Extended topological field theory and the 2-dimensional Ising model

Dan Freed
University of Texas at Austin

June 22, 2018

Joint work with Constantin Teleman

arXiv:1806.00008

1988–90: New Avenues into QFT (Examples)

Topological Quantum Field Theory

Edward Witten*

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Abstract. A twisted version of four dimensional supersymmetric gauge theory is formulated. The model, which refines a nonrelativistic treatment by Atiyah, appears to underlie many recent developments in topology of low dimensional manifolds; the Donaldson polynomial invariants of four manifolds and the Floer groups of three manifolds appear naturally. The model may also be interesting from a physical viewpoint; it is in a sense a generally covariant quantum field theory, albeit one in which general covariance is unbroken, there are no gravitons, and the only excitations are topological.

1. Introduction

One of the dramatic developments in mathematics in recent years has been the program initiated by Donaldson of studying the topology of low dimensional manifolds via nonlinear classical field theory [1, 2]. Donaldson's work uses heavily the self-dual Yang-Mills equations, which were first introduced by

1988–90: New Avenues into QFT (Examples)

Topological Quantum Field Theory

Edward Witten*
School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Abstract. A twisted version of four dimensional supersymmetric gauge theory is formulated. The model, which refines a nonrelativistic treatment by Atiyah, appears to underlie many recent developments in topology of low dimensional manifolds; the Donaldson polynomial invariants of four manifolds and the Floer groups of three manifolds appear naturally. The model may also be interesting from a physical viewpoint; it is in a sense a generally covariant quantum field theory, albeit one in which general covariance is unbroken, there are no gravitons, and the only excitations are topological.

1. Introduction

One of the dramatic developments in mathematics in recent years has been the
progran
manifol
heavily

Topological Sigma Models

Edward Witten*
School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Abstract. A variant of the usual supersymmetric nonlinear sigma model is described, governing maps from a Riemann surface Σ to an arbitrary almost complex manifold M. It possesses a fermionic BRST-like symmetry, conserved for arbitrary Σ, and obeying $Q^2 = 0$. In a suitable version, the quantum ground states are the $1+1$ dimensional Floer groups. The correlation functions of the BRST-invariant operators are invariants (depending only on the homotopy type of the almost complex structure of M) similar to those that have entered in recent work of Gromov on symplectic geometry. The model can be coupled to dynamical gravitational or gauge fields while preserving the fermionic symmetry; some observations by Atiyah suggest that the latter coupling may be related to the Jones polynomial of knot theory. From the point of view of string theory, the main novelty of this type of sigma model is that the graviton vertex operator is a BRST commutator. Thus, models of this type may correspond to a realization at the level of string theory of an unbroken phase of quantum gravity.

1. Introduction

In recent years, Yang–Mills instantons have played an important role in the study of four manifolds and three manifolds in the work of Donaldson [1] and Floer [2], respectively. More recently, Atiyah advocated an interpretation of Floer theory

1988–90: New Avenues into QFT (Examples)

Topological Quantum Field Theory

Edward Witten*
School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, U

Abstract. A twisted version of four dimensional supersymmetric gauge the
is formulated. The model, which refines a nonrelativistic treatment by Atiy
appears to underlie many recent developments in topology of low dimensio
manifolds; the Donaldson polynomial invariants of four manifolds and
Floer groups of three manifolds appear naturally. The model may also
interesting from a physical viewpoint; it is in a sense a generally covari
quantum field theory, albeit one in which general covariance is unbroken, th
are no gravitons, and the only excitations are topological.

1. Introduction

One of the dramatic developments in mathematics in recent years has been
progran
manifol
heavily

TOPOLOGICAL GRAVITY

Edward Witten*
School of Natural Sciences
Institute for Advanced Study
Princeton, NJ 08540

ABSTRACT

A version of conformal gravity is formulated with a local fermionic symmetry that is reminiscent of BRST invariance. It may have mathematical applications (gravitational counterpart of Donaldson theory) or physical ones (unbroken phase of general relativity).

Topological Sigma Models

Edward Witten*
School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 0

Abstract. A variant of the usual supersymmetric nonlinear sigma model is described, governing maps from a Riemann surface Σ to an arbitrary almost complex manifold M. It possesses a fermionic BRST-like symmetry, conserved for arbitrary Σ, and obeying $Q^2 = 0$. In a suitable version, the quantum ground states are the $1+1$ dimensional Floer groups. The correlation functions of the BRST-invariant operators are invariants (depending only on the homotopy type of the almost complex structure of M) similar to those that have entered in recent work of Gromov on symplectic geometry. The model can be coupled to dynamical gravitational or gauge fields while preserving the fermionic symmetry; some observations by Atiyah suggest that the latter coupling may be related to the Jones polynomial of knot theory. From the point of view of string theory, the main novelty of this type of sigma model is that the graviton vertex operator is a BRST commutator. Thus, models of this type may correspond to a realization at the level of string theory of an unbroken phase of quantum gravity.

1. Introduction

In recent years, Yang–Mills instantons have played an important role in the study of four manifolds and three manifolds in the work of Donaldson [1] and Floer [2], respectively. More recently, Atiyah advocated an interpretation of Floer theory

1988–90: New Avenues into QFT (Examples)

Topological Quantum Field Theory

Edward Witten*
School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, U

Abstract. A twisted version of four dimensional supersymmetric gauge the
is formulated. The model, which refines a nonrelativistic treatment by Atiy
appears to underlie many recent developments in topology of low dimensio
manifolds; the Donaldson polynomial invariants of four manifolds and
Floer groups of three manifolds appear naturally. The model may also
interesting from a physical viewpoint; it is in a sense a generally covari
quantum field theory, albeit one in which general covariance is unbroken, th
are no gravitons, and the only excitations are topological.

1. Introduction

One of the dramatic developments in mathematics in recent years has been
progran
manifol

TOPOLOGICAL GRAVITY

Edward Witten*
School of Natural Sciences
Institute for Advanced Study
Princeton, NJ 08540

Topological Sigma Models

Edward Witten*
School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 0

Abstract. A variant of the usual supersymmetric nonlinear sigma model is described, governing maps from a Riemann surface Σ to an arbitrary almost complex manifold M. It possesses a fermionic BRST-like symmetry, conserved for arbitrary Σ, and obeying $Q^2 = 0$. In a suitable version, the quantum ground states are the $1+1$ dimensional Floer groups. The correlation functions of the BRST-invariant operators are invariants (depending only on the homotopy type of the almost complex structure of M) similar to those that have entered in recent work of Gromov on symplectic geometry. The model can be coupled to dynamical gravitational or gauge fields while preserving the fermionic symmetry; some observations by Atiyah suggest that the latter coupling may be related to the Jones polynomial of knot theory. From the point of view of string theory, the main novelty of this type of sigma model is that the graviton vertex operator is a BRST commutator. Thus, models of this type may correspond to a realization at the level of string theory of an unbroken phase of quantum gravity.

1. Introduction

In recent years, Yang–Mills instantons have played an important role in the study of four manifolds and three manifolds in the work of Donaldson [1] and Floer [2], respectively. More recently, Atiyah advocated an interpretation of Floer theory

Quantum Field Theory and the Jones Polynomial*

Edward Witten**
School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Abstract. It is shown that $2+1$ dimensional quantum Yang-Mills theory, with an action consisting purely of the Chern-Simons term, is exactly soluble and gives a natural framework for understanding the Jones polynomial of knot theory in three dimensional terms. In this version, the Jones polynomial can be generalized from S^3 to arbitrary three manifolds, giving invariants of three manifolds that are computable from a surgery presentation. These results shed a surprising new light on conformal field theory in $1+1$ dimensions.

In a lecture at the Hermann Weyl Symposium last year [1], Michael Atiyah proposed two problems for quantum field theorists. The first problem was to give a physical interpretation to Donaldson theory. The second problem was to find an intrinsically three dimensional definition of the Jones polynomial of knot theory. These two problems might roughly be described as follows.

Donaldson theory is a key to understanding geometry in four dimensions.

Invariants of Knots and 3-Manifolds: A Puzzle

Two very distinct starting points—relationship between them?

Invariants of Knots and 3-Manifolds: A Puzzle

Two very distinct starting points—relationship between them?

Quantum Field Theory and the Jones Polynomial *

Edward Witten **

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Abstract. It is shown that $2+1$ dimensional quantum Yang-Mills theory, with an action consisting purely of the Chern-Simons term, is exactly soluble and gives a natural framework for understanding the Jones polynomial of knot theory in three dimensional terms. In this version, the Jones polynomial can be generalized from S^3 to arbitrary three manifolds, giving invariants of three manifolds that are computable from a surgery presentation. These results shed a surprising new light on conformal field theory in $1+1$ dimensions.

In a lecture at the Hermann Weyl Symposium last year [1], Michael Atiyah proposed two problems for quantum field theorists. The first problem was to give a physical interpretation to Donaldson theory. The second problem was to find an intrinsically three dimensional definition of the Jones polynomial of knot theory. These two problems might roughly be described as follows.

Donaldson theory is a key to understanding geometry in four dimensions.

Characteristic forms and geometric invariants

By Shiing-shen Chern and James Simons*

1. Introduction

This work, originally announced in [4], grew out of an attempt to derive a purely combinatorial formula for the first Pontrjagin number of a 4-manifold. The hope was that by integrating the characteristic curvature form (with respect to some Riemannian metric) simplex by simplex, and replacing the integral over each interior by another on the boundary, one

$$CS(A) = \int_M \langle A \wedge dA \rangle - \frac{1}{6} \langle A \wedge [A \wedge A] \rangle \qquad (\text{mod } 1)$$

Invariants of Knots and 3-Manifolds: A Puzzle

Two very distinct starting points—relationship between them?

Ribbon Graphs and Their Invariants Derived from Quantum Groups

N. Yu. Reshetikhin and V. G. Turaev
L.O.M.I., Fontanka 27, SU-191011 Leningrad, USSR

Abstract. The generalization of Jones polynomial of links to the case of graphs in R^3 is presented. It is constructed as the functor from the category of graphs to the category of representations of the quantum group.

1. Introduction

The present paper is intended to generalize the Jones polynomial of links and the related Jones-Conway and Kauffman polynomials to the case of graphs in R^3.

Originally the Jones polynomial was defined for links of circles in R^3 via an astonishing use of von Neumann algebras (see [Jo]). Later on it was understood that this and related polynomials may be constructed using the quantum R-matrices (see, for instance, [Tu_1]). This approach enables one to construct similar invariants for coloured links, i.e. links each of whose components is provided with a module over a fixed algebra (see [Re_1], where the role of the algebra is played by the quantized universal enveloping algebra $U_q(G)$ of a semisimple Lie algebra G).

Invariants of 3-manifolds via link polynomials and quantum groups

N. Reshetikhin[1] and V.G. Turaev[2]
[1] Department of Mathematics, Harvard University, Cambridge, MA 02138, USA
[2] LOMI, Fontanka 27, Leningrad 191011, USSR

Oblatum 20-XII-1989 & 7-VII-1990

1. Introduction

The aim of this paper is to construct new topological invariants of compact oriented 3-manifolds and of framed links in such manifolds. Our invariant of (a link in) a closed oriented 3-manifold is a sequence of complex numbers parametrized by complex roots of 1. For a framed link in S^3 the terms of the sequence are equale to the values of the (suitably parametrized) Jones polynomial of

$$XY - YX = \frac{K^2 - K^{-2}}{t^2 - t^{-2}}$$

$$XK = t^{-2}KX, \quad YK = t^2KY, \quad KK^{-1} = K^{-1}K = 1$$

$$K^{4r} = 1 \qquad X^r = Y^r = 0$$

1988–90: New Avenues into QFT (Axioms)

The definition of conformal field theory

Graeme Segal

I shall propose a definition of 2-dimensional conformal field theory which I believe is equivalent to that used by Friedan et al. The idea arises from joint work with Quillen.

§1 The definition

The category $\mathcal{C}$ is defined as follows. There is a sequence of objects $\{C_n\}_{n \geq 0}$, where C_n is the disjoint union of a set of n

1988–90: New Avenues into QFT (Axioms)

The definition of conformal field theory

Graeme Segal

I shall propose a definition of 2-dimensional conformal field theory which I believe is equivalent to that used by Friedan et al. The idea arises from joint work with Quillen.

