NON-KÄHLER RICCI FLOW SINGULARITIES MODELED ON KÄHLER–RICCI SOLITONS

A CHAPTER CONTRIBUTED TO A *PURE AND APPLIED MATHEMATICS QUARTERLY* VOLUME HONORING ROBERT BARTNIK'S CONTRIBUTIONS TO MATHEMATICAL PHYSICS AND MATHEMATICS

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Abstract. We investigate Riemannian (non-Kähler) Ricci flow solutions that develop finite-time Type-I singularities and present evidence in favor of a conjecture that parabolic rescalings at the singularities converge to singularity models that are shrinking Kähler–Ricci solitons. Specifically, the singularity model for these solutions is expected to be the "blowdown soliton" discovered in [FIK03]. Our partial results support the conjecture that the blowdown soliton is stable under Ricci flow, as well as the conjectured stability of the subspace of Kähler metrics under Ricci flow.

1. Introduction

While the behavior of Ricci flow is fairly well-understood for three-dimensional Riemannian geometries, significantly less is known about four-dimensional Ricci flow. In this work, we study Ricci flow for a certain family of four-dimensional geometries (defined in Section 1.3) that develop finite-time Type-I singularities. Our interest in these geometries is to illuminate two outstanding issues concerning four-dimensional Ricci flows: i) the stability of certain singularity models in such flows, and ii) the behavior of Ricci flows that start at non-Kähler Riemannian geometries which are nonetheless close to Kähler geometries. To motivate our work here, we discuss each of these issues in turn.

1.1. **Behavior of "generic Ricci flow".** One of the keys to understanding the nature of singularities that develop in solutions of n-dimensional Ricci flow is to adequately classify the set of singularity models that may arise. Singularity formation in 3-dimensional Ricci flow has been fairly well-understood since the work of Hamilton [Ham93] and of Perelman [Per02]. Indeed, it follows from the pinching estimate derived by Ivey [Ive93] and improved by Hamilton [Ham93] that the only possible 3-dimensional singularity models have nonnegative sectional curvature, which is a highly restrictive condition. By contrast, Máximo's results [Max14] imply that, starting in dimension $n = 4$, models of finite-time singularity formation can have Ricci curvature of mixed sign (even for Kähler solutions). As is well known, singularity models in every dimension have nonnegative scalar curvature

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(see, *e.g.,* [Che09]). However, as the only proven restriction on singularity models in dimensions $n \geq 4$, this condition is too weak to be very useful.

In dimensions $n \geq 4$, therefore, a classification of all singularity models is impractical. A more promising alternative is to try to classify those models that are generic, or at least stable. A singularity model developing from certain original data is labeled *stable* if flows starting from all sufficiently small perturbations of that data develop singularities with the same singularity model; it is labeled *generic* if flows that start from an open dense subset of all possible initial data develop singularities having the same singularity model. Clearly, a singularity model can be generic only if it is stable.

Important work of Colding and Minicozzi (see [CM12] and [CM15]) provides strong support in favor of the conjecture that the only generic singularities of Mean Curvature Flow are generalized cylinders $\mathbb{R}^m \times \mathcal{S}^{n-m}$. Although no analogous result is currently known for Ricci flow, a conjectural picture comes from the work of Cao, Hamilton, and Ilmanen [CHI04], who define the *central density* Θ and the *entropy* $\nu(\mathcal{M})$ of a shrinking Ricci soliton $\mathcal{M}$, using Perelman's reduced volume and entropy, respectively (see [Per02]). They observe that their central density imposes a partial order on shrinking solitons: monotonicity of the ν-functional in time means that if perturbations of a shrinking soliton develop singularities, these cannot be modeled on solitons of lower density. (Compare [CM12].)

