A time-reversal anomaly, bordism, and index theory

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University of Texas at Austin

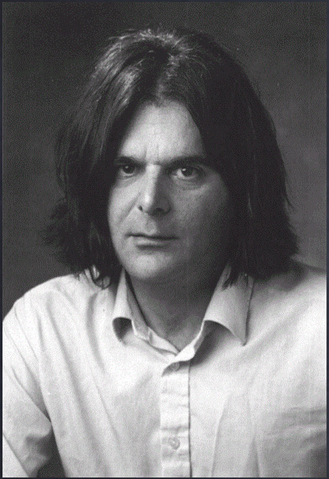
September 29, 2022

Joint work with Mike Hopkins

arXiv:1908.09916

The ideas in this talk touch on many periods of Graeme's mathematics:

Topological K-Theory $\rightsquigarrow$ Geometric Quantum Theory



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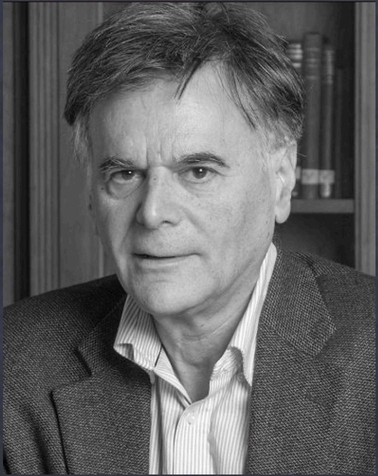

Theorems (joint with Hopkins)

Theorem A: The total anomaly $\alpha_{RS} \otimes \alpha_C$ of M-theory is trivializable

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Theorem B: The following six $\mathfrak{m}_c$-manifolds generate the group $\pi_{12} M\mathfrak{m}_c \otimes \mathbb{Z}_2$:

$$(W_0', \tilde{c}_0'), \quad (W_0'', 0), \quad (W_1, \lambda)$$

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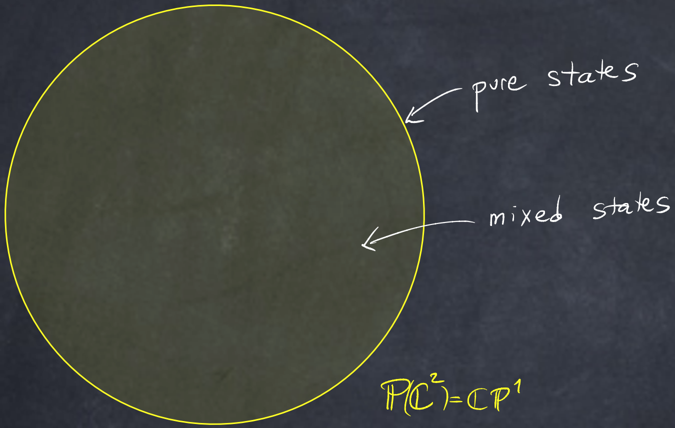

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So quantum geometry $(\mathbb{P}\mathcal{H}, p)$ is Fubini-Study geometry $(\mathbb{P}\mathcal{H}, d)$ of projective space

Recollection: the projectivity of a representation

A complex *projective* representation of a group G gives rise to a central extension

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A beautiful paper of Graeme tells what cohomology is appropriate for Lie groups G

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$\mathcal{F}$ can be a *collection* of fields; $\mathcal{F}(M)$ is the simplicial set of fields on an n-manifold M

Axiom System: An n-dimensional *field theory* F on background fields $\mathcal{F}$ is a homomorphism (symmetric monoidal functor)

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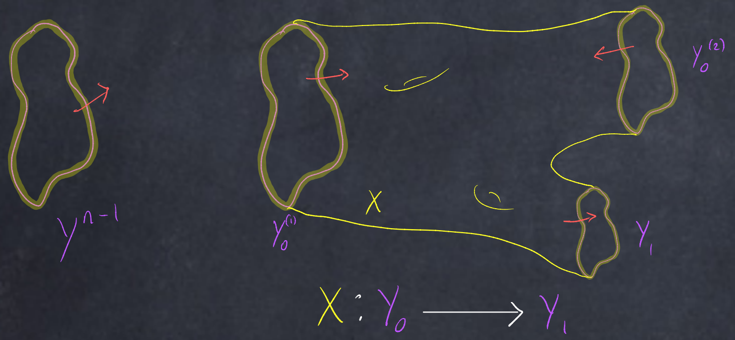