§1 The definition

The category $\mathcal{C}$ is defined as follows. There is a sequence of objects $\{C_n\}_{n\geq 0}$, where C_n is the disjoint union of a set of n

Great theorems are awesome; great definitions are transformational

1988–90: New Avenues into QFT (Axioms)

TOPOLOGICAL QUANTUM FIELD THEORIES

by MICHAEL ATIYAH

To René Thom on his 65th birthday.

1. Introduction

In recent years there has been a remarkable renaissance in the relation between Geometry and Physics. This relation involves the most advanced and sophisticated ideas on each side and appears to be extremely deep. The traditional links between the two subjects, as embodied for example in Einstein's Theory of General Relativity or in Maxwell's Equations for Electro-Magnetism are concerned essentially with classical fields of force, governed by differential equations, and their geometrical interpretation. The new feature of present developments is that links are being established between *quantum* physics and *topology*. It is no longer the purely *local* aspects that are involved but their *global* counterparts. In a very general sense this should not be too surprising. Both quantum theory and topology are characterized by discrete phenomena emerging from a continuous background. However, the realization that this vague philosophical view-point could be translated into reasonably precise and significant mathematical statements is mainly due to the efforts of Edward Witten who, in a variety of directions, has shown the insight that can be derived by examining the topological aspects of quantum field theories.

The best starting point is undoubtedly Witten's paper [11] where he explained

It will be clear how much this whole subject rests on the ideas of Witten. In formulating the axiomatic framework in § 2, I have also been following Graeme Segal who produced a very similar approach to conformal field theories [10]. Finally it seems appropriate to point out the major role that *cobordism* plays in these theories. Thus René Thom's most celebrated contribution to geometry has now a new and deeper relevance.

We come now to the promised axioms. A topological quantum field theory (QFT), in dimension d defined over a ground ring Λ, consists of the following data:

(A) A finitely generated Λ-module $Z(\Sigma)$ associated to each oriented closed smooth d-dimensional manifold Σ,

(B) An element $Z(M) \in Z(\partial M)$ associated to each oriented smooth $(d+1)$-dimensional manifold (with boundary) M.

These data are subject to the following axioms, which we state briefly and expand upon below:

(1) Z is *functorial* with respect to orientation preserving diffeomorphisms of Σ and M,

(2) Z is *involutory*, i.e. $Z(\Sigma^*) = Z(\Sigma)^*$ where Σ^* is Σ with opposite orientation and $Z(\Sigma)^*$ denotes the dual module (see below),

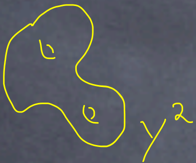
(3) Z is *multiplicative*.

We now elaborate on the precise meaning of the axioms. (1) means first that an orientation preserving diffeomorphism $f: \Sigma \to \Sigma'$ induces an isomorphism

3-Dimensional Finite Gauge Theory

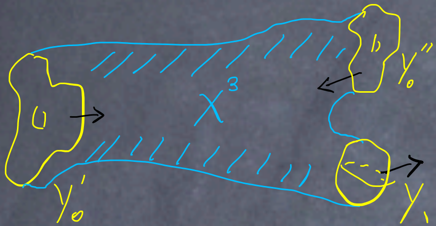
G	finite group
$\mathrm{Bord}_{\langle 2,3\rangle}$	(unoriented) bordism category
Vect	category of complex vector spaces
$\mathscr{G}_G\colon\ \mathrm{Bord}_{\langle 2,3\rangle} \longrightarrow \mathrm{Vect}$	symmetric monoidal functor

objects :



morphisms :

$Y_0' \sqcup Y_0'' \xrightarrow{X} Y_1$

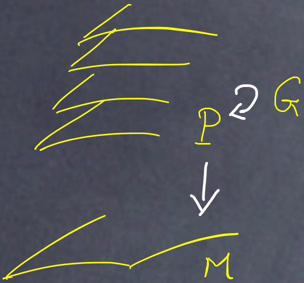


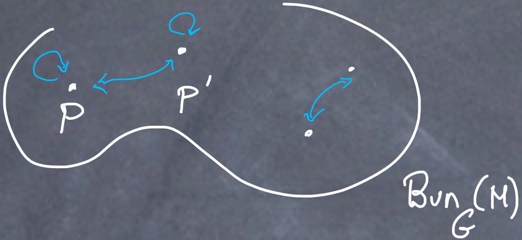
arrow of time on each component of ∂X

3-Dimensional Finite Gauge Theory

G	finite group
$\mathrm{Bord}_{\langle 2,3\rangle}$	(unoriented) bordism category
Vect	category of complex vector spaces
$\mathscr{G}_G\colon\ \mathrm{Bord}_{\langle 2,3\rangle} \longrightarrow \mathrm{Vect}$	symmetric monoidal functor

Finite path integral: $(P \to M) \in \mathrm{Bun}_G(M)$ fluctuating field





3-Dimensional Finite Gauge Theory

G	finite group
$\mathrm{Bord}_{\langle 2,3\rangle}$	(unoriented) bordism category
Vect	category of complex vector spaces
$\mathscr{G}_G\colon\ \mathrm{Bord}_{\langle 2,3\rangle} \longrightarrow \mathrm{Vect}$	symmetric monoidal functor

Finite path integral: $(P \to M) \in \mathrm{Bun}_G(M)$ fluctuating field

For X^3 closed sum the constant function 1:

$$\mathscr{G}_G(X) = \sum_{[P]\in\pi_0 \mathrm{Bun}_G(X)} \frac{1}{\#\,\mathrm{Aut}\,P}$$

3-Dimensional Finite Gauge Theory

G	finite group
$\mathrm{Bord}_{\langle 2,3\rangle}$	(unoriented) bordism category
Vect	category of complex vector spaces
$\mathscr{G}_G\colon\ \mathrm{Bord}_{\langle 2,3\rangle} \longrightarrow \mathrm{Vect}$	symmetric monoidal functor

Finite path integral: $(P \to M) \in \mathrm{Bun}_G(M)$ fluctuating field

For X^3 closed sum the constant function 1:

$$\mathscr{G}_G(X) = \sum_{[P]\in\pi_0 \mathrm{Bun}_G(X)} \frac{1}{\#\,\mathrm{Aut}\,P}$$

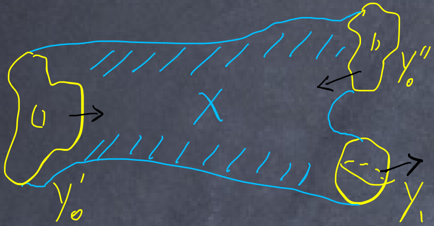
For Y^2 closed have canonical quantization:

$$\mathscr{G}_G(Y) = \mathrm{Fun}\big(\mathrm{Bun}_G(Y)\big)$$

$$\begin{tikzcd}
 & \mathrm{Bun}_G(X) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\
\mathrm{Bun}_G(Y_0) & & \mathrm{Bun}_G(Y_1)
\end{tikzcd}$$

Quantize (linearize) by push-pull:

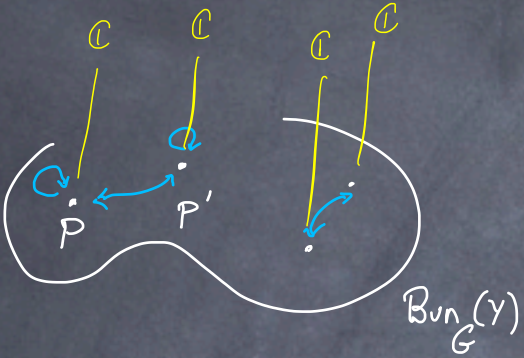
$$\mathscr{G}_G(X) = (r_1)_* \circ (r_0)^* \colon \mathscr{G}_G(Y_0) \longrightarrow \mathscr{G}_G(Y_1)$$



Extended Locality

$\mathscr{G}_G(Y)$ is a finite path integral... of the constant function $\mathbb{C}$:

$$\mathscr{G}_G(Y) \text{“}=\text{”} \int_{\mathrm{Bun}_G(Y)} \mathbb{C} \quad \cong \bigoplus_{[P]\in\pi_0 \mathrm{Bun}_G(Y)} \mathbb{C}$$



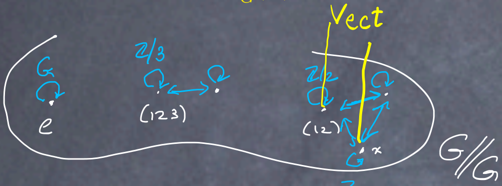
Extended Locality

$\mathscr{G}_G(Y)$ is a finite path integral. . . of the constant function $\mathbb{C}$:

$$\mathscr{G}_G(Y) \text{``}=\text{''} \int_{\mathrm{Bun}_G(Y)} \mathbb{C} \quad \cong \quad \bigoplus_{[P]\in\pi_0 \mathrm{Bun}_G(Y)} \mathbb{C}$$

Extend to lower dimensions: $\mathrm{Bun}_G(S^1) \simeq G/\!/G$

$$\mathscr{G}_G(S^1) \text{``}=\text{''} \int_{\mathrm{Bun}_G(Y)} \mathrm{Vect}$$



$G = \mathrm{Sym}_3$

Extended Locality

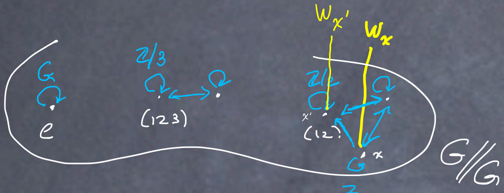
$\mathscr{G}_G(Y)$ is a finite path integral… of the constant function $\mathbb{C}$:

$$\mathscr{G}_G(Y) \text{“}=\text{”} \int_{\mathrm{Bun}_G(Y)} \mathbb{C} \quad \cong \quad \bigoplus_{[P]\in\pi_0 \mathrm{Bun}_G(Y)} \mathbb{C}$$

Extend to lower dimensions: $\mathrm{Bun}_G(S^1) \simeq G/\!/G$

$$\mathscr{G}_G(S^1) = \mathrm{Vect}_G(G)$$

$W_x \xrightarrow{\simeq} W_{x'}$



$G = \mathrm{Sym}_3$

Puzzle Solution

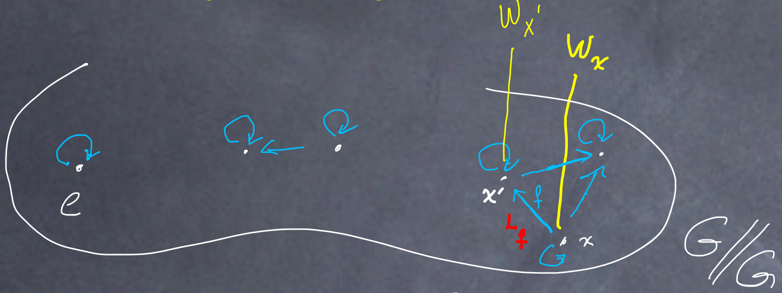
Cocycle for *level* $\lambda \in H^4(BG;\mathbb{Z})$ ... finite Chern-Simons theory

$$\mathscr{G}_G(X) = \sum_{[P]\in\pi_0\,\mathrm{Bun}_G(X)} \frac{\lambda(P)}{\#\,\mathrm{Aut}\,P}$$

Dijkgraaf-Witten

$$\mathscr{G}_G(Y) = \Gamma(L \to \mathrm{Bun}_G(Y))$$

$$\mathscr{G}_G(S^1) = \mathrm{Vect}^{\lambda}_G(G)$$



$$L_f \otimes W_x \xrightarrow{\cong} W_{x'}$$

Puzzle Solution

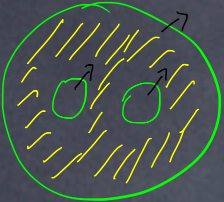
Cocycle for *level* $\lambda \in H^4(BG;\mathbb{Z})$ ... finite Chern-Simons theory

$$\mathscr{G}_G(X) = \sum_{[P]\in\pi_0 \operatorname{Bun}_G(X)} \frac{\lambda(P)}{\#\operatorname{Aut} P}$$