Motivated partly by [CHI04], it is conjectured by experts (see, e.g., [HHS14]) that the only generic singularity models in real dimension $n = 4$ are $\mathcal{S}^4$, $\mathcal{S}^3 \times \mathbb{R}$, $\mathcal{S}^2 \times \mathbb{R}^2$ (all with their canonical metrics), and $(\mathcal{L}^2_{-1}, h)$.[1] The manifold $(\mathcal{L}^2_{-1}, h)$, which is constructed and studied in [FIK03], is a U(2)-invariant gradient Kähler shrinking soliton on the complex line bundle[2] $\mathbb{C} \hookrightarrow \mathcal{L}^2_{-1} \twoheadrightarrow \mathbb{CP}^1$, which is the complex bundle $\mathcal{O}(-1)$; *i.e.,* it is the blow-up of $\mathbb{C}^2$ at the origin. The next manifold on the list in [CHI04], ordered by the central density Θ, is $\mathbb{CP}^2$ with its Fubini–Study metric.

As noted above, a pre-condition for a singularity model being generic is that it must be dynamically stable for Ricci flow regarded as a dynamical system on the space of Riemannian metrics. Dynamic stability of $\mathcal{S}^4$ is well established. (In fact, Brendle and Schoen [BS09] show that its basin of attraction includes all 1/4-pinched metrics.) Stability of generalized cylinders is strongly conjectured but not known for Ricci flow. Stability of the blowdown soliton is also not known, although Máximo's proof [Max14] shows that arbitrarily small U(2)-invariant Kähler perturbations of the unstable shrinking soliton on $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ (which was discovered independently by Koiso [Koi90] and by Cao [Cao96]) develop singularities modeled on $(\mathcal{L}^2_{-1}, h)$. (We remark that prior to Máximo's results, it was shown in [HM11] that the Koiso-Cao soliton is variationally unstable with respect to Perelman's functional ν. We note also that Máximo's results were extended to general dimensions by Guo and Song [GS16], who thus establish in full generality the conjecture made in part (3) of Example 2.2 in [FIK03].) $\mathbb{CP}^2$ is well known to be weakly variationally stable, and was expected by many to be dynamically stable. However, Kröncke [Kro13] has proven that it is unstable: there are conformal perturbations that increase ν, so Ricci flow starting from these initial data can never return to the Fubini–Study

[1] One reason this expectation is conjectural is that it is not known if there exist Type-II singularity models on which Perelman's ν-functional is undefined. However, such models, if they exist, are not expected to be generic.

[2] The bundle we label $\mathcal{L}^2_{-1}$ here is denoted by $\mathcal{L}^{-1}$ in [FIK03] and by $L(2, -1)$ in [CHI04].

metric (This has recently been independently verified by two of the authors [KS19].) That leaves $(\mathcal{L}^2_{-1}, h)$ as a critical "borderline" case. Our results in this chapter provide some evidence in favor of the conjectured dynamic stability of $(\mathcal{L}^2_{-1}, h)$. If true, this would indicate an incomplete analogy between Ricci flow and mean curvature flow, where only generalized cylinders are stable [CM12].

While the construction of the $(\mathcal{L}^2_{-1}, h)$ shrinker involves the blowup of a point on $\mathbb{C}^2$, following the authors of [CHI04]), we call $(\mathcal{L}^2_{-1}, h)$ the *blowdown soliton.* We do this because, as shown in Theorem 1.6 of [FIK03], there is a family of Riemannian manifolds $\mathcal{N}_t$, $-\infty < t < \infty$, with the following features: for $t < 0$, $\mathcal{N}_t$ is $(\mathcal{L}^2_{-1}, h(t))$; for $t = 0$, $\mathcal{N}_0$ is a Kähler cone on $\mathbb{C}^2$ with an isolated singularity at the origin; and for $t > 0$, $\mathcal{N}_t$ is an expanding soliton discovered by Cao [Cao97]. It is expected that according to most (if not all) of the definitions of a *weak solution of Ricci flow* which are currently being explored (e.g., see [HN18] and [Stu18]), the family $\mathcal{N}_t$ will qualify for such a designation. Consequently, in this weak sense, one sees that Ricci flow can carry out a blowdown, understood in the sense of algebraic geometry.

