Remarks on Fully Extended 3-Dimensional Topological Field Theories

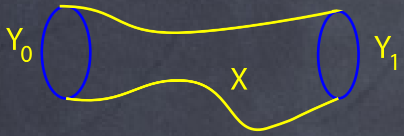
Dan Freed
University of Texas at Austin

June 6, 2011

Work in progress with Constantin Teleman

Manifolds and Algebra: Abelian Groups

Pontrjagin and Thom introduced abelian groups Ω_k of manifolds called bordism groups—equivalence classes of closed k-manifolds:



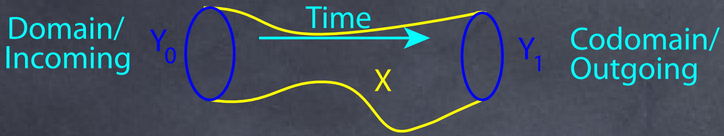
The abelian group operation is disjoint union. (Cartesian product defines a ring structure on $\bigoplus_k \Omega_k$)

Thom showed how to compute Ω_k via *homotopy theory*: $\Omega_k = \pi_k MO$. Different answers for different flavors of manifolds: oriented, spin, almost complex, framed, ...

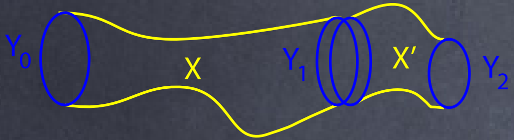
Many applications in homotopy theory: (i) framed bordism groups are stable homotopy groups of spheres (Pontrjagin-Thom construction); (ii) complex cobordism is universal among certain cohomology theories (Quillen)

Manifolds and Algebra: Symmetric Monoidal Categories

A more elaborate algebraic structure is obtained if we (i) do not identify bordant manifolds and (ii) remember the bordisms. So fix n and introduce a bordism category Bo_n whose objects are closed $(n-1)$-manifolds and morphisms are compact n-manifolds $X\colon Y_0 \to Y_1$



Identify diffeomorphic bordisms. Composition is gluing of manifolds:



$$Y_0 \xrightarrow{X} Y_1 \xrightarrow{X'} Y_2$$

Disjoint union gives a symmetric monoidal structure on Bo_n.

Recover the abelian group Ω_{n-1} by declaring all morphisms to be invertible=*iso*morphisms. New information from non-invertibility.

Topological Quantum Field Theory

Vect$_{\mathbb{C}}$ = symmetric monoidal category ($\otimes$) of complex vector spaces.

Definition: An n-dimensional TQFT is a homomorphism

$$F\colon \mathrm{Bo}_n \longrightarrow \mathbf{Vect}_{\mathbb{C}}$$

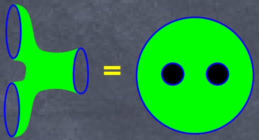
This slick definition encodes an algebraic understanding of field theory descended from Witten, Quillen, Segal, Atiyah, ... Segal defines conformal and more general QFTs via *geometric* bordism categories.

locality (compositions)

multiplicativity (monoidal structure)

E_n-algebra structure on $S^{n-1} \in \mathrm{Bo}_n$ via the generalized "pair of pants":

$$D^n \backslash (D^n \amalg D^n)\colon S^{n-1} \amalg S^{n-1} \longrightarrow S^{n-1}$$



Therefore, $F(S^{n-1}) \in \mathbf{Vect}_{\mathbb{C}}$ is also an E_n-algebra (OPE)

Manifolds and Algebra: Topological Categories

A more refined version of Bo_n is a *topological category* with a *space* of n-dimensional bordisms between fixed $(n-1)$-manifolds.

Algebraic topology provides a construction which inverts all morphisms: topological category → topological space. With abelian group structure: symmetric monoidal topological category → *spectrum*.

Galatius-Madsen-Tillmann-Weiss (2006) identify the spectrum $|\,\mathrm{Bo}_n\,| = MT$. For framed manifolds it is the sphere spectrum.

