

Math 392C: Riemannian Geometry

Fall Semester 2010: Unique Number 55765

2:00 Tuesdays/Thursdays in RLM 12.166

Introduction

This course is an introduction to Riemannian geometry. It is intended for those already familiar with topological and differentiable manifolds. The objects of Riemannian geometry are smooth manifolds equipped with extra structures that provide geometric information. In particular, we will equip a manifold with a *Riemannian metric* that allows measurement of quantities such as distance and angle, and an *affine connection* that allows comparison and differentiation of tensor (e.g. vector) fields. These structures can be constrained by the topology of the manifold, hence may not be prescribed arbitrarily. Such constraints often manifest themselves through the key idea of *curvature*, which measures how the geometry of a given smooth manifold differs from that of Euclidean space. Indeed, at sufficiently small scales, every manifold resembles Euclidean space. As the length scale increases, the deviation from a Euclidean model is reflected in the space's intrinsic curvature, which evinces a deep and beautiful relationship between geometry and topology, especially in low dimensions. A classical example is the Gauß-Bonnet theorem, which says that when one integrates the scalar curvature of a Riemannian metric on a compact surface, one recovers its Euler characteristic: $\int_{\mathcal{M}^2} R \, dA = 4\pi \cdot \chi(\mathcal{M}^2)$.

Text

Our recommended text is:

- John M. Lee. *Riemannian Manifolds: An Introduction to Curvature*.

Good alternate texts are:

- Jürgen Jost. *Riemannian Geometry and Geometric Analysis*.
- Peter Petersen. *Riemannian Geometry*.

Contact information

My office hours for this course are 1:00–2:00 Mondays in RLM 9.152, and by appointment. If I'm not busy with Graduate Adviser business, I'm also happy to discuss geometry during my GA office hours, which are 2:00–4:00 Mondays in RLM 8.146.

Workload

Like most of mathematics, Riemannian geometry is not readily learned without actively engaging the material. So to enhance learning, there will be a small number (5–8) of homework assignments during the semester, each comprising a small number of short problems. The main purpose of these exercises will be to enhance understanding of topics discussed in class. To keep the workload light, these exercises will be graded on a credit/no-credit basis: I'll give credit for a reasonable attempt at completion.

During the semester, I will also distribute a list of projects. Those desiring an A may be asked to prepare and present a short presentation on one of these assigned projects. These presentations may be done singly or in pairs.

Outline

We will cover the following topics, investigating the most important ones in detail and at least introducing the others. If there is interest and time remaining at the end of the semester, I may introduce other important topics, like some of the basic ideas behind Ricci flow.

Chapter 0 Differentiable manifolds

- Tensor fields, bundles
- Differential forms

Chapter 1 Riemannian metrics

- Induced metrics on tensor bundles
- Metric inverse

Chapter 2 Connections and covariant differentiation

- Differentiation of tensor fields
- Higher derivatives
- Lie and exterior derivatives

Chapter 3 Geodesics

- Variational approach; Euler–Lagrange equation
- Injectivity radius
- Completeness, Hopf–Rinow

Chapter 4 (a) Curvature

- Bianchi identities
- Ricci identities
- Bochner formulas
- Decompositions
- Examples

Chapter 4 (b) Integration

- Applications of Bochner formulas
- Eigenvalues, eigenfunctions, eigentensors
- Hodge theory

Chapter 5 Jacobi fields

- Riccati equations
- Differentiation of distance
- Laplacian comparison

Chapter 6 Isometric immersions

- Riemannian submersions
- Variation of area, volume

Chapter 7 Complete manifolds

- cut loci
- injectivity radius estimates

Chapter 8 Spaces of constant curvature

- Homogeneous spaces
- Symmetric spaces

Chapter 9 Variations of length, energy

- Applications to comparison geometry
- Stability of (long) geodesics

Chapter 10 Rauch comparison theorem

- Toponogov comparison theorem