Our results in this chapter provide significant, albeit incomplete, evidence that the blowdown soliton is a singularity model attractor for solutions of Ricci flow that originate from a set of compact Riemannian initial data defined by a structural (isometry) hypothesis and by a (weak) set of pinching conditions that we specify in Section 2 below. Metrics in this set are not Kähler. As noted above, these partial results provide some evidence in favor of the conjectured stability of $(\mathcal{L}^2_{-1}, h)$. What prevents this chapter from providing a complete proof is its reliance on two technical conjectures discussed below, for which we present formal arguments but thus far lack rigorous arguments.

1.2. **Behavior of Ricci flow near Kähler geometries.** As noted above, the $(\mathcal{L}^2_{-1}, h)$ shrinker is Kähler. Hence, the study of non-Kähler Ricci flows near the blowdown soliton provides information about the difficult issue of the behavior of Ricci flow solutions that start near, but not in, the subspace of Kähler metrics. Do those solutions stay near or (better) asymptotically approach that subspace, which is of infinite codimension? It is believed by many experts that the subspace of Kähler metrics *should* be dynamically stable for nearby solutions of Ricci flow. Evidence of favor of this conjecture is provided by the work of Streets and Tian [ST11], who prove that the Kähler subspace is an attractor for Hermitian curvature flow.

While our results fall far short of a general stability principle for Kähler geometries, the partial results and a formal argument presented later in this work do support a conjectural picture of non-Kähler solutions of Ricci flow that become asymptotically Kähler, in suitable space-time neighborhoods of developing singularities, at rates that break scaling invariance. We hope that the evidence we give here provides motivation for further study of this general question, particularly in (real) dimension $n = 4$.

1.3. **Organization.** The general class of Riemannian geometries that we study in this work are smooth cohomogeneity-one metrics on the closed manifold $\mathcal{S}^2 \tilde{\times} \mathcal{S}^2$ (the "twisted bundle" of $\mathcal{S}^2$ over $\mathcal{S}^2$). We describe these geometries (which we label "$[\mathcal{S}^2 \tilde{\times} \mathcal{S}^2]$-warped Berger geometries") in detail below in Section 2. Here, for the purposes of stating our main conjecture, we note that for these metrics, there are two distinguished fibers $\mathcal{S}^2_\pm$ (at either "pole"); by contrast, a generic fiber is diffeomorphic to $\mathcal{S}^3$.

In Section 2.3, we identify an open subset of the $[\mathcal{S}^2\tilde{\times}\mathcal{S}^2]$-warped Berger geometries by means of five pinching inequalities. These inequalities constitute our Closeness Assumptions, which we require the initial data for our Ricci flow solutions to satisfy. These assumptions ensure that our initial data, while not Kähler, are "not too far" from the subspace of Kähler metrics. In Section 2.4, we prove that our assumptions are not vacuous; *i.e.,* we show that the open subset of initial data satisfying the Closeness Assumptions is not empty.

We clarify the relationship between Kähler geometries and the $[\mathcal{S}^2\tilde{\times}\mathcal{S}^2]$-warped Berger geometries in Section 3. Also in that section, we provide some background information about the blowdown soliton.

In the remainder of this work, we prove a sequence of Lemmata and Corollaries that combine to yield an *almost complete* proof — modulo two technical conjectures discussed below — of the following result:

Main Conjecture. *There exists a nonempty open set of non-Kähler metrics on $\mathcal{S}^2\tilde{\times}\mathcal{S}^2$ (contained in the $[\mathcal{S}^2\tilde{\times}\mathcal{S}^2]$-warped Berger class, and satisfying the Closeness Assumptions) such that any Ricci flow solution originating from this set has the following properties:*

