Three Applications of Topology to Physics

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Problem 1: Invertible Phases of Matter

Fix discrete parameters for quantum system: dimension, symmetry type
(Quantum system: QFT, stat mech system, string theory . . .)

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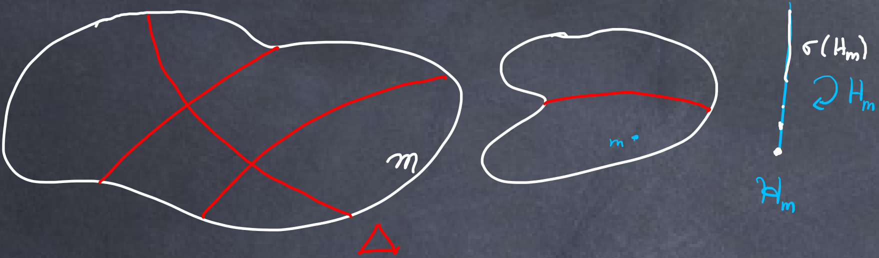
$\mathcal{M}$	moduli space
$\Delta \subset \mathcal{M}$	locus of phase transitions



Path components $\pi_0(\mathcal{M} \setminus \Delta)$ are *deformation classes* = *phases*

Warning: Often the quantum system, much less $\mathcal{M}$, has no rigorous mathematical definition/construction

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Analog in Geometry: $\mathcal{M}$ moduli space of 1d Riemannian manifolds M

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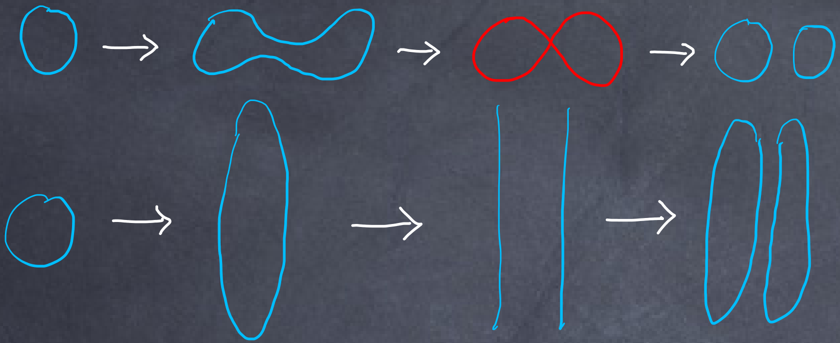
Two paths connecting 1 circle to 2 circles:



The first disallowed because *manifolds*; Δ = noncompact manifolds

Then $\pi_0(\mathcal{M} \setminus \Delta) \xrightarrow{\cong} \mathbb{Z}^{\geq 0}$; the map counts the components of M

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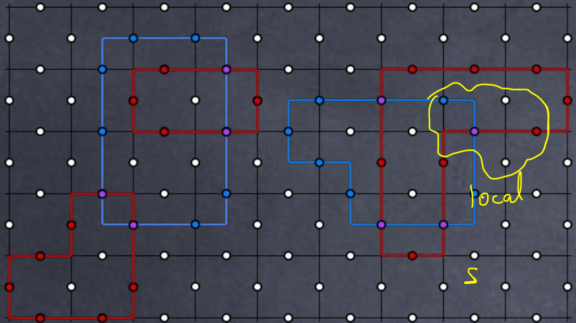
d dimension of space

I global symmetry group

Invertible ("short range entangled") gapped lattice systems:



$\mathcal{H} = \bigotimes_s \mathcal{H}_s$

$H = \sum_{\text{local}} H_{\text{local}} \otimes \text{id}$

Invertible: Unique ground state on each compact spatial manifold Y^d

Open Problem: Define moduli space $\mathcal{M}'(d, I)$ and compute π_0

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Two physical principles to move to QFT:

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- LEP of *gapped* system well-approximated by *topological** field theory

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We imagine a homotopy equivalence

$$\mathcal{M}'(d, I) \overset{\text{low energy approximation}}{\dashrightarrow} \mathcal{M}(n, H)$$

to a moduli space $\mathcal{M}(n, H)$ of invertible field theories

n	dimension of spacetime
$\rho_n \colon H_n \longrightarrow O_n$	symmetry type (to be explained)

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Problem 2: Parity Invariance of M-Theory

We investigate in low energy field theory approx: 11d supergravity + quantum correction. Parity invariance = time-reversal symmetry

Problem (Witten): Can we consistently formulate M-theory on *unorientable* manifolds? (Yes: M2-brane)

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Remark: Problems in string theory are more fun than those in condensed matter theory: higher dimensions!