Remark: Topological spaces give rise to (higher) categories in which all morphisms are invertible: the fundamental *groupoid*

Definition (F.-Moore): A field theory $\alpha\colon \mathrm{Bo}_n \to \mathbf{Vect}_{\mathbb{C}}$ is invertible if $\alpha(Y^{n-1}) \in \mathbf{Vect}_{\mathbb{C}}$ is a line and $\alpha(X^n)$ is an isomorphism between lines for all Y, X.

α factors through MT spectrum $\Rightarrow$ homotopy theory techniques

Extended Field Theories

The notion of extended QFT was explored in various guises in the early '90s by several mathematicians and has great current interest:

- 2-dimensional theories often include categories attached to a point: D-branes, Fukaya category, ...
- 4-dimensional supersymmetric gauge theories have categories of line operators. Also, the category attached to a surface plays a key role in the geometric Langlands story.
- Chern-Simons (1-2-3) theory F has a linear category $F(S^1)$. For gauge group G two descriptions of $F(S^1)$: positive energy representations of LG, or representations of a quantum group.

A *fully* extended theory (down to 0-manifolds) is *completely local* $\Rightarrow$ powerful computational techniques, simpler classification

Longstanding Question: Can 1-2-3 Chern-Simons theory be extended to a 0-1-2-3 theory? If so, what is attached to a point?

Partial results in special cases (F., Walker, F.-Hopkins-Lurie-Teleman, Kapustin-Saulina, Bartels-Douglas-Henriques).

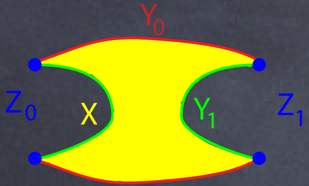
Manifolds and Algebra: (∞, n)-Categories

A new algebraic gadget: the bordism (∞, n)-category Bord_n.

Bo_n: $(n-1)$-manifolds and n-manifolds with boundary

Bord_n: 0-, 1-, ..., n-manifolds with corners

Objects are compact 0-manifolds, 1-morphisms are compact 1-manifolds with boundary, 2-morphisms are compact 2-manifolds with corners, ...



$$Z_0 \overset{Y_0}{\underset{Y_1}{\rightrightarrows}} Z_1 \quad \Downarrow X$$

Definition: An extended n-dimensional TQFT is a homomorphism

$$F\colon \mathrm{Bord}_n \longrightarrow \mathcal{C}$$

for some (∞, n)-category $\mathcal{C}$.

For example, if $n = 3$ then typically $F(S^1)$ is a $\mathbb{C}$-linear category, also an E_2-algebra. $E_2(\mathbf{Cat}_{\mathbb{C}}) = \beta\otimes\mathbf{Cat}_{\mathbb{C}}$ are braided tensor categories.

The Cobordism Hypothesis

A powerful theorem in topological field theory, conjectured by Baez-Dolan then elaborated and proved by Lurie (w/Hopkins for $n = 2$), asserts that an extended TQFT is determined by its value on a point.

Theorem: For framed manifolds the map

$$\begin{aligned} \mathrm{Hom}\big(\mathrm{Bord}_n, \mathcal{C}\big) &\longrightarrow \mathcal{C} \\ F \quad &\longmapsto F(\mathrm{pt}) \end{aligned}$$

is an isomorphism onto the fully dualizable objects in $\mathcal{C}$.

Remark: This is really a theorem about framed Bord_n, asserting that it is freely generated by a single generator.

The proof, only sketched heretofore, has at its heart a contractibility theorem in *Morse theory* (Igusa, Galatius). There are variations for other bordism categories of manifolds, also manifolds with singularities.

Full dualizable is a finiteness condition. For example, in a TQFT the vector spaces attached to closed $(n-1)$-manifolds are finite dimensional. In an extended theory $F(\mathrm{pt})$ satisfies analogous finiteness conditions.

Spheres and Invertibility

Theorem (F.-Teleman): Let $\alpha\colon \mathrm{Bord}_n \to \mathcal{C}$ be an extended TQFT such that $\alpha(S^k)$ is invertible. Then if $n \geqslant 2k$ the field theory α is invertible.