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- Field theories have a composition law (tensoring, stacking) with unit $\Longrightarrow$ *invertibility*

Invertible field theories and stable homotopy theory

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Takeaways: An invertible field theory is modeled as a spectrum map with domain a bordism spectrum (introduced by F-Hopkins-Teleman in `arXiv:0711.1909`)

An invertible field theory is a generalized "cocycle" on a bordism spectrum

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Note that $\mathcal{I}^{n+1}\big(\mathrm{Bord}_n(\mathcal{F})\big)$ consists of n-dimensional *once-categorifield theories*: a closed n-manifold maps to a complex line, the "categorification" of an invertible complex number

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These are called 't Hooft anomalies; their deformation classes contain powerful information

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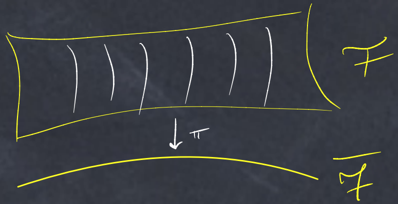

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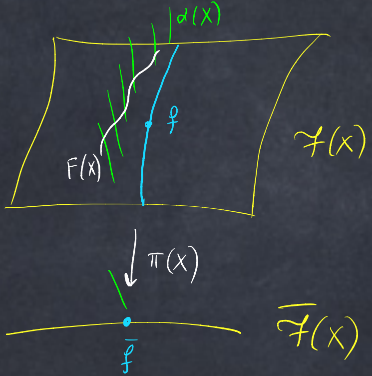

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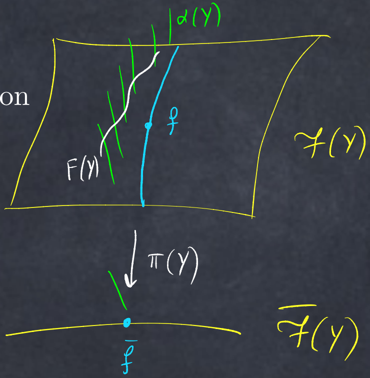

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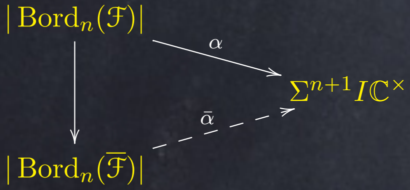
To carry out quantization we must *descend* the projectivity/anomaly α:



anomaly is obstruction to existence

descents form a torsor over n-dimensional theories

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- If the obstruction vanishes, one must specify descent data, which is a torsor over the abelian group of invertible field theories

Anomalies: summary

- Quantum theory is projective—the *'t Hooft anomaly* expresses the projectivity
- Quantization is linear—the *anomaly* obstructs quantization
- If the obstruction vanishes, one must specify descent data, which is a torsor over the abelian group of invertible field theories
- There is a well-developed theory of invertible field theories, so this part of quantum field theory is accessible using geometric and topological tools

Theorems (joint with Hopkins)

1 3 4 2

Theorem A: The total anomaly $\alpha_{RS} \otimes \alpha_C$ of M-theory is trivializable

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M-theory is a form of string theory; we treat it in a quantum field theoretic context

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The main work goes into the proof of the following bordism computation

Theorem B: The following six $\mathfrak{m}_c$-manifolds generate the group $\pi_{12} M\mathfrak{m}_c \otimes \mathbb{Z}_2$:

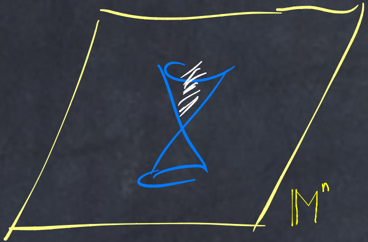
$$(W_0', \tilde{c}_0'), \quad (W_0'', 0), \quad (W_1, \lambda)$$

$$(K \times \mathbb{HP}^2, \lambda), \quad (\mathbb{RP}^4, \tilde{c}'_{\mathbb{RP}^4}) \times B, \quad (\mathbb{RP}^4 \# \mathbb{RP}^4, 0) \times B$$

I will explain how to pass from Theorem A to Theorem B and a bit more...