1. *Inequalities (a)–(d) in the Closeness Assumptions are preserved by the flow.*
2. *The solution develops a Type-I singularity at $T<\infty$, with $|\mathcal{S}^2_-(T)|=0$.*[3]
3. *Every blow-up sequence $\big(\mathcal{S}^2\tilde{\times}\mathcal{S}^2, G_k(t), p\big)$ with $p\in\mathcal{S}^2_-$ subconverges to a Kähler singularity model that is the blowdown shrinking soliton $(\mathcal{L}^2_{-1}, h)$.*

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2. The set-up

2.1. **Topology and geometry.** In [IKS16], we study "warped Berger" metrics which take the form

$$G=\mathrm{d}s\otimes\mathrm{d}s+\Big\{f^2\,\omega^1\otimes\omega^1+g^2\big(\omega^2\otimes\omega^2+\omega^3\otimes\omega^3\big)\Big\} \tag{1}$$

on $[s_-,s_+]\times\mathrm{SU}(2)$, where $\{\omega^1,\omega^2,\omega^3\}$ constitutes a one-form basis for $\mathrm{SU}(2)$, where $s(x)$ denotes arclength[4] from $x=0$, with $x\in[-1,1]$, and where we set $s_\pm:=s(\pm1)$. The functions f and g depend only on x (or equivalently on s); hence these metrics are cohomogeneity one. In [IKS16], we choose boundary conditions on f and g that result in these metrics inducing geometries on $\mathcal{S}^3\times\mathcal{S}^1$. Here, we instead choose boundary conditions on f and g that result in smooth cohomogeneity-one geometries on $\mathcal{S}^2\tilde{\times}\mathcal{S}^2$, thereby defining the class of $[\mathcal{S}^2\tilde{\times}\mathcal{S}^2]$-warped Berger geometries. We do this as follows.

It is a standard result in Riemannian geometry that one may smoothly close the boundary at s_-, provided that the functions $f_-(s):=f(s_-+s)$ and $g_-(s):=g(s_-+s)$ defined for $0\le s\le s_+-s_-$ satisfy

$$f_-^{(\text{even})}(0)=0,\quad f_-'(0)=1,\qquad\text{and}\qquad g_-(0)>0,\quad g_-^{(\text{odd})}(0)=0. \tag{2}$$

The topology then locally becomes that of the disc bundle $\mathcal{D}^2\hookrightarrow\mathcal{D}^4_1\twoheadrightarrow\mathcal{S}^2$ with Euler class 1 and boundary $\partial\mathcal{D}^4_1\approx\mathcal{S}^3$ that appears in the handlebody construction

[3] We denote the area of either exceptional fiber at any time $t\in[0,T]$ by $|\mathcal{S}^2_\pm(t)|$.

[4] For our initial data, s is a function of the spatial variable x alone. Once we flow $G(t)$, arclength $s(x,t)$ will depend on time t as well: it acts as a choice of gauge to make the flow strictly parabolic.

of $\mathbb{CP}^2$. Note that the 2-sphere here is the base of the Hopf fibration on $\mathbb{S}^3 \approx \mathrm{SU}(2)$. If one repeats this construction at s_+, with $f_+(s) := f(s_+ + s)$ and $g_+(s) := g(s_+ + s)$ defined for $s_- - s_+ \le s \le 0$ satisfying

$$f_+^{(\mathrm{even})}(0) = 0, \quad f_+'(0) = -1, \qquad \text{and} \qquad g_+(0) > 0, \quad g_+^{(\mathrm{odd})}(0) = 0, \tag{3}$$

one obtains a closed 4-manifold with the topology of $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$. We denote by $\mathbb{S}^2_\pm$ the distinguished 2-spheres that appear as the fibers in the closing construction at either "pole" $s_\pm$. We note that while $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ is diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, the Ricci flow evolutions we study are not Kähler.

The metrics $G = \mathrm{d}s \otimes \mathrm{d}s + f^2\,\omega^1 \otimes \omega^1 + g^2 \omega^2 \otimes \omega^2 + h^2 \omega^3 \otimes \omega^3$ described in Appendix A of [IKS16] are clearly SU(2)-invariant. The simplifying assumption $h \equiv g$ made here enlarges their symmetry group to U(2). However, although $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ admits Kähler metrics, including the U(2)-invariant Kähler–Ricci soliton mentioned above, we observe in Lemma 1 that metrics of the form (1) cannot be Kähler unless they satisfy the closed condition $f = g g_s$.