Thus $\alpha(X)$ is invertible for all manifolds X. This means α factors through the Madsen-Tillmann spectrum constructed from Bord_n, so is amenable to homotopy theory techniques.

Remark 1: We have only checked the details carefully for *oriented* manifolds; it is probably true for *stably framed* manifolds as well.

Remark 2: Again this is a theorem about Bord_n, asserting that if we *localize* by inverting S^k, then every manifold is inverted.

Remark 3: As I explain later we apply this to $n = 4$, $k = 2$, and $\mathcal{C} = \beta{\otimes}\mathbf{Cat}_{\mathbb{C}}$ the symmetric monoidal 4-category of braided tensor categories. Then α is the anomaly theory for Chern-Simons, and we construct Chern-Simons as a 0-1-2-3 *anomalous* theory.

Proof Sketch

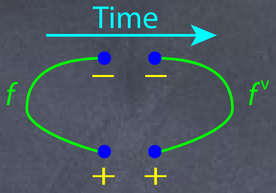
First, by the cobordism hypothesis (easy part) it suffices to prove that $\alpha(\mathrm{pt}_+)$ is invertible; '+' denotes the orientation. We omit 'α' and simply say 'pt_+ is invertible'.

We prove the 0-manifolds pt_+ and pt_- are inverse:

$$S^0 = \mathrm{pt}_+ \amalg \mathrm{pt}_- = \mathrm{pt}_+ \otimes \mathrm{pt}_- \cong \varnothing^0 = 1$$

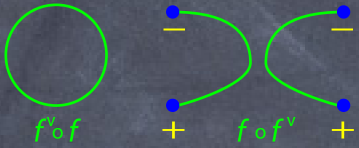
with inverse isomorphisms given by

$$f = D^1 : 1 \longrightarrow S^0$$
$$f^\vee = D^1 : S^0 \longrightarrow 1$$



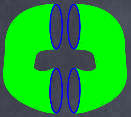
We arrive at a statement about 1-manifolds: the compositions

$$f^\vee \circ f = S^1 : 1 \longrightarrow 1$$
$$f \circ f^\vee \qquad : S^0 \longrightarrow S^0$$



are the identity.

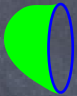
Let's now consider $n = 2$ where we assume that S^1 is invertible. We apply an easy algebraic lemma which asserts that invertible objects are dualizable and the dualization data is invertible. For S^1 these data are dual cylinders, and so the composition $S^1 \times S^1$ is also invertible.



Lemma: Suppose $\mathcal{D}$ is a symmetric monoidal category, $x \in \mathcal{D}$ is invertible, and $g\colon 1 \to x$ and $h\colon x \to 1$ satisfy $h \circ g = \mathrm{id}_1$. Then $g \circ h = \mathrm{id}_x$ and so each of g, h is an isomorphism.

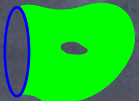
Proof sketch: x^{-1} is a dual of x, $\quad g^\vee = x^{-1}g\colon x^{-1} \to 1$, $h^\vee = x^{-1}h\colon 1 \to x^{-1}$, $\quad$ so the lemma follows from $(h \circ g)^\vee = \mathrm{id}_1$.

Apply the lemma to the 2-morphisms



$$g = D^2\colon 1 \longrightarrow S^1$$

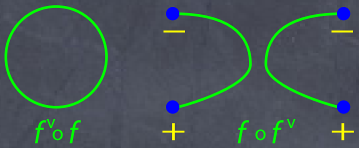
$$h = S^1 \times S^1 \backslash D^2\colon S^1 \longrightarrow 1$$



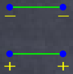
Conclude that $S^1 \cong 1$ and $S^2 = g^\vee \circ g$ is invertible.
Also, $g \circ g^\vee = \mathrm{id}_{S^1} \otimes S^2$, a simple surgery.