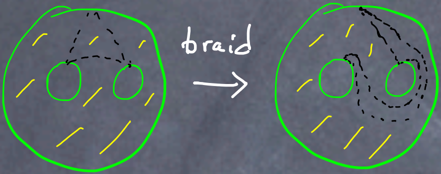
$$\mathscr{G}_G(Y) = \Gamma(L \to \operatorname{Bun}_G(Y))$$

$$\mathscr{G}_G(S^1) = \operatorname{Vect}^{\lambda}_G(G)$$

$\mathscr{G}_G(S^1)$: modular $\otimes$cat = module category for quasi-Hopf algebra



$\otimes$ structure



Puzzle Solution

Cocycle for *level* $\lambda \in H^4(BG;\mathbb{Z})$ ... finite Chern-Simons theory

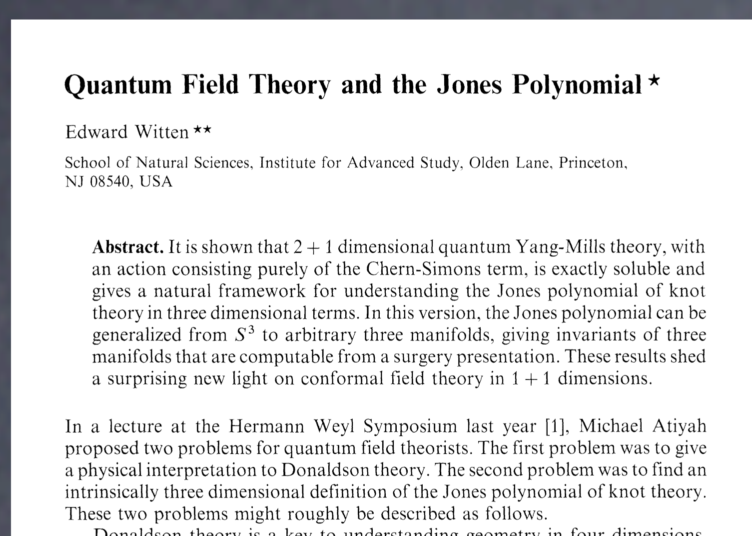
$$\mathscr{G}_G(X) = \sum_{[P]\in\pi_0 \operatorname{Bun}_G(X)} \frac{\lambda(P)}{\#\operatorname{Aut} P}$$

$$\mathscr{G}_G(Y) = \Gamma(L \to \operatorname{Bun}_G(Y))$$

$$\mathscr{G}_G(S^1) = \operatorname{Vect}^{\lambda}_G(G)$$

$\mathscr{G}_G(S^1)$: modular $\otimes$cat = module category for quasi-Hopf algebra

Solution: classical Chern-Simons $\xrightarrow{\text{quantize on } S^1}$ Hopf algebra



Quantum Field Theory and the Jones Polynomial ⋆

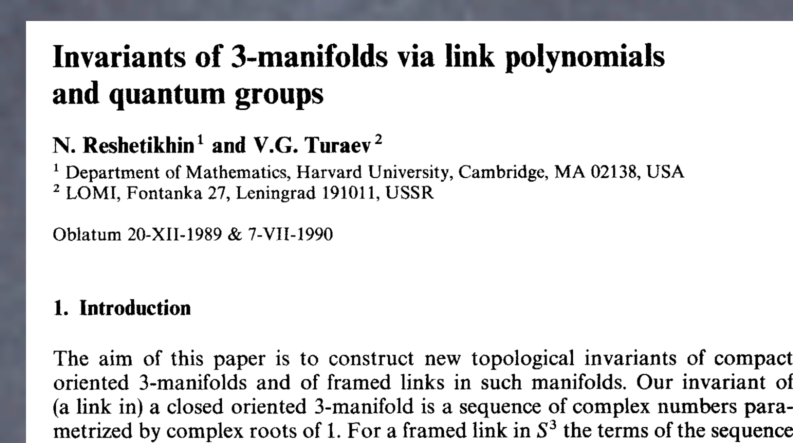
Edward Witten ⋆⋆

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Abstract. It is shown that $2+1$ dimensional quantum Yang-Mills theory, with an action consisting purely of the Chern-Simons term, is exactly soluble and gives a natural framework for understanding the Jones polynomial of knot theory in three dimensional terms. In this version, the Jones polynomial can be generalized from S^3 to arbitrary three manifolds, giving invariants of three manifolds that are computable from a surgery presentation. These results shed a surprising new light on conformal field theory in $1+1$ dimensions.

In a lecture at the Hermann Weyl Symposium last year [1], Michael Atiyah proposed two problems for quantum field theorists. The first problem was to give a physical interpretation to Donaldson theory. The second problem was to find an intrinsically three dimensional definition of the Jones polynomial of knot theory. These two problems might roughly be described as follows.

Donaldson theory is a key to understanding geometry in four dimensions.



Invariants of 3-manifolds via link polynomials and quantum groups

N. Reshetikhin[1] and V.G. Turaev[2]

[1] Department of Mathematics, Harvard University, Cambridge, MA 02138, USA
[2] LOMI, Fontanka 27, Leningrad 191011, USSR

Oblatum 20-XII-1989 & 7-VII-1990

1. Introduction

The aim of this paper is to construct new topological invariants of compact oriented 3-manifolds and of framed links in such manifolds. Our invariant of (a link in) a closed oriented 3-manifold is a sequence of complex numbers parametrized by complex roots of 1. For a framed link in S^3 the terms of the sequence are equale to the values of the (suitably parametrized) Jones polynomial of

Line operators

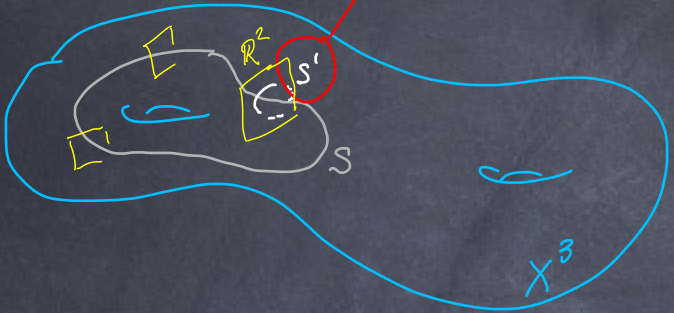
Extended field theory encodes extended operators (Kapustin)

Line operators

Extended field theory encodes extended operators (Kapustin)

$S \subset X$ coframed 1d submanifold of X^3 closed

Link S^1 used to label S by objects of $\mathscr{G}_G(S^1) = \mathrm{Vect}_G(G)$



Line operators

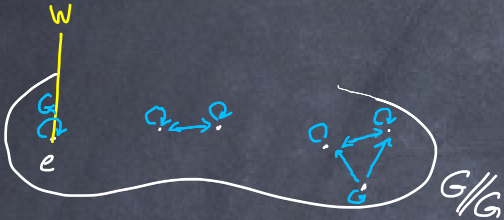
Extended field theory encodes extended operators (Kapustin)

$S \subset X$ coframed 1d submanifold of X^3 closed

Link S^1 used to label S by objects of $\mathscr{G}_G(S^1) = \mathrm{Vect}_G(G)$

Wilson loops: $\mathrm{Rep}(G) \approx$ full subcategory of $\mathrm{Vect}_G(G)$ with support at $e \in G$. Classical expression using holonomy with character χ:

$$\mathscr{G}_G\big(X; (S,\chi)_W\big) = \sum_{[P] \in \pi_0 \, \mathrm{Bun}_G(X)} \frac{h_{S,\chi}(P)}{\# \, \mathrm{Aut}\, P}.$$



Line operators

Extended field theory encodes extended operators (Kapustin)

$S \subset X$ coframed 1d submanifold of X^3 closed

Link S^1 used to label S by objects of $\mathscr{G}_G(S^1) = \mathrm{Vect}_G(G)$

Wilson loops: $\mathrm{Rep}(G) \approx$ full subcategory of $\mathrm{Vect}_G(G)$ with support at $e \in G$. Classical expression using holonomy with character χ:

$$\mathscr{G}_G\big(X; (S,\chi)_W\big) = \sum_{[P] \in \pi_0 \,\mathrm{Bun}_G(X)} \frac{h_{S,\chi}(P)}{\#\,\mathrm{Aut}\,P}.$$

't Hooft loops: Full subcategory of $\mathrm{Vect}_G(G)$ in which centralizers Z_x act trivially on fiber at $x \in G$. Classical model sums bundles on $X \setminus S$ with specified holonomy about S.

$$\mathscr{G}_G\colon\ \mathrm{Bord}_{\langle 1,2,3\rangle} \longrightarrow \mathrm{Cat}$$

2-categories

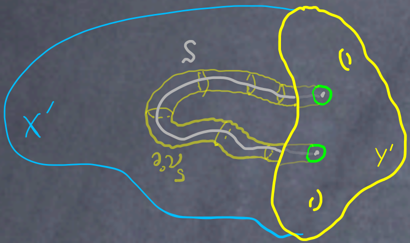
$$\mathscr{G}_G \colon \mathrm{Bord}_{\langle 1,2,3\rangle} \longrightarrow \mathrm{Cat}$$

If $\partial X \neq \emptyset$ there are line operators for neat 1d submanifolds $S \subset X$. Evaluate by cutting out tubular neighborhood ν_S.

$$\begin{tikzcd} S^1 \amalg S^1 \arrow[rr, bend left, "Y'"] \arrow[rr, bend right, "\partial_0 \nu_S"'] \arrow[rr, phantom, "\Downarrow X'"] & & \emptyset^1 \end{tikzcd}$$

$$X' = X \setminus \nu_S$$
$$Y' = X' \cap \partial X$$

Can evaluate explicitly on Wilson (parallel transport) and 't Hooft



Full Locality

$$\mathscr{G}_G : \ \mathrm{Bord}_{\langle 0,1,2,3\rangle} \longrightarrow ???$$

Open Question: Suitable codomain for general extended field theory?

Full Locality

$$\mathscr{G}_G \colon \mathrm{Bord}_{\langle 0,1,2,3\rangle} \longrightarrow ???$$

Open Question: Suitable codomain for general extended field theory?

Finite gauge theory: TensCat (complex linear tensor categories)

$$\mathscr{G}_G \colon \mathrm{Bord}_3 \longrightarrow \mathrm{TensCat}$$

Full Locality

$$\mathscr{G}_G : \ \mathrm{Bord}_{\langle 0,1,2,3\rangle} \longrightarrow ???$$

Open Question: Suitable codomain for general extended field theory?

Finite gauge theory: TensCat (complex linear tensor categories)

$$\mathscr{G}_G : \ \mathrm{Bord}_3 \longrightarrow \mathrm{TensCat}$$

General theory: Etingof-Gelaki-Nikshych-Ostrik
3-categorical aspects: Douglas-{Schommer-Pries}-Snyder

Full Locality

$$\mathscr{G}_G\colon\ \mathrm{Bord}_{\langle 0,1,2,3\rangle} \longrightarrow ???$$

Open Question: Suitable codomain for general extended field theory?

Finite gauge theory: TensCat (complex linear tensor categories)

$$\mathscr{G}_G\colon\ \mathrm{Bord}_3 \longrightarrow \mathrm{TensCat}$$

Quantize (finite path integral) $\mathrm{Bun}_G(\mathrm{pt}) \simeq \mathrm{pt}\,/\!/G$ to compute

$$\mathscr{G}_G(\mathrm{pt}) = \mathrm{Vect}[G]$$

$$(W' * W'')_x = \bigoplus_{x'x''=x} W'_{x'} \otimes W''_{x''}$$

Cobordism Hypothesis: Evaluation on a point is an equivalence

$$\mathrm{TFT}(\mathcal{C}) \longrightarrow \left[(\mathcal{C}^{\mathrm{fd}})^{\sim}\right]^{O_3}$$

$$\mathscr{F} \longmapsto \mathscr{F}(\mathrm{pt})$$

Baez-Dolan conjecture, Hopkins-Lurie in 2d, Lurie in general

On the Classification of Topological Field Theories

Jacob Lurie

Our goal in this article is to give an expository account of some recent work on the classification of topological field theories. More specifically, we will outline the proof of a version of the *cobordism hypothesis* conjectured by Baez and Dolan in [**2**].