2.2. **Ricci flow equations.** In this section, we investigate solutions $\big(\mathbb{S}^2 \tilde{\times} \mathbb{S}^2, G(t)\big)$ of Ricci flow that originate from smooth initial data $G(0)$ satisfying the closing conditions (2) and (3) for $[\mathbb{S}^2 \tilde{\times} \mathbb{S}^2]$-warped Berger geometries, as outlined above. For as long as such solutions remain smooth, the functions f and g continue to satisfy conditions (2) and (3), and hence remain $[\mathbb{S}^2 \tilde{\times} \mathbb{S}^2]$-warped Berger geometries.

Since the metrics studied in [IKS16] and those studied here are the same apart from boundary conditions, we may use formulas (10)–(13) of [IKS16] to obtain the sectional curvatures[5] of the metric G:

$$\kappa_{12} = \kappa_{31} = \frac{f^2}{g^4} - \frac{f_s g_s}{fg}, \tag{4a}$$

$$\kappa_{23} = \frac{4g^2 - 3f^2}{g^4} - \frac{g_s^2}{g^2}, \tag{4b}$$

$$\kappa_{01} = -\frac{f_{ss}}{f}, \tag{4c}$$

$$\kappa_{02} = \kappa_{03} = -\frac{g_{ss}}{g}. \tag{4d}$$

Writing the metric in coordinate form (1), we note that its evolution under Ricci flow is governed by the evolution equations for f and g, which (as shown in (14) of [IKS16]) take the following form:

$$f_t = f_{ss} + 2\frac{g_s}{g} f_s - 2\frac{f^3}{g^4}, \tag{5a}$$

$$g_t = g_{ss} + \left(\frac{f_s}{f} + \frac{g_s}{g}\right) g_s + 2\frac{f^2 - 2g^2}{g^3}. \tag{5b}$$

The variable $s = s(x,t)$, representing arclength from the $\mathbb{S}^3$ at $x = 0$, is a choice of gauge that results in this system being manifestly strictly parabolic. The cost one pays for this is the non-vanishing commutator,

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = -\left(\frac{f_{ss}}{f} + 2\frac{g_{ss}}{g}\right)\frac{\partial}{\partial s}. \tag{6}$$

[5] Using L'Hôpital's rule, it is straightforward to verify that all quantities appearing in this section are well defined at $\mathbb{S}^2_\pm$. We make this explicit below.

2.3. **Closeness Assumptions.** The Riemannian Ricci flow solutions we study originate from an open set of cohomogeneity-one metrics that is defined by certain mild hypotheses, which effectively guarantee that at least initially, the metrics are "somewhat close" to the subspace of Kähler metrics.

Closeness Assumptions. At time $t = 0$, the metric G of the form (1) determined by the pair (f, g) satisfies the following:

(a) $f \leq g$;
(b) $gg_s \leq f$;
(c) $|f_s| \leq 2/\sqrt{3}$;
(d) $g^2(s_+) - 3g^2(s_-) \geq \delta^2$ for some $\delta > 0$;
(e) $g_s \geq 0$, with strict inequality off $\mathcal{S}^2_\pm$.

It follows from Lemma 26 of [IKS16] that condition (a) is preserved under the flow. We prove in Section 4.2 that condition (b) — which, as we show there, may be regarded as a "Kähler pinching condition" — and condition (c) are preserved by the flow. We prove in Section 5 that (d) is preserved. We explain the motivation for condition (e), which we do not prove is preserved, in the discussion of Conjecture A.