Time-reversal symmetry

The arena for relativistic quantum field theory is Minkowski spacetime $\mathbb{M}^n$: affine space with a Lorentz metric and time orientation



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Wick rotation: the theory is defined on (1) oriented manifolds, or (2) unoriented manifolds

M-Theory from 11d supergravity

Fields in M-theory ($\mathcal{F}$):

	ρ	pin$^+$ structure
	g	Riemannian metric
	ψ	Rarita-Schwinger field
	C	local 3-form, field strength is global closed 4-form

SUPERGRAVITY THEORY IN 11 DIMENSIONS

E. CREMMER, B. JULIA and J. SCHERK

Laboratoire de Physique Théorique de l'Ecole Normale Supérieure[+]
Paris, France

Abstract : We present the action and transformation laws of supergravity in 11 dimensions which is expected to be closely related to the O(8) theory in 4 dimensions after dimensional reduction.

LPTENS 78/10
March 1978

The Lagrangian we find is the following :

$$\mathcal{L} = -\frac{V}{4\kappa^2} R(\omega) - \frac{iV}{2}\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu\left(\frac{\omega+\hat{\omega}}{2}\right)\psi_\rho - \frac{V}{48} F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}$$

$$+\frac{\kappa V}{192}\left(\bar{\psi}_\mu \Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu + 12\,\bar{\psi}^\alpha\Gamma^{\gamma\delta}\psi^\beta\right)\left(F_{\alpha\beta\gamma\delta}+\hat{F}_{\alpha\beta\gamma\delta}\right)$$

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Definition: Let M be a pin^+ manifold. An $\mathfrak{m}_c$ *structure* on M is a w_1-twisted integer lift of $w_4(M)$. Compare: spin^c structure = integer lift of $w_2(M)$

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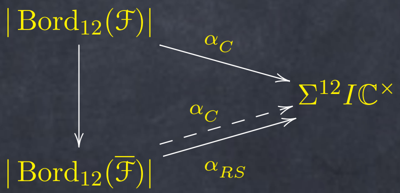

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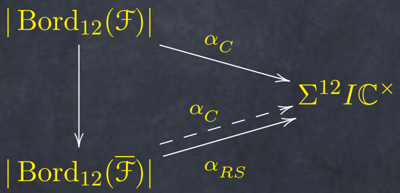

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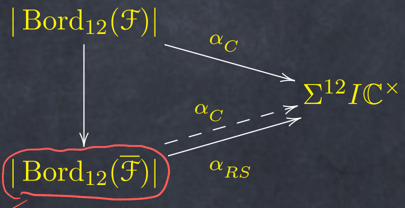

Fact: Anomaly theories α_{RS}, α_C are *topological* and *unitary*, so factor through $\mathcal{F}_{\mathrm{top}} = \{\mathrm{pin}^+, \mathfrak{m}_c\}$ and hence through the Thom spectrum $M\mathfrak{m}_c$

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General formula for the anomaly theory:

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The isomorphism class is a *differential* refinement; the formula is for the deformation class

Index theory on pin manifolds

Use the embeddings

$$\mathrm{Pin}_n^+ \hookrightarrow \mathrm{Spin}_{n,1}$$

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$\mathbb{RP}^{12}$ generates the first summand of *tangential* pin$^+$ bordism

$$\pi_{12}MT\mathrm{Pin}^+ \cong \mathbb{Z}/2^8\mathbb{Z} \oplus \mathbb{Z}/2^4\mathbb{Z} \oplus \mathbb{Z}/2^2\mathbb{Z}$$

and (Gilkey-Stolz)

$$\tau_{\mathbb{RP}^{12}} = \exp\left(\frac{2\pi i}{2^8}\right)$$

More generally, for $V \to W$ a real vector bundle over a closed pin$^+$ 12-manifold set

