of Noether charges can be computed in different ways, and is in fact the spatial integral of a Poisson bracket of Noether currents. We use the explicit formulas (7.38) and note that the nontrivial contributions come from the brackets $\{\psi_+, \psi_+\}$ and $\{\psi_-, \psi_-\}$. Note that we compute on-shell, that is, assuming the equations of motion to be satisfied. The result is

(7.45)
$$\begin{aligned}\frac{1}{2}\{Q_+,Q_-\} &= \frac{1}{2}\int_{x^0=\text{const}}\Big\{\langle\partial_+\phi,\phi^*\operatorname{grad}h\rangle - \langle\partial_-\phi,\phi^*\operatorname{grad}h\rangle\Big\}\,|dx^1| \\ &= \int_{x^0=\text{const}}\langle\partial_1\phi,\phi^*\operatorname{grad}h\rangle\,|dx^1| \\ &= \int_{x^0=\text{const}}\partial_1(\phi^*h)\,|dx^1| \\ &= Z.\end{aligned}$$

So we obtain a central extension of the supersymmetry algebra, as promised. The energy inequality (7.43) is the classical version of (6.32), and it follows from the supersymmetry algebra as in (6.30). Field configurations which saturate this classical *BPS inequality* are annihilated by the supersymmetry $\tau_{Q_+}+\tau_{Q_-}$ or by $\tau_{Q_+}-\tau_{Q_-}$. This provides an alternative derivation—using supersymmetry—of the first order equation (7.44) for minimal energy bosonic field configurations. Namely, we look at the supersymmetry transformation (7.37) and ask that the vector field $\hat{\zeta}$ vanish. Inspecting the variation of the fermions ψ_+,ψ_- we learn that such *BPS field configurations* satisfy

(7.46)
$$\begin{aligned}\partial_+\phi \pm \phi^*\operatorname{grad}h &= 0\\ \pm\partial_-\phi - \phi^*\operatorname{grad}h &= 0.\end{aligned}$$

These equations are equivalent to the equations

(7.47)
$$\begin{aligned}\partial_0\phi &= 0\\ \partial_1\phi \mp \phi^*\operatorname{grad}h &= 0,\end{aligned}$$

which are the equations for a static flow line. We also obtain an equation on fermions by requiring that the variation of the boson ϕ vanish in (7.37):

(7.48)
$$\psi_+ \pm \psi_- = 0.$$

To analyze this we must also bring in the equations of motion for the fermions, which are

(7.49)
$$\begin{aligned}-\partial_+\psi_- &= R(\psi_+,\psi_-)\psi_+ + \phi^*(\nabla\operatorname{grad}h)(\psi_+)\\ \partial_-\psi_+ &= -R(\psi_-,\psi_+)\psi_- + \phi^*(\nabla\operatorname{grad}h)(\psi_-).\end{aligned}$$

Combining (7.47) and (7.48) we see that the curvature terms vanish by the Bianchi identity and we are left with equations for a single fermion ψ_+:

(7.50)
$$\begin{aligned}\partial_0\psi_+ &= 0\\ \partial_1\psi_+ \mp \phi^*(\nabla\operatorname{grad}h)(\psi_+) &= 0.\end{aligned}$$

Equations (7.50) are the variations of equations (7.46) in the direction ψ_+. Thus the BPS fermion field ψ_+ is an odd tangent vector to the manifold of flow lines.

Recapping: The BPS condition leads to a first order equation on bosons which implies the second order equation of motion. This demonstrates that classical fermionic fields, which are difficult to visualize in a geometric manner, can impact classical bosonic fields.

Dimensional reduction to $n = 1$

Classically, we reduce to a mechanical model by requiring the fields in the $n = 3$-dimensional model to be invariant under all spatial translations. We obtain a model with supersymmetry group $P^{1|2}$. It has a superspacetime formulation on $M^{1|2}$, which is the dimensional reduction of the superspacetime formulation in $n = 3$ dimensions. This theory has twice the minimal amount of supersymmetry. In particular, it has twice as much supersymmetry as the superparticle model considered in Lecture 4. There we indicated that the quantum Hilbert space of that model is the space of spinor fields on X. With twice as much supersymmetry we obtain instead the space of differential forms on X as the quantum Hilbert space; the $\mathbb{Z}/2$-grading is by the parity of the degree. In the minimal superparticle there is a single supercharge, and its quantization is the Dirac operator. Here there are two supercharges (7.38) and the corresponding quantum operators are, up to factors, the first order differential operators $d + d^*$ and $\sqrt{-1}(d - d^*)$. This is if we have no potential term. In case there is a potential term, then the second term in the supercharges (7.38) gives an additional term which is a combination of exterior and interior multiplication by dh. In this way we obtain the modification of the de Rham complex used by Witten in his study of Morse theory. The hamiltonian in this model is the Hodge laplacian on differential forms modified by lower order terms involving dh. The solitons we discussed in $n = 2$ dimensions are *instantons* in $n = 1$ dimension, and they enter into the Morse theory discussion.