Remark 1. *Even for Kähler–Ricci flow solutions, condition (d) is necessary for the* $g^2\big(\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3\big)$ *factor to vanish before the* $\big(\mathrm{d}s \otimes \mathrm{d}s + f^2\,\omega^1 \otimes \omega^1\big)$ *factor does. This is necessary for the development of a local singularity on* $\mathcal{S}^2_-$ *(see Theorem 1.1 of* [SW11] *and Remark 4 below).*

2.4. **Construction of metrics satisfying the Closeness Assumptions.** We choose f to be any smooth function that is defined for $s \in [s_-, s_+]$, is strictly positive except at $s_\pm$, satisfies $|f_s| \leq 1$ with equality only at $s_\pm$, and satisfies the closing conditions (2) and (3). For each such function, we now construct an infinite-dimensional family $\mathcal{G}_{\alpha,\delta,\varepsilon}$ of initial metrics which satisfy our Closeness Assumptions. The family depends on parameters α, δ, and ε, to be chosen below. We define

$$A^2 := 2\int_{s_-}^{s_+} f(s)\,\mathrm{d}s,$$

noting that we are free to let the difference $s_+ - s_-$, and hence A^2, be as large as we wish. We then choose α and δ to be any positive parameters satisfying

$$\alpha^2 + \delta^2 \leq \frac{A^2}{2}. \tag{7}$$

To define g, and hence a metric $(f, g) \in \mathcal{G}_{\alpha,\delta,\varepsilon}$, we choose φ to be any smooth function satisfying $1 - \varepsilon \leq \varphi \leq 1$, requiring that it be nonconstant unless $\varepsilon = 0$. Clearly ε controls how much φ can stray from being constant. We then set

$$g^2(s) := \alpha^2 + 2\int_{s_-}^{s} \varphi(\bar{s}) f(\bar{s})\,\mathrm{d}\bar{s}.$$

We readily verify that g defined in this way satisfies closing conditions (2) and (3).

To verify that part (a) of our Closeness Assumptions is satisfied, we observe that the gradient restriction $|f_s| \leq 1$ implies that

$$\begin{aligned} f^2(s) &= 2\int_{s_-}^{s} f(\bar{s}) f_{\bar{s}}(\bar{s})\,\mathrm{d}\bar{s} \\ &\leq 2\int_{s_-}^{s} \big\{\varphi + (1-\varphi)\big\} f(\bar{s})\,\mathrm{d}\bar{s} \\ &\leq g^2 - \alpha^2 + \varepsilon A^2 \\ &\leq g^2, \end{aligned}$$

provided that

$$\varepsilon \leq \frac{\alpha^2}{A^2}. \tag{8}$$

It immediately follows that part (b) holds for all $\varepsilon \in [0,1)$, with equality — which Lemma 1 (below) shows is equivalent to the metric being Kähler — if and only if $\varepsilon = 0$. Part (c) holds as a consequence of the gradient restriction $|f_s| \leq 1$.

To verify that part (d) holds, we observe that one has

$$\begin{aligned} g^2(s_+) - 3g^2(s_-) &\geq \big\{\alpha^2 + (1-\varepsilon)A^2\big\} - 3\alpha^2 \\ &= (A^2 - 2\alpha^2) - \varepsilon A^2 \\ &\geq 2\delta^2 - \varepsilon A^2, \end{aligned}$$

with the last inequality following from the restrictions on α and δ that we have imposed in (7). Hence we satisfy part (d) so long as

$$\varepsilon \leq \frac{\delta^2}{A^2}. \tag{9}$$

Because $gg_s \geq (1-\varepsilon)f \geq 0$, it is clear that part (e) is satisfied.

3. Characterizing Kähler metrics

In this work, we provide evidence that as they become singular, solutions originating from initial data satisfying our Closeness Assumptions asymptotically approach the blowdown soliton. For the reader's convenience, we include here a brief review of metrics related to that singularity model.