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4d QCD is a gauge theory with fermionic fields and internal symmetry group $K = G \times G$ where $G = SU_{N_f}$

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Solution introduces a refined WZW factor in exponentiated action

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Can often develop theories separately from construction of examples.
But examples are important: they are what Mathematics is about!

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Need for foundations (definitions and axioms) arose from concrete problems and crises

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Symmetry $\longrightarrow$ background fields. No fluctuating fields. No lagrangians. No quantization. Axiomatize quantum structure.

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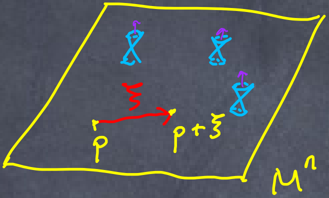

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$\mathcal{H}_n \xrightarrow{\rho_n} \mathcal{I}^{\uparrow}_{1,n-1}$	unbroken global relativistic symmetry group
$H_{1,n-1}$	$\mathcal{H}_n$/translations
$K := \ker(\rho_n)$	internal symmetry group (compact)

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Remark: The internal symmetry group K can also include supersymmetries and higher symmetries

Wick Rotation

Positive energy $\Longrightarrow$ correlation functions are boundary values of holomorphic fns on a complex domain $\mathcal{D}$. Restrict to Euclidean space $\mathbb{E}^n$

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Schematic notation for Wick rotation: $M^n \rightsquigarrow \mathcal{D} \rightsquigarrow \mathbb{E}^n$

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  - There is a splitting $\mathfrak{h}_n \cong \mathfrak{o}_n \oplus \mathfrak{k}$ (recall Coleman-Mandula)
  - $(n \geq 3)$ There exists central element $k_0 \in K$ with $(k_0)^2 = 1$ and a canonical homomorphism $\mathrm{Spin}_n \to H_n$ mapping -1 to k_0
  - There exists a canonical stabilization

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states/symmetry	H_n	K	k_0
bosons only	SO_n	$\{1\}$	1
bosons, time-reversal (T)	O_n	$\{1\}$	1
fermions allowed	Spin_n	$\{\pm 1\}$	-1
fermions, $T^2 = (-1)^F$	Pin_n^+	$\{\pm 1\}$	-1
fermions, $T^2 = \mathrm{id}$	Pin_n^-	$\{\pm 1\}$	-1

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Including translations the Euclidean symmetry group is an extension

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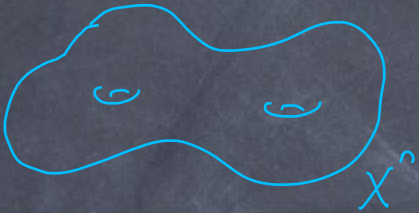

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Nontrivial step: Restrict to *compact* manifolds. Not at all obvious that we retain IR physics, but will see so in examples

The Axiom System

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Axioms capture properties of state spaces and correlation functions

Discrete data: spacetime dimension n
symmetry type (H, ρ)

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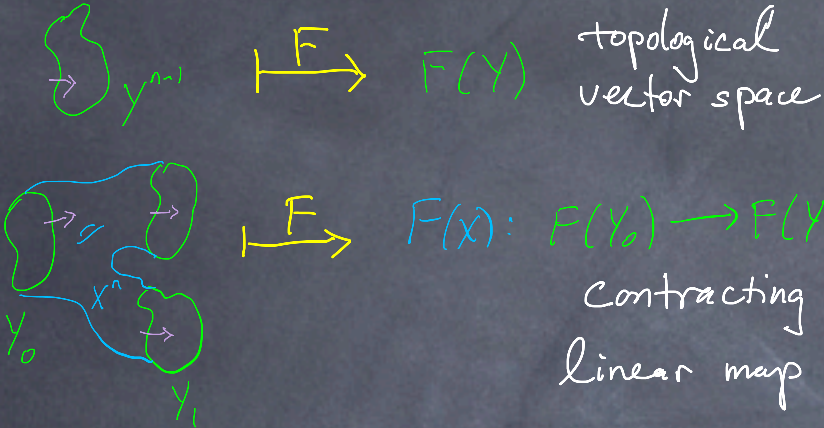