The dimensional reduction from $n = 3$ dimensions to $n = 1$ has a global $SO(2)$ symmetry from spatial rotations. From the $n = 3$-dimensional point of view these symmetries are part of the Poincaré group; from the $n = 1$-dimensional point of view they are an example of an R-symmetry. In the quantum mechanical theory this global $SO(2)$ symmetry gives a $\mathbb{Z}$-grading on the quantum Hilbert space; it is simply the usual $\mathbb{Z}$-grading on differential forms.

Exercises

1. The Lorentz group in dimension 4 is isomorphic to $SL(2;\mathbb{C})$, so there are two complex conjugate two-dimensional spin representations S', S''. The vector representation V has complexification

$$V_{\mathbb{C}} \cong S' \otimes S''.$$

There is a unique minimal real spinor representation, which is the underlying real representation of S' (or of S''); its complexification is $S' \oplus S''$. Each of S', S'' has a skew form ϵ. Let $M^{4|4}$ be the super Minkowski spacetime built on the spin representation S.

(a) Take Q_1, Q_2 as a (complex) basis of S' and then the complex conjugate basis $\overline{Q}_{\dot 1}, \overline{Q}_{\dot 2}$ for S''. The corresponding left-invariant complex odd vector fields on $M^{4|4}$ are denoted D_a and $\overline{D}_{\dot a}$. The odd coordinates on $M^{4|4}$ corresponding to the basis are denoted θ^1, θ^2; the complex conjugates $\bar\theta^{\dot 1}, \bar\theta^{\dot 2}$ Use the notation $\partial_{a\dot b}$ for the complex even vector fields, using the isomorphism above. Write formulas for the D_a and their complex conjugates in terms of the coordinate vector fields. Write all of the brackets of the odd vector fields.

(b) Is there an infinitesimal R-symmetry?

(c) Write the vector fields $\partial_{a\dot{b}}$ in terms of the standard vector fields $\partial/\partial x^\mu$ (for some nice choice of basis; recall similar formulas (6.22) in the 3-dimensional case).

(d) Write the kinetic term for a spinor field with values in S. (Schematically the kinetic term, ignoring the factor $1/2$, is $\bar{\psi}\not{D}\psi$.) Your answer should involve indices, the symbols $\epsilon^{ab}, \epsilon^{\dot{a}\dot{b}}$, etc.

(e) Integration over all four odd coordinates is defined by the first line of the formula:

$$\begin{aligned}\int d^4\theta &= i^* \frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\bar{\theta}^{\dot{2}}}\frac{\partial}{\partial\bar{\theta}^{\dot{1}}} \\ &= \frac{1}{2}\, i^*\, \{D_1\overline{D}_{\dot{1}}\overline{D}_{\dot{2}}D_2 + D_2\overline{D}_{\dot{2}}\overline{D}_{\dot{1}}D_1\} - \Box i^* \\ &= \frac{1}{2}\, i^* \left\{D^2\overline{D}^2 + \overline{D}^2 D^2\right\} + \Box i^*.\end{aligned}$$

Verify the equality of the first line with the remaining two. Here $\Box$ is the wave operator (3.7), which is Poincaré-invariant. Finally, $D^2 = \frac{1}{2}\epsilon^{ab}D_aD_b$ and $\overline{D}^2 = \frac{1}{2}\epsilon^{\dot{a}\dot{b}}\overline{D}_{\dot{a}}\overline{D}_{\dot{b}}$ is the complex conjugate.

(f) How does all of this reduce to 3 dimensions?

(g) How does it reduce to 2 dimensions? (You should recover what you did in Problem Set 6.)

2. (a) Write the geometric data (5.30) and (5.34) which leads to the supersymmetric σ-model lagrangian (7.31).

(b) What happens in case $X = \mathbb{R}$ and the superpotential h is quadratic? Do we get a free theory? What is the particle content (masses and spins)?

(c) Consider the previous questions in the dimensionally reduced models as well.

3. (a) Derive (7.7). To do so let $\exp(-t\eta\tau_{Q_a})$ be the one-parameter group of diffeomorphisms of $M^{n|s}$ induced from the vector field $-\eta\tau_{Q_a}$, and consider its action by pullback on (7.6). Note that it preserves the left-invariant vector fields D^r. Now differentiate at $t = 0$, use the fact that right-invariant and left-invariant vector fields commute, and the fact that τ_{Q_a} and D_a agree on $M^n \subset M^{n|s}$.

(b) Apply (7.7) to compute the supersymmetry transformation laws (7.8).

(c) Compute the commutator of infinitesimal supersymmetry transformations $\hat{\zeta}, \hat{\zeta}'$. Explain why the curvature terms enter and why they are expected.

4. (a) Complete the derivation of the component lagrangian (7.22) from the superspacetime lagrangian (7.10).

(b) Compute the equations of motion.

(c) Compute the supercurrent (7.25).

(d) Fill in the details of the computation (7.45). Compute the Poisson bracket as the action of the vector field corresponding to Q_+ on Q_-. That vector field can be read off from (7.37), though to obtain its action on $\phi^*\operatorname{grad} h$ it is easier to use the dimensional reduction of the last equation in (7.8), supplemented by (7.30). You'll need the Bianchi identity and the equations of motion as well.

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