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Theorem (Zhang): Let $V \to W$ be a real vector bundle over a closed pin$^+$ 12-manifold W. Let $L \to W$ be the orientation real line bundle, $H \to \mathbb{RP}^{20}$ the tautological line bundle, and $\gamma\colon W \to \mathbb{RP}^{20}$ a map such that $\gamma^* H \cong L$. Then

$$\gamma_*([V]) = 2^{11}\,\frac{\eta_W(V)}{4}\,(1 - [H]) \qquad \text{in } \widetilde{KO}^0(\mathbb{RP}^{20})$$

The group $\widetilde{KO}^0(\mathbb{RP}^{20})$ is cyclic of order 2^{11} with generator $1 - [H]$

The Rarita-Schwinger anomaly: summary

- The partition function $\hat{\alpha}_{RS} \colon \pi_{12} MT\mathrm{Pin}^+ \to \mathbb{C}^\times$ of $\alpha_{RS} \colon MT\mathrm{Pin}^+ \to \Sigma^{12} I\mathbb{C}^\times$ is

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- There are several formulas/techniques to compute this topological invariant
- It turns out that the composition

$$\pi_{12}M\mathfrak{m}_c \longrightarrow \pi_{12}MT\mathrm{Pin}^+ \xrightarrow{\hat{\alpha}_{RS}} \mathbb{C}^\times$$

factors through $\mu_2 = \{\pm 1\} \subset \mathbb{C}^\times$

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The anomalous term in the action

The anomaly $\alpha_C \colon M\mathfrak{m}_c \to \Sigma^{12} I\mathbb{C}^\times$ is of order 2

It arises from the inhomogeneous skew-symmetric cubic form

$$\kappa(c) = \frac{c^3 - p \cdot c}{48}$$

in the action—the division by 24 is not anomalous

The Lagrangian we find is the following :

$$\mathcal{L} = -\frac{V}{4\kappa^2} R(\omega) - \frac{iV}{2} \bar{\Psi}_\mu \Gamma^{\mu\nu\rho} D_\nu\left(\frac{\omega+\hat{\omega}}{2}\right)\Psi_\rho - \frac{V}{48} F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}$$

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ELEVEN DIMENSIONAL ORIGIN OF STRING/STRING DUALITY: A ONE LOOP TEST[1]

M. J. Duff, James T. Liu and R. Minasian

Thus membrane/fivebrane duality predicts a spacetime correction to the $D = 11$ supermembrane action

$$I_{11}(\text{Lorentz}) = T_3 \int C_3 \wedge \frac{1}{(2\pi)^4}\left[-\frac{1}{768}(\mathrm{tr}R^2)^2 + \frac{1}{192}\mathrm{tr}R^4\right]. \qquad (3.14)$$

Unfortunately, since the correct quantization of the supermembrane is unknown, this prediction is difficult to check. However, by simultaneous dimensional reduction [33] of ($d = 3, D =$

Recollection: algebraic theory of a quadratic form

L	finitely generated free abelian group
$\langle -, - \rangle \colon L \times L \to \mathbb{Z}$	nondegenerate (i.e., unimodular) symmetric bilinear form
$\Downarrow$	
$\bar{c} \in L \otimes \mathbb{Z}/2\mathbb{Z}$	unique element such that $\langle \bar{x}, \bar{x} \rangle \equiv \langle \bar{c}, \bar{x} \rangle \pmod{2}, \qquad \bar{x} \in L \otimes \mathbb{Z}/2\mathbb{Z}$

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$\langle c, c \rangle \pmod 8$ is independent of $c \in L_{\mathrm{char}}$, so for any integer lift $\sigma \in \mathbb{Z}$ of $\langle c, c \rangle \pmod 8$,

$$\kappa_2(c) = \frac{\langle c, c \rangle - \sigma}{8}$$

is an integer $\qquad$ (σ may be chosen to be the signature of $\langle -, - \rangle$ on $L \otimes \mathbb{Q}$)