Recall that we must prove that the compositions

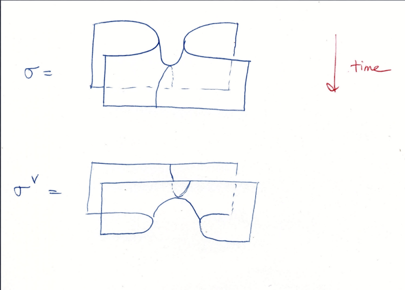
$$f^\vee \circ f = S^1 : 1 \longrightarrow 1$$
$$f \circ f^\vee \qquad : S^0 \longrightarrow S^0$$



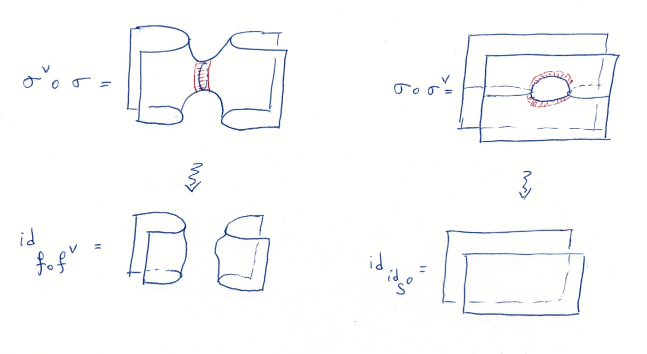
are the identity. We just did the first.

For the second the identity is 

and we will show that the saddle $\sigma\colon f \circ f^\vee \to \mathrm{id}_{S^0}$ is an isomorphism with inverse $\sigma^\vee \otimes S^2$.



The saddle σ is diffeomorphic to $D^1 \times D^1$, which is a manifold with corners. Its dual $\sigma^\vee$ is the time-reversed bordism.



Inside each composition $\sigma^\vee \circ \sigma$ and $\sigma \circ \sigma^\vee$ we find a cylinder $\mathrm{id}_{S^1} = D^1 \times S^1$, which is $(S^2)^{-1} \otimes g \circ g^\vee = (S^2)^{-1} \otimes (S^0 \times D^2)$ by a previous argument. Making the replacement we get the desired isomorphisms to identity maps.

This completes the proof of the theorem in $n = 2$ dimensions.

In higher dimensions we see a kind of Poincaré duality phenomenon: we prove invertibility by assuming it in the middle dimension. A new ingredient—a dimensional reduction argument—also appears.

Application to Modular Tensor Categories

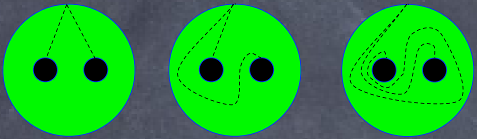
Let F denote the usual quantum Chern-Simons 1-2-3 theory for some gauge group G. It was introduced by Witten and constructed by Reshetikhin-Turaev from quantum group data. The latter construction works for any modular tensor category A, a braided tensor category which satisfies finiteness conditions (semisimple with finitely many simples, duality, etc.) and a nondegeneracy condition (the S matrix is invertible). Then F is a 1-2-3 theory with $F(S^1) = A$.

Let A be a braided tensor category with braiding $\beta(x, y)\colon x \otimes y \to y \otimes x$.

Theorem: The nondegeneracy condition on A is equivalent to

$$\{x \in A : \beta(y, x) \circ \beta(x, y) = \mathrm{id}_{x \otimes y} \text{ for all } y \in A\} = \{\text{multiples of } 1 \in A\}.$$

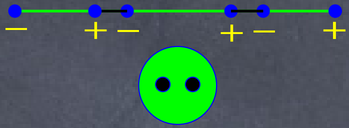
This is proved by Müger and others.



Recall that $\beta\otimes\mathbf{Cat}_{\mathbb{C}} = E_2(\mathbf{Cat}_{\mathbb{C}})$.

Braided tensor categories form the objects of a 4-category!

object	category number
element of $\mathbb{C}$	-1
$\mathbb{C}$-vector space	0
$\mathbf{Vect}_{\mathbb{C}}$	1
$\mathbf{Cat}_{\mathbb{C}}$	2
$\otimes\mathbf{Cat}_{\mathbb{C}} = E_1(\mathbf{Cat}_{\mathbb{C}})$	3
$\beta\otimes\mathbf{Cat}_{\mathbb{C}} = E_2(\mathbf{Cat}_{\mathbb{C}})$	4



Morita: Morphisms of tensor categories are bimodules and morphisms of braided tensor categories are tensor categories which are bimodules.