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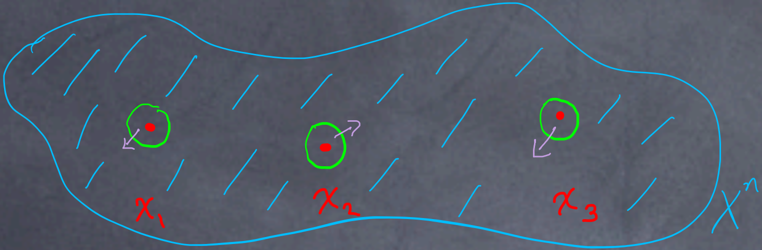

$$F(X)\colon\; F(Y_1)\otimes F(Y_2)\otimes F(Y_3) \longrightarrow \mathbb{C}$$

correlation functions

Reconstruction Question: Reverse $M^n \rightsquigarrow \mathcal{D} \rightsquigarrow \mathbb{E}^n \rightsquigarrow \text{cpt } X^n$?
In essence, we assume that the answer is "yes" and work with field theories using this Axiom System.

Locality and Unitarity

The state space $F(Y^{n-1})$ depends *locally* on Y

$$F(Y) \text{ "=" } \langle F(Y_-), F(Y_+) \rangle_{F(Z)}$$

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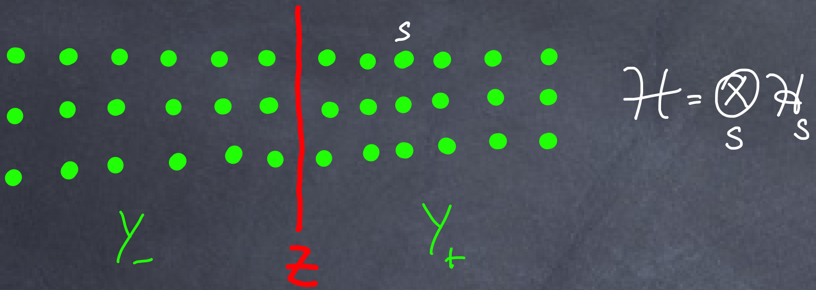

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Powerful classification theorem for *topological* theories (Lurie)

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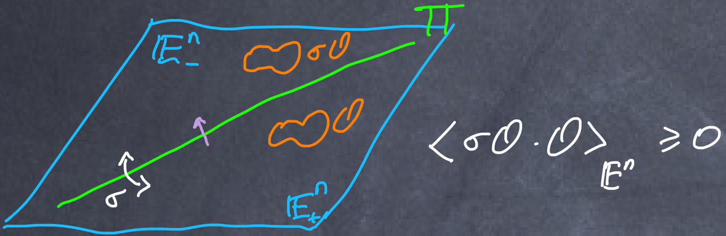
Extended field theory: invariants for manifolds of dimension $\leq n$
brings in higher categorical ideas

Powerful classification theorem for *topological* theories (Lurie)

Wick-rotated unitarity: *reflection positivity*



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Open Question: What is *extended* reflection positivity?

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We propose a solution for invertible topological theories

Invertibility and Homotopy Theory

Field theories have a composition law $F \otimes F'$ and a trivial theory $\mathbf{1}$

Definition: A field theory F is *invertible* if there exists F' such that $F \otimes F'$ is isomorphic to $\mathbf{1}$

F invertible $\Longrightarrow \dim F(Y) = 1$ for all closed Y^{n-1} $(\partial Y = \emptyset)$

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Invertible theories are maps in stable homotopy theory:

$$\begin{array}{ccc} \mathrm{Bord}_n(H_n) & \xrightarrow{\;F\;} & \mathcal{C} \\ \downarrow & \searrow & \uparrow \\ \overline{\mathrm{Bord}_n(H_n)} & \xrightarrow{\;\widetilde{F}\;} & \mathcal{C}^{\times} \end{array}$$