Algebraic theory of a cubic form

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$\langle -,-,-\rangle\colon L \times L \to \mathbb{Z}$	symmetric trilinear form
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Lemma 1: There exists a unique $\hat{p} \in L^* \otimes \mathbb{Z}/24\mathbb{Z}$ such that

$$\hat{p} \cdot \hat{x} \equiv 4\hat{x}^3 + 6\hat{c}\hat{x}^2 + 3\hat{c}^2\hat{x} \pmod{24}$$

for all $\hat{x} \in L \otimes \mathbb{Z}/24\mathbb{Z}$ and mod 24 reductions $\hat{c}$ of characteristic elements c.

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Lemma 2: Let $p \in L^*$ satisfy $p \equiv \hat{p} \pmod{24}$. Then

$$\frac{c^3 - p \cdot c}{24} \pmod{2}$$

lies in $\mathbb{Z}/2\mathbb{Z}$ and is independent of $c \in L_{\mathrm{char}}$. Furthermore, there exist lifts $p \in L^*$ of $\hat{p}$ such that this invariant vanishes, in which case

$$\kappa_3(c) = \frac{c^3 - p \cdot c}{48}$$

is an integer. Also, $\kappa_3(-c) = -\kappa_3(c)$.

Cubic form on a closed $\mathfrak{m}_c$ 12-manifold W

$$
\begin{aligned}
L &= H^4(W;\mathbb{Z}_{w_1})/\text{torsion} \\
L^* &= H^8(W;\mathbb{Z})/\text{torsion} \\
\langle x, y, z\rangle &= (x \smile y \smile z)[W] \\
\bar{c} &= w_4(W)
\end{aligned}
$$

Cubic form on a closed $\mathfrak{m}_c$ 12-manifold W

$$L = H^4(W; \mathbb{Z}_{w_1})/\text{torsion}$$
$$L^* = H^8(W; \mathbb{Z})/\text{torsion}$$
$$\langle x, y, z\rangle = (x \smile y \smile z)[W]$$
$$\bar{c} = w_4(W)$$

Proposition 1: In $B\mathrm{Pin}^+$ there is a unique characteristic class $\bar{p} \in H^8(B\mathrm{Pin}^+; \mathbb{Z})/\text{torsion}$ whose lift to $B\mathrm{Spin}$ satisfies

$$2p = p_2 - \lambda^2$$

where $2\lambda = p_1$. Then $\bar{c}$ and $\bar{p} \pmod{24}$ satisfy the previous conditions

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Proposition 2: Let W be a closed $\mathfrak{m}_c$ 12-manifold and $\tilde{c} \in H^4(W;\tilde{\mathbb{Z}})$ a w_1-twisted integer lift of $w_4(W)$. Then

$$\frac{\tilde{c}^3 - \bar{p}(W)\tilde{c}}{48} \pmod{\mathbb{Z}}$$

lies in $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$, is independent of the choice of $\tilde{c}$, and is a bordism invariant of $\mathfrak{m}_c$-manifolds

The C-field anomaly: summary

- The partition function $\hat{\alpha}_C\colon \pi_{12}M\mathfrak{m}_c \to \mathbb{C}^\times$ of $\alpha_C\colon M\mathfrak{m}_c \to \Sigma^{12}I\mathbb{C}^\times$ is

$$\hat{\alpha}_C(W) = \exp\left(2\pi i\,\frac{\tilde{c}^3 - \bar{p}(W)\tilde{c}}{48}\right)$$

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- $\hat{\alpha}_C$ is a very easy invariant to compute

Theorems (joint with Hopkins)

1 3 4 2

Theorem A: The total anomaly $\alpha_{RS} \otimes \alpha_C$ of M-theory is trivializable

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M-theory is a form of string theory; we treat it in a quantum field theoretic context

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The main work goes into the proof of the following bordism computation

Theorem B: The following six $\mathfrak{m}_c$-manifolds generate the group $\pi_{12} M\mathfrak{m}_c \otimes \mathbb{Z}_2$:

$$(W_0', \tilde{c}_0'), \quad (W_0'', 0), \quad (W_1, \lambda)$$

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I will explain how to pass from Theorem A to Theorem B and a bit more. . .