So, given sufficient finiteness, a braided tensor category determines (using the cobordism hypothesis) an extended 4-dimensional TQFT

$$\alpha\colon \mathrm{Bord}_4 \to \beta\otimes\mathbf{Cat}_{\mathbb{C}}$$

In the theory α we compute

$$\alpha(S^2) = \{x \in A : \beta(y,x) \circ \beta(x,y) = \mathrm{id}_{x\otimes y} \text{ for all } y \in A\} \in \mathbf{Cat}_{\mathbb{C}}$$

Recall that for a modular tensor category this "higher center" of A is the tensor unit $1 = \mathbf{Vect}_{\mathbb{C}}$, which in particular is invertible.

Thus, modulo careful verification of finiteness conditions, we have

Corollary: A modular tensor category $A \in \beta^{\otimes}\mathbf{Cat}_{\mathbb{C}}$ is invertible, so determines an invertible field theory $\alpha\colon \mathrm{Bord}_4 \to \beta^{\otimes}\mathbf{Cat}_{\mathbb{C}}$.

Remark: This is a theorem in algebra, proved using the universal "algebra", rather (∞, n)-category, of manifolds with corners.

We believe that this is the anomaly theory for a 0-1-2-3 extension of the 1-2-3 theory F with $F(S^1) = A$. In the remainder of the lecture I will explain this idea.

Anomalous Field Theories

The top-level values of an n-dimensional field theory $F\colon \mathrm{Bo}_n \to \mathbf{Vect}_{\mathbb{C}}$ are complex numbers $F(X^n) \in \mathbb{C}$, the *partition function* of a closed n-manifold. In an anomalous field theory f there is a complex line L_X associated to X and the partition function $f(X) \in L_X$ lies in that line.

The lines L_X obey locality and multiplicativity laws, so typically belong to an $(n+1)$-dimensional *invertible* field theory $\alpha\colon \mathrm{Bo}_{n+1} \to \mathbf{Vect}_{\mathbb{C}}$.

f is an n-dimensional theory with values in the $(n+1)$-dimensional theory α. We write $f\colon 1 \to \alpha$ in the sense that $f(X)\colon 1 \to \alpha(X)$ for all X. (1 is the trivial theory.) If α is invertible we say f is anomalous with anomaly α. The same ideas apply in *extended* field theories.

Remark: The notion of α-valued field theory makes sense even if α is not invertible and also for non-topological field theories. Examples: (i) $n = 2$ chiral WZW valued in topological Chern-Simons, (ii) $n = 6$ (0,2)-(super)conformal field theory valued in a 7-dimensional theory.

Remark: This is a specialization of the notion of a domain wall.

Fully Extended Chern-Simons

Recall that a modular tensor category A determines an invertible extended field theory $\alpha\colon \mathrm{Bord}_4 \to \beta_{\otimes}\mathbf{Cat}_{\mathbb{C}}$ with values in the 4-category of braided tensor categories, or equivalently E_2-algebras in $\mathbf{Cat}_{\mathbb{C}}$.

An ordinary algebra A is in a natural way a left A-module. This holds for E_2-algebras, and in that context the module defines a morphism $A\colon 1 \to A$ in the 4-category $\beta_{\otimes}\mathbf{Cat}_{\mathbb{C}}$.

Let A be a modular tensor category. Modulo careful verification of finiteness conditions, a version of the cobordism hypothesis constructs from the module A a 0-1-2-3-dimensional anomalous field theory $f\colon 1 \to \alpha$ with anomaly α.

Claim: On 1-, 2-, and 3-dimensional manifolds we can trivialize the anomaly α and so identify f with the Reshetikhin-Turaev 1-2-3 theory F associated to the modular tensor category A.

For example, the composition $1 \xrightarrow{f(S^1)} \alpha(S^1) \xrightarrow{\alpha(D^2)} 1$ is $F(S^1) = A$, where the bordism $D^2\colon S^1 \to 1$ is used to trivialize the anomaly on S^1.