Contents

Terminology 130
Disclaimer 130
Acknowledgements 130
1. Topological field theories and higher categories 131
1.1. Classical definitions 131
1.2. Extending down: lower dimensional manifolds 137
1.3. Higher category theory 148
1.4. Extending up: diffeomorphism groups 152
2. Formulation of the cobordism hypothesis 160
2.1. Complete Segal spaces 161
2.2. Bordism categories as Segal spaces 174
2.3. Fully dualizable objects 179
2.4. The cobordism hypothesis 186
2.5. The Mumford conjecture 194
3. Proof of the cobordism hypothesis 198
3.1. Inductive formulation 199

Cobordism Hypothesis: Evaluation on a point is an equivalence

$$\mathrm{TFT}(\mathcal{C}) \longrightarrow \left[(\mathcal{C}^{\mathrm{fd}})^{\sim}\right]^{O_3}$$
$$\mathscr{F} \longmapsto \mathscr{F}(\mathrm{pt})$$

Baez-Dolan conjecture, Hopkins-Lurie in 2d, Lurie in general

Warning: Need O_3-invariance data for unoriented theories

Cobordism Hypothesis: Evaluation on a point is an equivalence

$$\mathrm{TFT}(\mathcal{C}) \longrightarrow \left[(\mathcal{C}^{\mathrm{fd}})^{\sim}\right]^{O_3}$$
$$\mathscr{F} \longmapsto \mathscr{F}(\mathrm{pt})$$

Baez-Dolan conjecture, Hopkins-Lurie in 2d, Lurie in general

New approaches by Ayala-Francis and others

Cobordism Hypothesis: Evaluation on a point is an equivalence

$$\mathrm{TFT}(\mathcal{C}) \longrightarrow \left[(\mathcal{C}^{\mathrm{fd}})^{\sim}\right]^{O_3}$$
$$\mathscr{F} \longmapsto \mathscr{F}(\mathrm{pt})$$

Baez-Dolan conjecture, Hopkins-Lurie in 2d, Lurie in general

New approaches by Ayala-Francis and others

Construct a theory $\mathscr{R}_G \colon \mathrm{Bord}_3 \to \mathrm{TensCat}$ with

$$\mathscr{R}_G(\mathrm{pt}) = \mathrm{Rep}(G)$$

No classical model in general, Turaev-Viro state sum

Cobordism Hypothesis: Evaluation on a point is an equivalence

$$\mathrm{TFT}(\mathcal{C}) \longrightarrow \Big[(\mathcal{C}^{\mathrm{fd}})^{\sim}\Big]^{O_3}$$
$$\mathscr{F} \longmapsto \mathscr{F}(\mathrm{pt})$$

Baez-Dolan conjecture, Hopkins-Lurie in 2d, Lurie in general

New approaches by Ayala-Francis and others

Construct a theory $\mathscr{R}_G \colon \mathrm{Bord}_3 \to \mathrm{TensCat}$ with

$$\mathscr{R}_G(\mathrm{pt}) = \mathrm{Rep}(G)$$

No classical model in general, Turaev-Viro state sum

Pontrjagin dual group

A finite abelian: $\mathrm{Rep}(A) \simeq \mathrm{Vect}[A^\vee] \Longrightarrow \mathscr{R}_G \simeq \mathscr{G}_{A^\vee}$

(Nonabelian) Electromagnetic Duality

Theorem: There is a Morita equivalence $\mathrm{Vect}[G] \simeq \mathrm{Rep}(G)$, and iso

$$\mathcal{F} \colon \mathscr{G}_G \xrightarrow{\;\cong\;} \mathscr{R}_G$$

of field theories of *oriented* manifolds

(Nonabelian) Electromagnetic Duality

Theorem: There is a Morita equivalence $\mathrm{Vect}[G] \simeq \mathrm{Rep}(G)$, and iso

$$\mathfrak{F}\colon \mathscr{G}_G \xrightarrow{\ \cong\ } \mathscr{R}_G$$

of field theories of *oriented* manifolds

$\mathfrak{F}\colon \mathscr{F}_A \xrightarrow{\ \cong\ } \mathscr{F}_{A^\vee}$ is Fourier transform on states for Y^2 closed oriented:

$$\mathfrak{F}\colon\ \mathrm{Fun}\big(H^1(Y;A)\big) \xrightarrow{\ \cong\ } \mathrm{Fun}\big(H^1(Y;A^\vee)\big)$$

(Nonabelian) Electromagnetic Duality

Theorem: There is a Morita equivalence $\mathrm{Vect}[G] \simeq \mathrm{Rep}(G)$, and iso

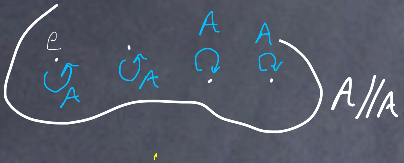
$$\mathcal{F}\colon \mathscr{G}_G \xrightarrow{\ \cong\ } \mathscr{R}_G$$

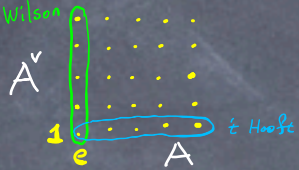
of field theories of *oriented* manifolds

$\mathcal{F}\colon \mathscr{F}_A \xrightarrow{\ \cong\ } \mathscr{F}_{A^\vee}$ is Fourier transform on states for Y^2 closed oriented:

$$\mathcal{F}\colon\ \mathrm{Fun}\big(H^1(Y;A)\big) \xrightarrow{\ \cong\ } \mathrm{Fun}\big(H^1(Y;A^\vee)\big)$$

Line operators: $\mathscr{G}_A(S^1) = \mathrm{Vect}_A(A) \approx \mathrm{Vect}(A \times A^\vee)$ duality $A \longleftrightarrow A^\vee$





Extended field theory ideas appear in many places in geometry, topology, and quantum field theory

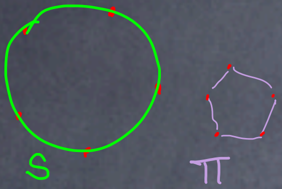
Now onto a new application to lattice models (w/Constantin Teleman)

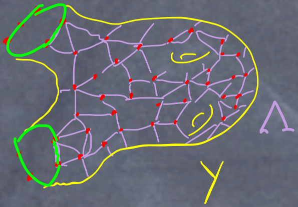
Latticed 1- and 2-manifolds

Definition:

(i) A *latticed 1-manifold* (S, Π) is a closed 1-manifold S equipped with a finite subset; $\Pi \subset S$ is an embedded graph, each component of which is a polygon with ≥ 2 sides.

(ii) A *latticed 2-manifold* (Y, Λ) is a compact 2-manifold Y equipped with a smoothly embedded finite graph $\Lambda \subset Y$ such that the closure of each face (component of $Y \setminus \Lambda$) is a smoothly embedded solid n-gon with $n \geq 2$. Furthermore, if e is an edge of Λ, then either (a) $e \cap \partial Y = \emptyset$, (b) $e \cap \partial Y$ is a single boundary vertex of e, or (c) $e \subset \partial Y$.





Latticed 1- and 2-manifolds

Definition:

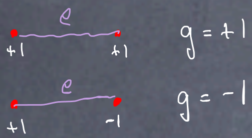
(i) A *latticed 1-manifold* (S, Π) is a closed 1-manifold S equipped with a finite subset; $\Pi \subset S$ is an embedded graph, each component of which is a polygon with ≥ 2 sides.

(ii) A *latticed 2-manifold* (Y, Λ) is a compact 2-manifold Y equipped with a smoothly embedded finite graph $\Lambda \subset Y$ such that the closure of each face (component of $Y \setminus \Lambda$) is a smoothly embedded solid n-gon with $n \geq 2$. Furthermore, if e is an edge of Λ, then either (a) $e \cap \partial Y = \emptyset$, (b) $e \cap \partial Y$ is a single boundary vertex of e, or (c) $e \subset \partial Y$.

- No choice of embedding of n-gons
- Loops are disallowed by the conditions
- Faces may share multiple edges

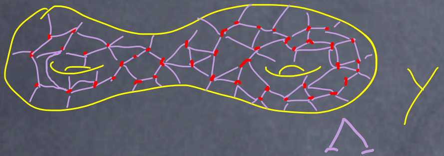
Ising model

$A = \mu_2 = \{\pm 1\}$	abelian group of "spins"
$\beta \in \mathbb{R}^{>0}$	inverse temperature
$\begin{aligned} \theta_\beta \colon A &\longrightarrow \mathbb{R}^{\geq 0} \\ \pm 1 &\longmapsto e^{\pm\beta} \end{aligned}$	weight function
$A^{\mathrm{Vert}(\Lambda)} = \mathrm{Map}\big(\mathrm{Vert}(\Lambda), A\big)$	configuration space of spins
$g \colon A^{\mathrm{Vert}(\Lambda)} \times \mathrm{Edge}(\Lambda) \to A$	ratio of boundary spins



Ising model

$A = \mu_2 = \{\pm 1\}$	abelian group of "spins"
$\beta \in \mathbb{R}^{>0}$	inverse temperature
$\theta_\beta \colon A \longrightarrow \mathbb{R}^{\geq 0}$ $\pm 1 \longmapsto e^{\pm\beta}$	weight function
$A^{\mathrm{Vert}(\Lambda)} = \mathrm{Map}\big(\mathrm{Vert}(\Lambda), A\big)$	configuration space of spins
$g \colon A^{\mathrm{Vert}(\Lambda)} \times \mathrm{Edge}(\Lambda) \to A$	ratio of boundary spins



Y closed:

$$I(Y,\Lambda) = \sum_{s \in A^{\mathrm{Vert}(\Lambda)}} \prod_{e \in \mathrm{Edge}(\Lambda)} \theta_\beta\big(g(s;e)\big)$$

This is the Ising *partition function*. Note limits $\beta \to \infty$, $\beta \to 0$.

Ising model

$A = \mu_2 = \{\pm 1\}$ — abelian group of "spins"

$\beta \in \mathbb{R}^{>0}$ — inverse temperature

$$\begin{aligned} \theta_\beta \colon A &\longrightarrow \mathbb{R}^{\geq 0} \\ \pm 1 &\longmapsto e^{\pm\beta} \end{aligned}$$ — weight function

$A^{\mathrm{Vert}(\Lambda)} = \mathrm{Map}\big(\mathrm{Vert}(\Lambda), A\big)$ — configuration space of spins

$g \colon A^{\mathrm{Vert}(\Lambda)} \times \mathrm{Edge}(\Lambda) \to A$ — ratio of boundary spins

The model can be defined for more general data:

G — finite group

$\theta \colon G \longrightarrow \mathbb{R}^{\geq 0}$ — admissible function

Ising model

$A = \mu_2 = \{\pm 1\}$	abelian group of "spins"
$\beta \in \mathbb{R}^{>0}$	inverse temperature
$\begin{aligned} \theta_\beta \colon A &\longrightarrow \mathbb{R}^{\geq 0} \\ \pm 1 &\longmapsto e^{\pm\beta} \end{aligned}$	weight function
$A^{\mathrm{Vert}(\Lambda)} = \mathrm{Map}\big(\mathrm{Vert}(\Lambda), A\big)$	configuration space of spins
$g\colon A^{\mathrm{Vert}(\Lambda)} \times \mathrm{Edge}(\Lambda) \to A$	ratio of boundary spins

Definition: Let G be a finite group. A function $\theta\colon G \to \mathbb{R}$ is *admissible* if (i) $\theta(g) \geq 0$ for all $g \in G$; (ii) $\theta(g^{-1}) = \theta(g)$ for all $g \in G$; and (iii) $\theta^\vee(\rho)$ is a nonnegative operator for each irreducible unitary representation $\rho\colon G \to \mathrm{Aut}(W)$.