(Grothendieck) $\widetilde{F}$ "is" an ∞-loop map of ∞-loop spaces (spectra)

Main Theorem

$\mathcal{M}_{\text{top}}(n, H_n) :=$ moduli space of reflection positive invertible n-dimensional extended topological field theories with symmetry group H_n

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Extended positivity

Spacetime dimension

symmetry type

Discrete Data

Thom spectrum

Stabilized symmetry group

Anderson dual to sphere spectrum

torsion subgroup

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Theorem and Conjecture determine entire homotopy type, not just π_0

Now apply Theorem and Conjecture to Problem 1 (Phases of matter) and Problem 2 (Parity invariance of M-theory)

Problem 3 (WZW factor): different application of invertible field theory

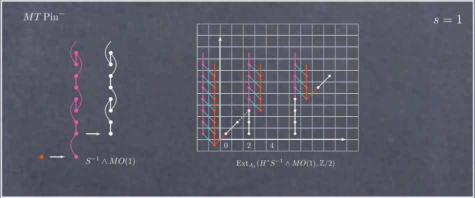
Solution 1: `arXiv:1604.06527 (revised version soon)`
Solution 2: `to appear`
Solution 3: `arXiv:hep-th/0607134`

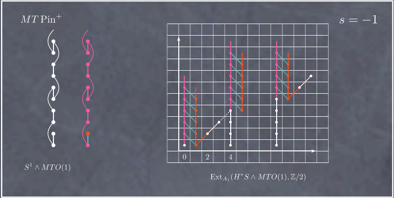
Relativistic 10-fold way

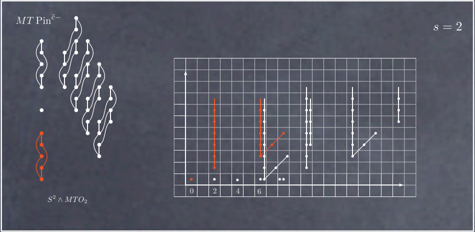
- For electron systems expect $K = U(1) = \mathbb{T}$
- Spin/charge relation: $-1 \in \mathbb{T}$ is central element of Spin_n $(= (-1)^F)$
- Particle-hole symmetry: "breaks" $K = \mathbb{T}$ to $K = \{\pm 1\}$ or $K = SU_2$

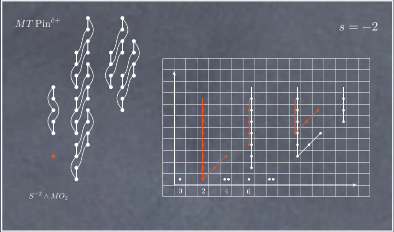
Theorem: There are 10 stable symmetry groups H of this type:

$K = \mathbb{T}$	Spin^c, Pin^c,
	$\mathrm{Pin}^{\tilde{c}+} := \mathrm{Pin}^+ \ltimes_{\{\pm 1\}} \mathbb{T}$
	$\mathrm{Pin}^{\tilde{c}-} := \mathrm{Pin}^- \ltimes_{\{\pm 1\}} \mathbb{T}$
$K = \{\pm 1\}$	Spin, Pin^+, Pin^-
$K = SU_2$	$\mathrm{Spin} \times_{\{\pm 1\}} SU_2$
	$\mathrm{Pin}^+ \times_{\{\pm 1\}} SU_2$
	$\mathrm{Pin}^- \times_{\{\pm 1\}} SU_2$


$MT\,\mathrm{Pin}^-$
$s = 1$
$S^{-1} \wedge MO(1)$
0 2 4
$\mathrm{Ext}_{A_1}(H^*S^{-1} \wedge MO(1), \mathbb{Z}/2)$


$MT\,\mathrm{Pin}^+$
$s = -1$
$S^1 \wedge MTO(1)$
0 2 4
$\mathrm{Ext}_{A_1}(H^*S \wedge MTO(1), \mathbb{Z}/2)$