Anomaly cancellation for *spin* manifolds

1n 1996, Witten gave a conceptual argument for spin manifolds

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x	class in $H^4(W;\mathbb{Z})$
$V(x) \longrightarrow W$	real adjoint vector bundle to principal E_8-bundle with characteristic class
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We could not find a conceptual proof for the pin case, so we turned to computation

Strategy

Theorem A: The total anomaly $\alpha_{RS} \otimes \alpha_C$ of M-theory is trivializable

Recall that each anomaly theory α is determined by a bordism invariant $\hat{\alpha}\colon \pi_{12} M\mathfrak{m}_c \to \mathbb{C}^\times$

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We apply the Adams spectral sequence together with much computer aid, in particular using a program by Rob Bruner as well as Mathematica

Then we searched for 12-dimensional $\mathfrak{m}_c$-manifolds which represent the algebraic generators

Generators of the $\mathfrak{m}_c$ bordism group

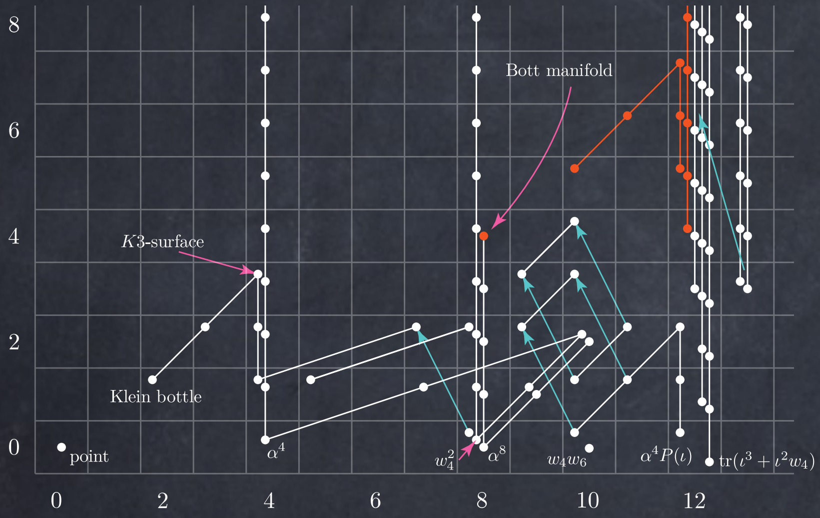
Theorem B: The following six $\mathfrak{m}_c$-manifolds generate the group $\pi_{12} M\mathfrak{m}_c \otimes \mathbb{Z}_2$:

$$(W_0', c_0), \quad (W_0'', 0), \quad (W_1, \lambda)$$
$$(K \times \mathbb{HP}^2, \lambda), \quad (\mathbb{RP}^4, c_{\mathbb{RP}^4}) \times B, \quad (\mathbb{RP}^4 \# \mathbb{RP}^4, 0) \times B.$$

K	$K3$ surface
$\mathbb{HP}^2$	quaternionic projective plane
B	Bott manifold
$\mathbb{HP}^2 \# \mathbb{HP}^2 \longrightarrow W_0' \longrightarrow \mathbb{RP}^4$	$S^4 \times (\mathbb{HP}^2 \# \mathbb{HP}^2) \xrightarrow{2:1} W_0'$
$\mathbb{RP}^8 \longrightarrow W_0'' = \mathbb{P}(K_{\mathbb{R}}^{\oplus 2} \oplus \underline{\mathbb{R}}) \xrightarrow{\rho} S^4$	$K_{\mathbb{R}} \to S^4$ generating $\mathbb{H}$-line bundle
$\mathbb{HP}^2 \longrightarrow W_1 \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$	$\mathcal{B}_{SO}\big(\mathcal{O}(1,1)_{\mathbb{R}} \oplus \underline{\mathbb{R}} \to \mathbb{CP}^1 \times \mathbb{CP}^1\big)$ $SO_3 \cong \mathbb{P}\,Sp_1 \circlearrowright \mathbb{HP}^2$

Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathbf{A}}^{s,t}(H^* M\mathfrak{m}_c, \mathbb{Z}/2\mathbb{Z}) \Rightarrow \pi_{t-s} M\mathfrak{m}_c \otimes \mathbb{Z}_2$$



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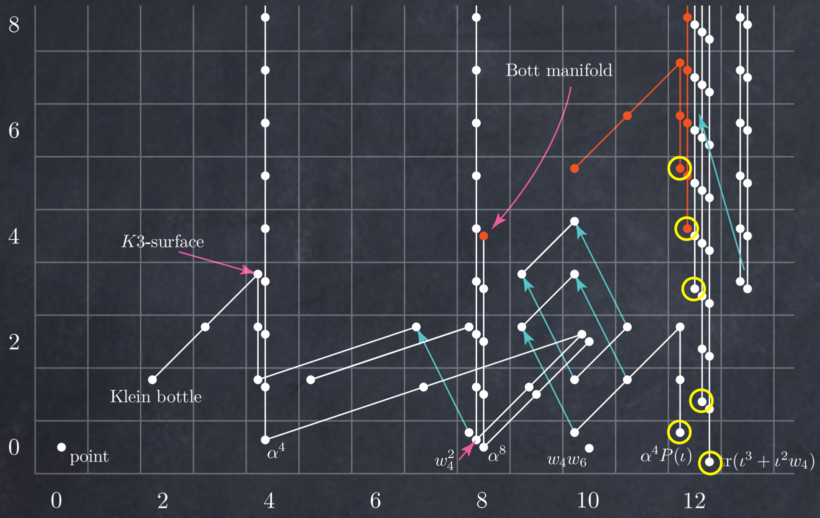

The invariants on the generators

$(W, \tilde{c})$	$\alpha_{RS}(W)$	$\alpha_C(W)$
$(W_0', \tilde{c}_0')$	$+1$	$+1$
$(W_0'', 0)$	$+1$	$+1$
(W_1, λ)	$+1$	$+1$
$(K \times \mathbb{HP}^2, \lambda)$	-1	-1
$(\mathbb{RP}^4, \tilde{c}'_{\mathbb{RP}^4})$	$+1$	$+1$
$(\mathbb{RP}^4 \# \mathbb{RP}^4, 0)$	$+1$	$+1$

Uniqueness

Trivializations of $\alpha_{RS} \otimes \alpha_C$ form a torsor over 11-dimensional invertible theories

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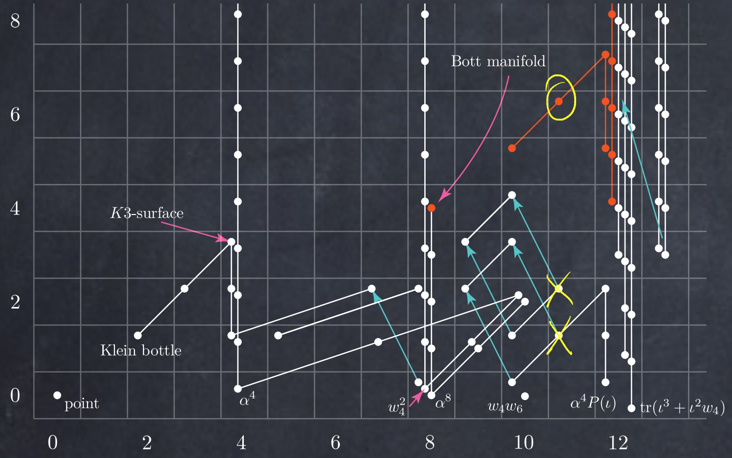
Theorem (Guo–Hopkins): There are two trivializations of $\alpha_{RS} \otimes \alpha_C$



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analytic algebraic topological

Happy Birthday, Graeme!