Ising model

$A = \mu_2 = \{\pm 1\}$	abelian group of “spins”
$\beta \in \mathbb{R}^{>0}$	inverse temperature
$\theta_\beta \colon A \longrightarrow \mathbb{R}^{\geq 0}$ $\quad \pm 1 \longmapsto e^{\pm\beta}$	weight function
$A^{\mathrm{Vert}(\Lambda)} = \mathrm{Map}\big(\mathrm{Vert}(\Lambda), A\big)$	configuration space of spins
$g \colon A^{\mathrm{Vert}(\Lambda)} \times \mathrm{Edge}(\Lambda) \to A$	ratio of boundary spins

The group G acts by constant translation on $A^{\mathrm{Vert}(\Lambda)}$

$$s^h(v) = hs(v), \qquad s \in A^{\mathrm{Vert}(\Lambda)}, \quad h \in G, \quad v \in \mathrm{Vert}(\Lambda)$$

preserving the function g, so as a *symmetry* of the Ising model

Probabilistic interpretation:

$$\delta_s = \frac{\prod_{e \in \mathrm{Edge}(\Lambda)} \theta_\beta\big(g(s;e)\big)}{I(Y,\Lambda)}$$

is a probability measure on $A^{\mathrm{Vert}(\Lambda)}$

Probabilistic interpretation:

$$\delta_s = \frac{\prod_{e \in \mathrm{Edge}(\Lambda)} \theta_\beta\big(g(s;e)\big)}{I(Y,\Lambda)}$$

is a probability measure on $A^{\mathrm{Vert}(\Lambda)}$

$\beta \to 0$	uniform measure	paramagnetic
$\beta \to \infty$	support at 2 points	ferromagnetic

Probabilistic interpretation:

$$\delta_s = \frac{\prod_{e \in \mathrm{Edge}(\Lambda)} \theta_\beta\big(g(s;e)\big)}{I(Y,\Lambda)}$$

is a probability measure on $A^{\mathrm{Vert}(\Lambda)}$

$\beta \to 0$	uniform measure	paramagnetic
$\beta \to \infty$	support at 2 points	ferromagnetic

Expectation value of a function

$$f\colon A^{\mathrm{Vert}(\Lambda)} \longrightarrow \mathbb{C}$$

such as $f(s) = s(v_1)s(v_2)$ for vertices v_1, v_2 (*order operator*):

$$\langle f \rangle = \sum_{s \in A^{\mathrm{Vert}(\Lambda)}} f(s)\delta_s$$

Quantum mechanical interpretation (Wick-rotated time):

Construct a functor

$$I:\ \mathrm{Bord}^{\mathrm{latticed}}_{\langle 1,2\rangle} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

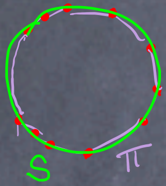
Quantum mechanical interpretation (Wick-rotated time):

Construct a functor

$$I\colon\ \mathrm{Bord}^{\mathrm{latticed}}_{\langle 1,2\rangle} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

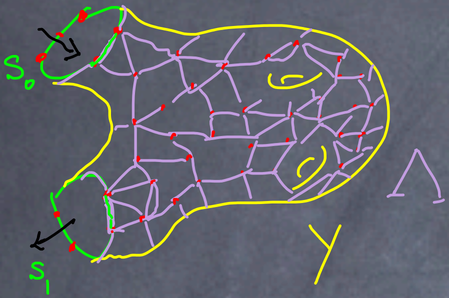
Objects: closed latticed 1-manifold (S,Π) maps to the vector space

$$I(S,\Pi)=\mathrm{Fun}(A^{\mathrm{Vert}\,(\Pi)})=\mathrm{Map}(A^{\mathrm{Vert}\,(\Pi)},\mathbb{C})$$



Morphisms: 2d latticed bordism $(Y, \Lambda)\colon (S_0, \Pi_0) \to (S_1, \Pi_1)$ gives a correspondence diagram of spin configuration spaces

$$\begin{tikzcd}[column sep=small] & A^{\mathrm{Vert}(\Lambda)} \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\ A^{\mathrm{Vert}(\Pi_0)} & & A^{\mathrm{Vert}(\Pi_1)} \end{tikzcd}$$



Morphisms: 2d latticed bordism $(Y,\Lambda)\colon (S_0,\Pi_0)\to(S_1,\Pi_1)$ gives a correspondence diagram of spin configuration spaces

$$\begin{tikzcd} & A^{\mathrm{Vert}(\Lambda)} \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\ A^{\mathrm{Vert}(\Pi_0)} & & A^{\mathrm{Vert}(\Pi_1)} \end{tikzcd}$$

Define the linear map by push-pull

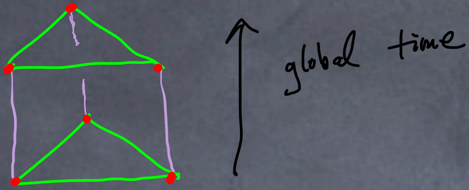
$$I(Y,\Lambda)=(r_1)_*\circ K\circ(r_0)^*\colon I(S_0,\Pi_0)\longrightarrow I(S_1,\Pi_1)$$

where the “integral kernel” K is the weight function

$$K(s)=\prod_e \theta_\beta\big(g(s;e)\big),\qquad e \text{ incoming or interior}$$

Wick-rotated discrete time evolution via product bordism ("prism")

$$(Y, \Lambda) = [0,1] \times (S, \Pi)$$



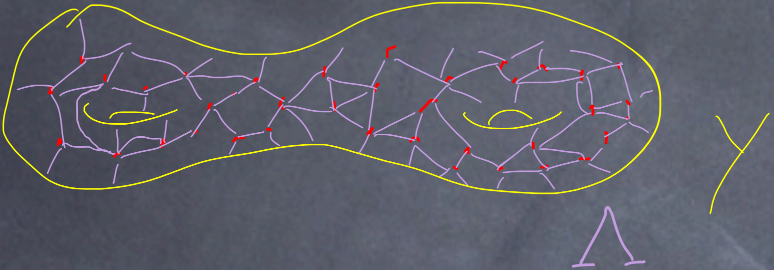
The resulting endomorphism of $I(S,\Pi)$ is called the *transfer matrix*. We write it as e^{-H}, where H is the *Hamiltonian*. Eigenvalues of H are energies (possibly infinite).

Fourier-Kramers-Wannier Duality

(Y, Λ) closed latticed surface

$$C^0(\Lambda; A) = A^{\mathrm{Vert}(\Lambda)}$$
$$C^1(\Lambda; A) = A^{\mathrm{Edge}(\Lambda)}$$

$$\begin{aligned} C^0(\Lambda; A) &\xrightarrow{\ \delta\ } C^1(\Lambda; A) \\ s &\longmapsto \big(e \mapsto g(s; e)\big) \end{aligned}$$



Fourier-Kramers-Wannier Duality

(Y, Λ) closed latticed surface

$$C^0(\Lambda; A) = A^{\mathrm{Vert}(\Lambda)}$$
$$C^1(\Lambda; A) = A^{\mathrm{Edge}(\Lambda)}$$

$$\begin{aligned} C^0(\Lambda; A) &\xrightarrow{\ \delta\ } C^1(\Lambda; A) \\ s &\longmapsto \big(e \mapsto g(s; e)\big) \end{aligned}$$

Two functions on $C^1(\Lambda; A)$:

$$\Theta = \prod_{e \in \mathrm{Edge}(\Lambda)} \theta$$

$$\delta_* 1 = \# H^0(\Lambda; A)\, \Delta_{B^1(\Lambda; A)}$$

Fourier-Kramers-Wannier Duality

(Y, Λ) closed latticed surface

$$C^0(\Lambda; A) = A^{\mathrm{Vert}(\Lambda)}$$
$$C^1(\Lambda; A) = A^{\mathrm{Edge}(\Lambda)}$$

$$\begin{aligned} C^0(\Lambda; A) &\xrightarrow{\ \delta\ } C^1(\Lambda; A) \\ s &\longmapsto \big(e \mapsto g(s; e)\big) \end{aligned}$$

Two functions on $C^1(\Lambda; A)$:

$$\Theta = \prod_{e \in \mathrm{Edge}(\Lambda)} \theta$$

$$\delta_* 1 = \# H^0(\Lambda; A)\, \Delta_{B^1(\Lambda; A)}$$

Ising partition function:

$$I(Y, \Lambda) = \sum_{s \in A^{\mathrm{Vert}(\Lambda)}} \prod_{e \in \mathrm{Edge}(\Lambda)} \theta_\beta\big(g(s; e)\big)$$

Fourier-Kramers-Wannier Duality

(Y, Λ) closed latticed surface

$$C^0(\Lambda; A) = A^{\mathrm{Vert}(\Lambda)}$$
$$C^1(\Lambda; A) = A^{\mathrm{Edge}(\Lambda)}$$

$$\begin{aligned} C^0(\Lambda; A) &\xrightarrow{\ \delta\ } C^1(\Lambda; A) \\ s &\longmapsto \big(e \mapsto g(s; e)\big) \end{aligned}$$

Two functions on $C^1(\Lambda; A)$:
$$\Theta = \prod_{e \in \mathrm{Edge}(\Lambda)} \theta$$
$$\delta_* 1 = \# H^0(\Lambda; A)\, \Delta_{B^1(\Lambda; A)}$$

Ising partition function as inner product of functions on $C^1(\Lambda; A)$:

$$I(Y, \Lambda) = \frac{1}{\# H^0(\Lambda; A)} \langle \Theta, \delta_* 1 \rangle = c \,\langle \Theta, \Delta_{B^1} \rangle$$

Pontrjagin dual groups and maps:

$$C^0(\Lambda; A) \xrightarrow{\delta} C^1(\Lambda; A)$$

$$C_0(\Lambda; A^\vee) \xleftarrow{\partial} C_1(\Lambda; A^\vee)$$

Pontrjagin dual groups and maps:

$$C^0(\Lambda; A) \xrightarrow{\delta} C^1(\Lambda; A)$$

$$C_0(\Lambda; A^\vee) \xleftarrow{\partial} C_1(\Lambda; A^\vee)$$

Parsevel's formula (Fourier transform $f \mapsto f^\vee$ is an L^2-isometry):

$$I(Y, \Lambda) = c \, \langle \Theta, \Delta_{B^1} \rangle = c' \, \langle \Theta^\vee, \Delta^\vee_{B^1} \rangle$$

Pontrjagin dual groups and maps:

$$C^0(\Lambda; A) \xrightarrow{\delta} C^1(\Lambda; A)$$

$$C_0(\Lambda; A^\vee) \xleftarrow{\partial} C_1(\Lambda; A^\vee)$$

Parsevel's formula (Fourier transform $f \mapsto f^\vee$ is an L^2-isometry):

$$I(Y, \Lambda) = c\,\langle \Theta, \Delta_{B^1} \rangle = c'\,\langle \Theta^\vee, \Delta^\vee_{B^1} \rangle$$

Rewrite in terms of the dual triangulation (Y oriented) $(Y, \Lambda^\vee)$ as an inner product of functions in $C^1(\Lambda^\vee; A^\vee)$; then

$$\Theta^\vee = \prod_{e^\vee \in \mathrm{Edge}(\Lambda^\vee)} \theta^\vee$$

$$\Delta^\vee_{B^1(\Lambda; A)} = c''\,\Delta_{Z^1(\Lambda^\vee; A^\vee)}$$

Pontrjagin dual groups and maps:

$$\begin{array}{ccc} C^0(\Lambda; A) & \xrightarrow{\delta} & C^1(\Lambda; A) \\ C_0(\Lambda; A^\vee) & \xleftarrow{\partial} & C_1(\Lambda; A^\vee) \end{array}$$

Parsevel's formula (Fourier transform $f \mapsto f^\vee$ is an L^2-isometry):

$$I(Y, \Lambda) = c \, \langle \Theta, \Delta_{B^1} \rangle = c' \, \langle \Theta^\vee, \Delta^\vee_{B^1} \rangle$$

Rewrite in terms of the dual triangulation (Y oriented) $(Y, \Lambda^\vee)$ as an inner product of functions in $C^1(\Lambda^\vee; A^\vee)$; then

$$\Theta^\vee = \prod_{e^\vee \in \mathrm{Edge}(\Lambda^\vee)} \theta^\vee$$

$$\Delta^\vee_{B^1(\Lambda;A)} = c'' \, \Delta_{Z^1(\Lambda^\vee;A^\vee)}$$

For $A = \mu_2$, Fourier transform: $\beta \leftrightarrow \beta^\vee$ where $\sinh(2\beta) \sinh(2\beta^\vee) = 1$

Features/Problems

Features/Problems

1. Kramers-Wannier duality for $G = A$ abelian relates theories $I_{(A,\theta)}$ and $I_{(A^\vee,\theta^\vee)}$, but off by homology groups