$MT\,\mathrm{Pin}^{\tilde{c}-}$
$s = 2$
0 2 4 6
$S^2 \wedge MTO_2$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0 2 4 6
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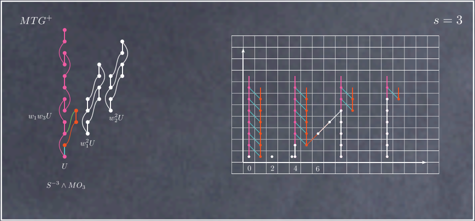
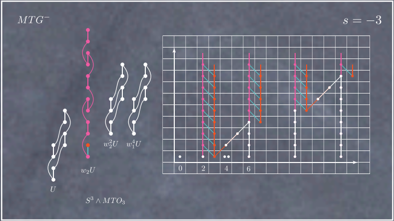

MTG^+
$s = 3$
w_1w_3U
w_2^2U
w_1^2U
U
0 2 4 6
$S^{-3} \wedge MO_3$


MTG^-
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U
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Computations

Class DIII (Pin^+):

n	$\ker\Phi \longrightarrow$	$FF_n(\mathrm{Pin}^+) \xrightarrow{\Phi}$	$TP_n(\mathrm{Pin}^+) \longrightarrow$	$\operatorname{coker}\Phi$
4	$16\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/16\mathbb{Z}$	0
3	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
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- Φ is the map described above (essentially ABS)
- The FF_n groups are well-known. Many TP_n appear in the condensed matter literature (together with Φ) via other methods

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- Metlitski asked about $TP_4(\mathrm{Pin}^{\tilde{c}+})$ vs. bordism computation
- The results in 3 dimensions are also known via non-bordism means

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n	$\ker\Phi \longrightarrow$	$FF_n(G^+) \xrightarrow{\Phi}$	$TP_n(G^+) \longrightarrow$	$\mathrm{coker}\,\Phi$
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- Unsure if $TP_{2,3}(G^+)$ are in the CM literature (predictions)

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Other hty computations: Kapustin et. al., Campbell, Guo-Putrov-Wang

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- Wick-rotated theory on *compact* X^n detects long-range behavior
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Open Problems: Extended positivity for general field theories
Relation of lattice system and field theory

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Theorem (FH): Trivializations of α form a torsor over $(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$.

Computer/hand computations (Adams spectral sequence) to find generators of the bordism group and compute partition fns of α_{RS}, α_C

Theorem (FH): Let

$$\begin{aligned} A &= A_0' \oplus A_0'' \oplus A_1 \oplus A_3 \oplus A_4 \oplus A_5 \\ &= \mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \end{aligned}$$

Then there is a surjective homomorphism

$$\rho\colon A \longrightarrow \pi_{12}\mathcal{B}$$

under which the indicated manifolds and twisted integral lifts of w_4 represent images of generators:

$$\begin{aligned} \rho(a_0') &= [(W_0', \tilde{c}_0')] \\ \rho(a_0'') &= [(W_0'', 0)] \\ \rho(a_1) &= [(W_1, \lambda z)] \\ \rho(a_3) &= [(K \times \mathbb{HP}^2, \lambda)] \\ \rho(a_4) &= [(\mathbb{RP}^4 \times B, \tilde{c}_{\mathbb{RP}^4})] \\ \rho(a_5) &= [((\mathbb{RP}^4 \# \mathbb{RP}^4) \times B, 0)] \end{aligned}$$

Solution 3: WZW Factor in Theory of Pions

4d QCD is a gauge theory with fermionic fields and internal symmetry group $K = G \times G$ where $G = SU_{N_f}$, so $H_4 = \mathrm{Spin}_4 \times G \times G$

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Witten: studed on S^4 to normalize coefficient: WZW class in $H^5(G;\mathbb{Z})$

Solution: WZW term lives in $E^5(G)$, *generalized cohomology theory* E

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^5(SU_N;\mathbb{Z}) & \longrightarrow & E^5(SU_N) & \longrightarrow & H^3(SU_N;\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\;2\;} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \end{array}$$

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$$

1. Integration of classes in E requires a spin structure
2. Evaluation on Y^3 naturally leads to a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space

Solution: WZW term lives in $E^5(G)$, *generalized cohomology theory* E

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1. Integration of classes in E requires a spin structure
2. Evaluation on Y^3 naturally leads to a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space
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4. $E^5(SU_2) \cong \mathbb{Z}/2\mathbb{Z}$