Features/Problems

1. Kramers-Wannier duality for $G = A$ abelian relates theories $I_{(A,\theta)}$ and $I_{(A^\vee,\theta^\vee)}$, but off by homology groups
2. Need to see how order operators map under duality; usual story with disorder operators not cleanly matching

Features/Problems

1. Kramers-Wannier duality for $G = A$ abelian relates theories $I_{(A,\theta)}$ and $I_{(A^\vee,\theta^\vee)}$, but off by homology groups

2. Need to see how order operators map under duality; usual story with disorder operators not cleanly matching

3. Missing dual for G nonabelian

Features/Problems

1. Kramers-Wannier duality for $G = A$ abelian relates theories $I_{(A,\theta)}$ and $I_{(A^\vee,\theta^\vee)}$, but off by homology groups

2. Need to see how order operators map under duality; usual story with disorder operators not cleanly matching

3. Missing dual for G nonabelian

4. Mismatch in low energy states under duality

Features/Problems

1. Kramers-Wannier duality for $G = A$ abelian relates theories $I_{(A,\theta)}$ and $I_{(A^\vee,\theta^\vee)}$, but off by homology groups

2. Need to see how order operators map under duality; usual story with disorder operators not cleanly matching

3. Missing dual for G nonabelian

4. Mismatch in low energy states under duality

Key Idea: Use the full strength of the *symmetry* group G

Features/Problems

1. Kramers-Wannier duality for $G = A$ abelian relates theories $I_{(A,\theta)}$ and $I_{(A^\vee,\theta^\vee)}$, but off by homology groups

2. Need to see how order operators map under duality; usual story with disorder operators not cleanly matching

3. Missing dual for G nonabelian

4. Mismatch in low energy states under duality

Key Idea: Use the full strength of the *symmetry* group G

Settles these issues and much more:

Features/Problems

1. Kramers-Wannier duality for $G = A$ abelian relates theories $I_{(A,\theta)}$ and $I_{(A^\vee,\theta^\vee)}$, but off by homology groups

2. Need to see how order operators map under duality; usual story with disorder operators not cleanly matching

3. Missing dual for G nonabelian

4. Mismatch in low energy states under duality

Key Idea: Use the full strength of the *symmetry* group G

Settles these issues and much more:

- prediction for low energy behavior (all G)

Features/Problems

1. Kramers-Wannier duality for $G = A$ abelian relates theories $I_{(A,\theta)}$ and $I_{(A^\vee,\theta^\vee)}$, but off by homology groups

2. Need to see how order operators map under duality; usual story with disorder operators not cleanly matching

3. Missing dual for G nonabelian

4. Mismatch in low energy states under duality

Key Idea: Use the full strength of the *symmetry* group G

Settles these issues and much more:

- prediction for low energy behavior (all G)
- more general classes of models

Features/Problems

1. Kramers-Wannier duality for $G = A$ abelian relates theories $I_{(A,\theta)}$ and $I_{(A^\vee,\theta^\vee)}$, but off by homology groups

2. Need to see how order operators map under duality; usual story with disorder operators not cleanly matching

3. Missing dual for G nonabelian

4. Mismatch in low energy states under duality

Key Idea: Use the full strength of the *symmetry* group G

Settles these issues and much more:

- prediction for low energy behavior (all G)
- more general classes of models
- whole story in context of extended *topological* field theory

Features/Problems

1. Kramers-Wannier duality for $G = A$ abelian relates theories $I_{(A,\theta)}$ and $I_{(A^\vee,\theta^\vee)}$, but off by homology groups
2. Need to see how order operators map under duality; usual story with disorder operators not cleanly matching
3. Missing dual for G nonabelian
4. Mismatch in low energy states under duality

Key Idea: Use the full strength of the *symmetry* group G

Settles these issues and much more:

- prediction for low energy behavior (all G)
- more general classes of models
- whole story in context of extended *topological* field theory
- higher dimensional abelian models (stable homotopy theory)

Fibering over BG

If a group G acts as a symmetry on mathematical object M (condition), we can try to extend (data) to a fibering

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d] & \mathcal{M} \arrow[d] \\ \mathrm{pt} \arrow[r, hook] & BG \end{tikzcd}$$

Fibering over BG

If a group G acts as a symmetry on mathematical object M (condition), we can try to extend (data) to a fibering

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d] & \mathcal{M} \arrow[d] \\ \mathrm{pt} \arrow[r, hook] & BG \end{tikzcd}$$

The precise nature of BG and ‘fibering’ vary

Fibering over BG

If a group G acts as a symmetry on mathematical object M (condition), we can try to extend (data) to a fibering

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d] & \mathcal{M} \arrow[d] \\ \mathrm{pt} \arrow[r, hook] & BG \end{tikzcd}$$

The precise nature of BG and ‘fibering’ vary

In geometry/topology $\mathcal{M}$ is the Borel quotient

Fibering over BG

If a group G acts as a symmetry on mathematical object M (condition), we can try to extend (data) to a fibering

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d] & \mathcal{M} \arrow[d] \\ \mathrm{pt} \arrow[r, hook] & BG \end{tikzcd}$$

The precise nature of BG and 'fibering' vary

In geometry/topology $\mathcal{M}$ is the Borel quotient

In general there may be obstructions ("anomalies") which are important features of the symmetry; in any case $\mathcal{M}$ yields a richer picture

Fibering over BG

If a group G acts as a symmetry on mathematical object M (condition), we can try to extend (data) to a fibering

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d] & \mathcal{M} \arrow[d] \\ \mathrm{pt} \arrow[r, hook] & BG \end{tikzcd}$$

The precise nature of BG and 'fibering' vary

In geometry/topology $\mathcal{M}$ is the Borel quotient

In general there may be obstructions ("anomalies") which are important features of the symmetry; in any case $\mathcal{M}$ yields a richer picture

Equivariance $\longrightarrow$ Families

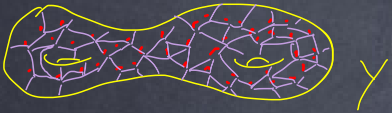
‘Fibering over BG’ in Ising Model

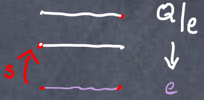
G-Ising model on Y^2: *background* lattice $\Lambda \subset Y$ and G-bundle $Q \to Y$
fluctuating field a “discrete gauged σ-model”

$$Q^{\mathrm{Vert}(\Lambda)} = \text{sections of } Q \to Y \text{ over } \mathrm{Vert}(\Lambda)$$

The ratio of spins defined via parallel transport

$$g\colon Q^{\mathrm{Vert}(\Lambda)} \times \mathrm{Edge}(\Lambda) \longrightarrow G$$





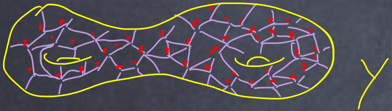
‘Fibering over BG’ in Ising Model

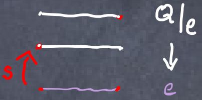
G-Ising model on Y^2: *background* lattice $\Lambda \subset Y$ and G-bundle $Q \to Y$
fluctuating field a “discrete gauged σ-model”

$$Q^{\mathrm{Vert}(\Lambda)} = \text{sections of } Q \to Y \text{ over } \mathrm{Vert}(\Lambda)$$

The ratio of spins defined via parallel transport

$$g\colon Q^{\mathrm{Vert}(\Lambda)} \times \mathrm{Edge}(\Lambda) \longrightarrow G$$





The partition function of $I = I_{(G,\theta)}$ is now a function of a G-bundle:

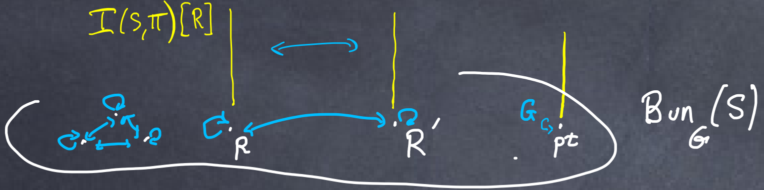
$$I(Y,\Lambda)\colon \mathrm{Bun}_G(Y) \longrightarrow \mathbb{C}$$

The old partition function is the value at the trivial bundle

To a latticed 1-manifold (S, Π) we obtain a vector bundle

$$I(S, \Pi) \longrightarrow \mathrm{Bun}_G(S)$$

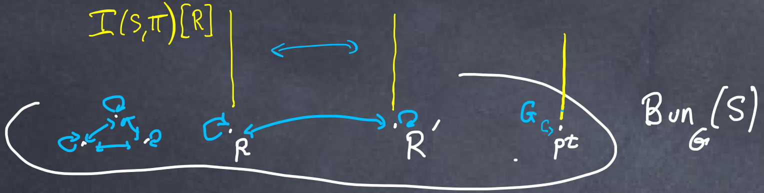
These are "twisted sectors"; the old state space is the fiber at pt $\in BG$



To a latticed 1-manifold (S, Π) we obtain a vector bundle

$$I(S, \Pi) \longrightarrow \mathrm{Bun}_G(S)$$

These are "twisted sectors"; the old state space is the fiber at pt $\in BG$



Upshot: I is a *boundary theory* for $\mathscr{G}_G$:

$$I(Y, \Lambda) \in \mathscr{G}_G(Y)$$
$$I(S, \Pi) \in \mathscr{G}_G(S)$$

We learned recently that we were anticipated by Pavol Ševera (2002) in some of our pictures of the Ising model and topological field theory, though he works in a non-extended context: `arXiv:hep-th/0206162`

Boundary theories

Definition: A *topological boundary theory* for $\mathscr{G}_G\colon \mathrm{Bord}_3 \to \mathrm{TensCat}$ is

$$\mathscr{B}\colon 1 \longrightarrow \tau_{\leq 2}\mathscr{G}_G,$$

a map of functors $\mathrm{Bord}_2 \to \mathrm{TensCat}$

Boundary theories

Definition: A *topological boundary theory* for $\mathscr{G}_G \colon \mathrm{Bord}_3 \to \mathrm{TensCat}$ is

$$\mathscr{B} \colon 1 \longrightarrow \tau_{\leq 2}\mathscr{G}_G,$$

a map of functors $\mathrm{Bord}_2 \to \mathrm{TensCat}$

Cobordism hypothesis: $\mathscr{B}$ determined by $\mathscr{B}(\mathrm{pt})$, a left $\mathrm{Vect}[G]$-module

Boundary theories

Definition: A *topological boundary theory* for $\mathscr{G}_G\colon \mathrm{Bord}_3 \to \mathrm{TensCat}$ is

$$\mathscr{B}\colon 1 \longrightarrow \tau_{\leq 2}\mathscr{G}_G,$$

a map of functors $\mathrm{Bord}_2 \to \mathrm{TensCat}$

Cobordism hypothesis: $\mathscr{B}$ determined by $\mathscr{B}(\mathrm{pt})$, a left $\mathrm{Vect}[G]$-module

Ising is a boundary theory on *latticed* manifolds (finite path integral)

$$\begin{tikzcd} \mathrm{Bord}^{\mathrm{latticed}}_{\langle 1,2\rangle} \arrow[r] \arrow[rrr, bend left, "1"] & \mathrm{Bord}_{\langle 1,2\rangle} \arrow[r] \arrow[r, phantom, shift left=4ex, "\Downarrow I = I(G,\theta)"] & \mathrm{Bord}_{\langle 1,2,3\rangle} \arrow[r, "\mathscr{G}_G"'] & \mathrm{Cat} \end{tikzcd}$$

Boundary theories

Definition: A *topological boundary theory* for $\mathscr{G}_G\colon \mathrm{Bord}_3 \to \mathrm{TensCat}$ is

$$\mathscr{B}\colon 1 \longrightarrow \tau_{\leq 2}\mathscr{G}_G,$$

a map of functors $\mathrm{Bord}_2 \to \mathrm{TensCat}$

Cobordism hypothesis: $\mathscr{B}$ determined by $\mathscr{B}(\mathrm{pt})$, a left $\mathrm{Vect}[G]$-module

Ising is a boundary theory on *latticed* manifolds (finite path integral)

$$\begin{tikzcd} \mathrm{Bord}^{\text{latticed}}_{\langle 1,2\rangle} \arrow[r] \arrow[rrr, bend left, "1"] & \mathrm{Bord}_{\langle 1,2\rangle} \arrow[r] \arrow[u, phantom, "\Downarrow I = I_{(G,\theta)}"] & \mathrm{Bord}_{\langle 1,2,3\rangle} \arrow[r, "\mathscr{G}_G"'] & \mathrm{Cat} \end{tikzcd}$$

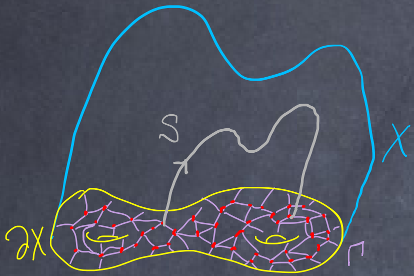
For (Y,Λ) closed obtain a function on $\mathrm{Bun}_G(Y)$:

$$I(Y,\Lambda)[Q] = \sum_{s\in Q^{\mathrm{Vert}(\Lambda)}} \prod_{e\in\mathrm{Edge}(\Lambda)} \theta\big(g(s;e)\big)$$

Line operators for neat 1d submanifolds $S \subset X^3$ with $\partial S \subset (\partial X, \Gamma)$

Wilson/order operators: $\chi\colon G \to \mathbb{T}$ character, S ends at vertices

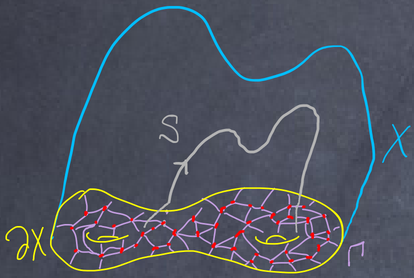
$$(F,I)(X,\Gamma) = \sum_{[P]\in\pi_0 \operatorname{Bun}_G(X)} \frac{1}{\#\operatorname{Aut} P} \sum_{s\in\mathcal{S}_{(\partial X,\Gamma)}[\partial P]} h_{S,\chi}(P,s) \prod_{e\in\operatorname{Edges}(\Gamma)} \theta\big(g(s;e)\big)$$



Line operators for neat 1d submanifolds $S \subset X^3$ with $\partial S \subset (\partial X, \Gamma)$

Wilson/order operators: $\chi\colon G \to \mathbb{T}$ character, S ends at vertices

't Hooft/disorder operators: conjugacy class in G, S ends in faces



Revisit problems

1. Kramers-Wannier duality for $G = A$ abelian relates theories $I_{(A,\theta)}$ and $I_{(A^\vee,\theta^\vee)}$, but off by a sum over homology

   ✓ Kramers-Wannier duality is part of electromagnetic duality

Revisit problems

1. Kramers-Wannier duality for $G = A$ abelian relates theories $I_{(A,\theta)}$ and $I_{(A^\vee,\theta^\vee)}$, but off by a sum over homology
   ✓ Kramers-Wannier duality is part of electromagnetic duality

2. Need to see how order operators map under duality; usual story with disorder operators not cleanly matching
   ✓ Order/Disorder special case of Wilson/'t Hooft

Revisit problems

1. Kramers-Wannier duality for $G = A$ abelian relates theories $I_{(A,\theta)}$ and $I_{(A^\vee,\theta^\vee)}$, but off by a sum over homology
   ✓ Kramers-Wannier duality is part of electromagnetic duality

2. Need to see how order operators map under duality; usual story with disorder operators not cleanly matching
   ✓ Order/Disorder special case of Wilson/'t Hooft

3. Missing dual for G nonabelian
   ✓ Can construct using Turaev-Viro for $\mathscr{R}_G$

Revisit problems

1. Kramers-Wannier duality for $G = A$ abelian relates theories $I_{(A,\theta)}$ and $I_{(A^\vee,\theta^\vee)}$, but off by a sum over homology
   ✓ Kramers-Wannier duality is part of electromagnetic duality

2. Need to see how order operators map under duality; usual story with disorder operators not cleanly matching
   ✓ Order/Disorder special case of Wilson/'t Hooft

3. Missing dual for G nonabelian
   ✓ Can construct using Turaev-Viro for $\mathscr{R}_G$

4. Mismatch in low energy states under duality
   Discuss next

Low energy behavior; phase diagram

$\mathcal{M}$	moduli space of quantum theories
$\Delta \subset \mathcal{M}$	locus of phase transitions
$(\mathcal{M} \setminus \Delta)_{\text{gapped}} \subset \mathcal{M} \setminus \Delta$	systems with spectral gap
$\pi_0(\mathcal{M} \setminus \Delta)_{\text{gapped}}$	set of *phases*

Low energy behavior; phase diagram

$\mathcal{M}$	moduli space of quantum theories
$\Delta \subset \mathcal{M}$	locus of phase transitions
$(\mathcal{M} \setminus \Delta)_{\text{gapped}} \subset \mathcal{M} \setminus \Delta$	systems with spectral gap
$\pi_0(\mathcal{M} \setminus \Delta)_{\text{gapped}}$	set of *phases*

- Points in $(\mathcal{M} \setminus \Delta)_{\text{gapped}}$ have a low energy effective *topological* field* theory, thought to be a complete invariant of its path component

Low energy behavior; phase diagram

$\mathcal{M}$	moduli space of quantum theories
$\Delta \subset \mathcal{M}$	locus of phase transitions
$(\mathcal{M} \setminus \Delta)_{\text{gapped}} \subset \mathcal{M} \setminus \Delta$	systems with spectral gap
$\pi_0(\mathcal{M} \setminus \Delta)_{\text{gapped}}$	set of *phases*

- Points in $(\mathcal{M} \setminus \Delta)_{\text{gapped}}$ have a low energy effective *topological* field* theory, thought to be a complete invariant of its path component
- Renormalization group flow on $(\mathcal{M} \setminus \Delta)_{\text{gapped}}$

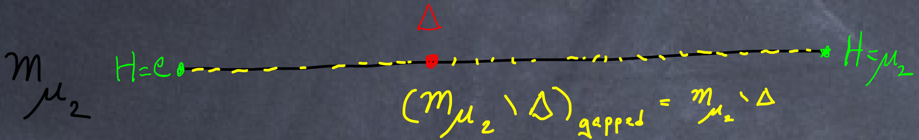
Low energy behavior; phase diagram

$\mathcal{M}$	moduli space of quantum theories
$\Delta \subset \mathcal{M}$	locus of phase transitions
$(\mathcal{M} \setminus \Delta)_{\text{gapped}} \subset \mathcal{M} \setminus \Delta$	systems with spectral gap
$\pi_0(\mathcal{M} \setminus \Delta)_{\text{gapped}}$	set of *phases*

- Points in $(\mathcal{M} \setminus \Delta)_{\text{gapped}}$ have a low energy effective *topological* field* theory, thought to be a complete invariant of its path component
- Renormalization group flow on $(\mathcal{M} \setminus \Delta)_{\text{gapped}}$

In our case take

$\mathcal{M}_G = \{\text{admissible } \theta\}/\text{rescaling}$

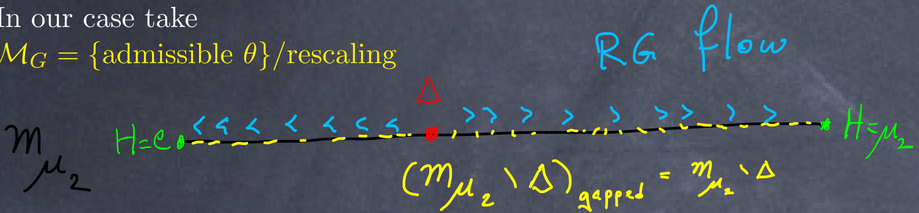


Low energy behavior; phase diagram

$\mathcal{M}$	moduli space of quantum theories
$\Delta \subset \mathcal{M}$	locus of phase transitions
$(\mathcal{M} \setminus \Delta)_{\text{gapped}} \subset \mathcal{M} \setminus \Delta$	systems with spectral gap
$\pi_0(\mathcal{M} \setminus \Delta)_{\text{gapped}}$	set of *phases*

- Points in $(\mathcal{M} \setminus \Delta)_{\text{gapped}}$ have a low energy effective *topological* field* theory, thought to be a complete invariant of its path component
- Renormalization group flow on $(\mathcal{M} \setminus \Delta)_{\text{gapped}}$

In our case take
$\mathcal{M}_G = \{\text{admissible } \theta\}/\text{rescaling}$

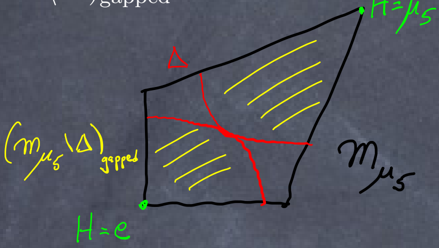


Low energy behavior; phase diagram

$\mathcal{M}$	moduli space of quantum theories
$\Delta \subset \mathcal{M}$	locus of phase transitions
$(\mathcal{M} \setminus \Delta)_{\text{gapped}} \subset \mathcal{M} \setminus \Delta$	systems with spectral gap
$\pi_0(\mathcal{M} \setminus \Delta)_{\text{gapped}}$	set of *phases*

- Points in $(\mathcal{M} \setminus \Delta)_{\text{gapped}}$ have a low energy effective *topological* field* theory, thought to be a complete invariant of its path component
- Renormalization group flow on $(\mathcal{M} \setminus \Delta)_{\text{gapped}}$

In our case take
$\mathcal{M}_G = \{\text{admissible } \theta\}/\text{rescaling}$



Apply full force of G-symmetry: *low energy effective field theory is a topological boundary theory for* $\mathscr{G}_G$

Apply full force of G-symmetry: *low energy effective field theory is a topological boundary theory for* $\mathscr{G}_G$

Theorem [EGNO]: Irreducible $\mathrm{Vect}[G]$-modules are parametrized by central extensions $1 \longrightarrow \mathbb{T} \longrightarrow \widetilde{H} \longrightarrow H \longrightarrow 1$ of subgroups $H \subset G$

Apply full force of G-symmetry: *low energy effective field theory is a topological boundary theory for* $\mathscr{G}_G$

Theorem [EGNO]: Irreducible Vect$[G]$-modules are parametrized by central extensions $1 \longrightarrow \mathbb{T} \longrightarrow \widetilde{H} \longrightarrow H \longrightarrow 1$ of subgroups $H \subset G$

Classical model: boundary field a section of associated G/H-bundle

Apply full force of G-symmetry: *low energy effective field theory is a topological boundary theory for* $\mathscr{G}_G$

Theorem [EGNO]: Irreducible Vect$[G]$-modules are parametrized by central extensions $1 \longrightarrow \mathbb{T} \longrightarrow \widetilde{H} \longrightarrow H \longrightarrow 1$ of subgroups $H \subset G$

Classical model: boundary field a section of associated G/H-bundle

Prediction: No central extensions; irreducible modules
$\Longrightarrow$ Phases classified by symmetry breaking (Landau)

Apply full force of G-symmetry: *low energy effective field theory is a topological boundary theory for* $\mathscr{G}_G$

Theorem [EGNO]: Irreducible $\mathrm{Vect}[G]$-modules are parametrized by central extensions $1 \longrightarrow \mathbb{T} \longrightarrow \widetilde{H} \longrightarrow H \longrightarrow 1$ of subgroups $H \subset G$

Classical model: boundary field a section of associated G/H-bundle

Prediction: No central extensions; irreducible modules
$\Longrightarrow$ Phases classified by symmetry breaking (Landau)

Uses twisted sectors—low energy states form a vector bundle

$$W \longrightarrow G/\!/G$$

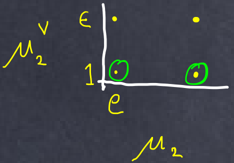
Apply full force of G-symmetry: *low energy effective field theory is a topological boundary theory for* $\mathscr{G}_G$

Theorem [EGNO]: Irreducible $\mathrm{Vect}[G]$-modules are parametrized by central extensions $1 \longrightarrow \mathbb{T} \longrightarrow \widetilde{H} \longrightarrow H \longrightarrow 1$ of subgroups $H \subset G$

Classical model: boundary field a section of associated G/H-bundle

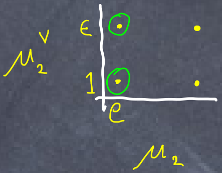
Prediction: No central extensions; irreducible modules
$\Longrightarrow$ Phases classified by symmetry breaking (Landau)

Uses twisted sectors—low energy states form a vector bundle



paramagnetic ($\beta \to 0$)

$W \longrightarrow G//G$



ferromagnetic ($\beta \to \infty$)

EM duality
($G = \mu_2$)

Topological construction; general theories

Two canonical topological boundary theories: *Dirichlet* and *Neumann*

Topological construction; general theories

Two canonical topological boundary theories: *Dirichlet* and *Neumann*

Dirichlet: subgroup $e \subset G$, so trivialization of G-bundle on boundary
module is $\mathrm{Vect}(G)$ (*regular boundary theory*)

Topological construction; general theories

Two canonical topological boundary theories: *Dirichlet* and *Neumann*

Dirichlet: subgroup $e \subset G$, so trivialization of G-bundle on boundary
module is $\mathrm{Vect}(G)$ (*regular boundary theory*)

Neumann: subgroup $G \subset G$, so no new boundary field
module is Vect (fiber functor for $\mathrm{Vect}[G]$)

Topological construction; general theories

Two canonical topological boundary theories: *Dirichlet* and *Neumann*

Dirichlet: subgroup $e \subset G$, so trivialization of G-bundle on boundary
module is $\mathrm{Vect}(G)$ (*regular boundary theory*)

Neumann: subgroup $G \subset G$, so no new boundary field
module is Vect (fiber functor for $\mathrm{Vect}[G]$)

Quartet of data:

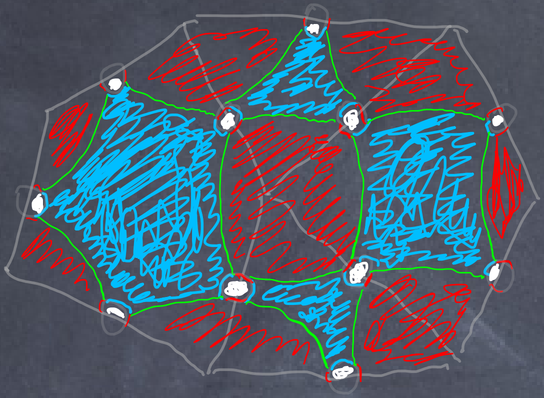
$\mathcal{T} = \mathrm{Vect}[G]$	categorified group algebra
$\mathcal{B}_1 = \mathrm{Vect}(G)$	Neumann boundary theory
$\mathcal{B}_2 = \mathrm{Vect}$	Dirichlet boundary theory
δ	generator of $\mathrm{Hom}_{\mathcal{T}}(\mathcal{B}_1, \mathcal{B}_2)$

Quartet: 3d TFT $\mathscr{G}_G$, boundary theories $\mathscr{B}_1$, $\mathscr{B}_2$, and domain wall $\mathscr{D}$

Quartet: 3d TFT $\mathscr{G}_G$, boundary theories $\mathscr{B}_1$, $\mathscr{B}_2$, and domain wall $\mathscr{D}$

Replace lattice Λ by a coloring via Morse function with critical points:

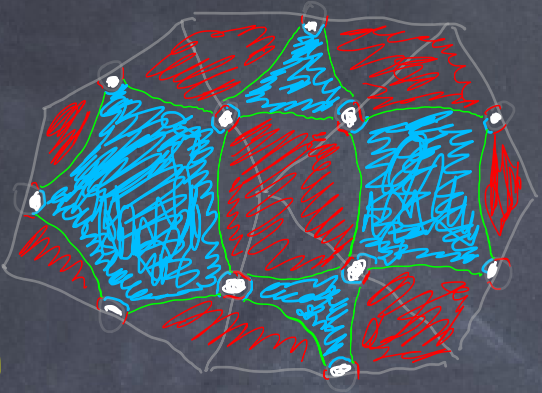
index 0	vertices
index 1	edges
index 2	faces

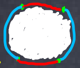


Quartet: 3d TFT $\mathscr{G}_G$, boundary theories $\mathscr{B}_1$, $\mathscr{B}_2$, and domain wall $\mathscr{D}$

Replace lattice Λ by a coloring via Morse function with critical points:

index 0	vertices
index 1	edges
index 2	faces



$F_G\Big($  $\Big) = \mathrm{Fun}(G)$

Quartet: 3d TFT $\mathscr{G}_G$, boundary theories $\mathscr{B}_1$, $\mathscr{B}_2$, and domain wall $\mathscr{D}$

Replace lattice Λ by a coloring via Morse function with critical points:

index 0	vertices
index 1	edges
index 2	faces

Use TFT to construct Ising state space, loop operators, etc.

Quartet: 3d TFT $\mathscr{G}_G$, boundary theories $\mathscr{B}_1$, $\mathscr{B}_2$, and domain wall $\mathscr{D}$

Replace lattice Λ by a coloring via Morse function with critical points:

index 0	vertices
index 1	edges
index 2	faces

Use TFT to construct Ising state space, loop operators, etc.

Ising entirely in world of extended TFT—cobordism hypothesis applies

Quartet: 3d TFT $\mathscr{G}_G$, boundary theories $\mathscr{B}_1$, $\mathscr{B}_2$, and domain wall $\mathscr{D}$

Replace lattice Λ by a coloring via Morse function with critical points:

index 0	vertices
index 1	edges
index 2	faces

Use TFT to construct Ising state space, loop operators, etc.

Ising entirely in world of extended TFT—cobordism hypothesis applies

More general theories: spherical fusion category and fiber functor
= Frobenius-Hopf algebra

Morita equivalence:

$$\begin{aligned} \mathrm{Vect}[G] &\longleftrightarrow \mathrm{Rep}(G) \qquad && \text{(tensor categories)} \\ \mathrm{Vect}(G/H) &\longleftrightarrow \mathrm{Rep}(H) \qquad && \text{(left modules)} \end{aligned}$$

Morita equivalence:

$$\mathrm{Vect}[G] \longleftrightarrow \mathrm{Rep}(G) \qquad \text{(tensor categories)}$$
$$\mathrm{Vect}(G/H) \longleftrightarrow \mathrm{Rep}(H) \qquad \text{(left modules)}$$

Exchanges theories specified by quartets Dirichelet $\longleftrightarrow$ Neumann:

$$(\mathrm{Vect}[G], \mathrm{Vect}[G], \mathrm{Vect}, \mathrm{Vect}) \longleftrightarrow (\mathrm{Rep}(G), \mathrm{Vect}, \mathrm{Rep}(G), \mathrm{Vect})$$

Morita equivalence:

$$\mathrm{Vect}[G] \longleftrightarrow \mathrm{Rep}(G) \qquad \text{(tensor categories)}$$
$$\mathrm{Vect}(G/H) \longleftrightarrow \mathrm{Rep}(H) \qquad \text{(left modules)}$$

Exchanges theories specified by quartets Dirichelet $\longleftrightarrow$ Neumann:

$$(\mathrm{Vect}[G], \mathrm{Vect}[G], \mathrm{Vect}, \mathrm{Vect}) \longleftrightarrow (\mathrm{Rep}(G), \mathrm{Vect}, \mathrm{Rep}(G), \mathrm{Vect})$$

Cobordism hypothesis $\Longrightarrow$ Electromagnetic/Kramers-Wannier duality

Theorem: On oriented manifolds there is an equivalence of G-gauge theory and the Turaev-Viro $\mathrm{Rep}(G)$ theory which exchanges their lattice boundary theories, and exchanges Wilson/Order and 't Hooft/Disorder operators.

Abelian duality in higher dimensions

S	pointed space, finite homotopy type
$\mathcal{F}_X$	$\mathrm{Map}(X_+, S)$

n-dimensional theory F_S (finite path integral) with partition function

$$F_S(X) = \sum_{[\varphi]\in\pi_0\mathcal{F}_X} \frac{1}{\#\pi_1(\mathcal{F}_X,\varphi)}\,\frac{\#\pi_2(\mathcal{F}_X,\varphi)}{\#\pi_3(\mathcal{F}_X,\varphi)}\cdots$$

Abelian duality in higher dimensions

S	pointed space, finite homotopy type
$\mathcal{F}_X$	$\mathrm{Map}(X_+, S)$

n-dimensional theory F_S (finite path integral) with partition function

$$F_S(X) = \sum_{[\varphi]\in\pi_0\mathcal{F}_X} \frac{1}{\#\pi_1(\mathcal{F}_X,\varphi)}\,\frac{\#\pi_2(\mathcal{F}_X,\varphi)}{\#\pi_3(\mathcal{F}_X,\varphi)}\cdots$$

Canonical Dirichlet and Neumann boundary theories

$$\begin{tikzcd} * \arrow[dr] & & S \arrow[dl] \\ & S & \end{tikzcd}$$

Abelian duality in higher dimensions

S	pointed space, finite homotopy type
$\mathcal{F}_X$	$\mathrm{Map}(X_+, S)$

n-dimensional theory F_S (finite path integral) with partition function

$$F_S(X) = \sum_{[\varphi]\in\pi_0\mathcal{F}_X} \frac{1}{\#\pi_1(\mathcal{F}_X,\varphi)}\,\frac{\#\pi_2(\mathcal{F}_X,\varphi)}{\#\pi_3(\mathcal{F}_X,\varphi)}\cdots$$

Canonical Dirichlet and Neumann boundary theories

$$\begin{tikzcd}
\partial X \arrow[r, "\text{bdy}", "\text{field}"'] \arrow[d, hook] & * \arrow[dr] & & S \arrow[dl] \\
X \arrow[rr, "\varphi"] & & S &
\end{tikzcd}$$

Abelian duality in higher dimensions

S pointed space, finite homotopy type

$\mathcal{F}_X$ $\quad \mathrm{Map}(X_+, S)$

n-dimensional theory F_S (finite path integral) with partition function

$$F_S(X) = \sum_{[\varphi]\in\pi_0\mathcal{F}_X} \frac{1}{\#\pi_1(\mathcal{F}_X,\varphi)}\,\frac{\#\pi_2(\mathcal{F}_X,\varphi)}{\#\pi_3(\mathcal{F}_X,\varphi)}\cdots$$

Canonical Dirichlet and Neumann boundary theories

[unique domain wall]

$$\begin{tikzcd}
 & & \ast \arrow[dl] \arrow[dr] & \\
\partial X \arrow[r, "\text{bdy}", "\text{field}"'] \arrow[d] & \ast \arrow[dr] & & S \arrow[dl] \\
X \arrow[rr, "\varphi"] & & S &
\end{tikzcd}$$

Abelian duality in higher dimensions

S	pointed space, finite homotopy type
$\mathcal{F}_X$	$\mathrm{Map}(X_+, S)$

n-dimensional theory F_S (finite path integral) with partition function

$$F_S(X) = \sum_{[\varphi]\in\pi_0\mathcal{F}_X} \frac{1}{\#\pi_1(\mathcal{F}_X,\varphi)}\,\frac{\#\pi_2(\mathcal{F}_X,\varphi)}{\#\pi_3(\mathcal{F}_X,\varphi)}\cdots$$

Canonical Dirichlet and Neumann boundary theories

If S is an ∞ loop space, the 0-space of a spectrum $\mathcal{T}$, then there is a (Pontrjagin) dual spectrum $\mathcal{T}^\vee$. Electromagnetic duality:

$$F_{\mathcal{T}} \approx F_{\Sigma^{n-1}\mathcal{T}^\vee}$$

Abelian duality in higher dimensions

S	pointed space, finite homotopy type
$\mathcal{F}_X$	$\mathrm{Map}(X_+, S)$

n-dimensional theory F_S (finite path integral) with partition function

$$F_S(X) = \sum_{[\varphi]\in\pi_0\mathcal{F}_X} \frac{1}{\#\pi_1(\mathcal{F}_X,\varphi)}\,\frac{\#\pi_2(\mathcal{F}_X,\varphi)}{\#\pi_3(\mathcal{F}_X,\varphi)}\cdots$$

Canonical Dirichlet and Neumann boundary theories

If S is an ∞ loop space, the 0-space of a spectrum $\mathcal{T}$, then there is a (Pontrjagin) dual spectrum $\mathcal{T}^\vee$. Electromagnetic duality:

$$F_{\mathcal{T}} \approx F_{\Sigma^{n-1}\mathcal{T}^\vee}$$

The abelian Ising story is $n = 3$ and $\mathcal{T} = \Sigma HA$.