3.1. **The Calabi construction.** We call a metric on $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ or $\mathcal{L}^2_{-1}$ a *Calabi metric* if it is both Kähler and U(2)-invariant. As part of a much more general construction [Cal82], Calabi has observed that any U(2)-invariant Kähler metric on $\mathbb{C}^2\backslash(0,0)$ has the form

$$h_{\mathbb{C}^2\backslash(0,0)} = \Big\{e^{-r}\phi\,\delta_{\alpha\beta} + e^{-2r}(\phi_r - \phi)\,\bar{z}^\alpha z^\beta\Big\}\,\mathrm{d}z^\alpha \otimes \mathrm{d}\bar{z}^\beta. \tag{10}$$

Here $r := \log(|z_1|^2 + |z_2|^2)$ is Calabi's coordinate, and $\phi(r) = P_r(r)$, where P is the Kähler potential. The metric closes smoothly at the origin, hence induces a smooth metric on the total space of the bundle $\mathcal{L}^2_{-1}$ (or on a neighborhood of $\mathcal{S}^2_-$ in $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$) if and only if there are $a_0, a_1 > 0$ such that

$$\phi(r) = a_0 + a_1 e^r + a_2 e^{2r} + \mathcal{O}(e^{3r}) \quad \text{as} \quad r \to -\infty. \tag{11}$$

The metric closes smoothly at spatial infinity, hence induces a smooth Kähler metric with respect to the unique complex structure on $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, if and only if two conditions hold: i) $\phi_r > 0$ everywhere, and ii) there are $b_0 > a_0$ and $b_1 < 0$ such that

$$\phi(r) = b_0 + b_1 e^{-r} + b_2 e^{-2r} + \mathcal{O}(e^{-3r}) \quad \text{as} \quad r \to \infty.$$

Alternatively, one may obtain a complete Calabi metric on the noncompact space $\mathcal{L}^2_{-1}$ by imposing conditions at spatial infinity that guarantee completeness; see, *e.g.,* [FIK03]. As noted in equation (19) of that paper, any U(2)-invariant metric on $\mathbb{C}^2\backslash(0,0)$ can be written in real coordinates on $\mathbb{R}^4\backslash(0,0,0,0)$ as

$$h_{\mathbb{R}^4\backslash(0,0,0,0)} = \phi_r\big(\frac{1}{4}\mathrm{d}r\otimes\mathrm{d}r + \omega^1\otimes\omega^1\big) + \phi\big(\omega^2\otimes\omega^2 + \omega^3\otimes\omega^3\big). \tag{12}$$

3.2. **A coordinate transformation.** A comparison of equations (1) and (12) shows that a coordinate transformation is needed to write a Calabi metric in the s-coordinate system. We implement this as follows. Recalling that $s(x,t)$ denotes arclength from the $\mathcal{S}^3$ at the "interior" point $x = 0$ and time t, and motivated by Calabi's (fixed) r-coordinate introduced in Section 3.1, we define here a function

$$\varrho(s,t) := 2\int_0^s \frac{\mathrm{d}\bar{s}}{f(\bar{s},t)}. \tag{13}$$

The closing conditions then show that $\varrho \to \pm\infty$ at $\mathcal{S}^2_\pm$. Moreover, one has

$$\mathrm{d}s = \frac{1}{2} f\,\mathrm{d}\varrho, \tag{14}$$

so that equation (1) may be re-expressed in the form

$$G = f^2\big(\frac{1}{4}\mathrm{d}\varrho\otimes\mathrm{d}\varrho + \omega^1\otimes\omega^1\big) + g^2\big(\omega^2\otimes\omega^2 + \omega^3\otimes\omega^3\big), \tag{15}$$

where we emphasize that the coordinate ϱ is allowed to depend on time. We note that ϱ and its time evolution depend only on $s(x,t)$ and $f(s(x,t),t)$, neither of which depend on ϱ.

We observe that equation (15) has the form (12) of a Calabi metric h if and only if $g^2 = \phi$ and $f^2 = (g^2)_\varrho$, in which case one has

$$f = g g_s \qquad \text{and} \qquad f_s = g g_{ss} + g_s^2.$$

We summarize this simple observation, which is crucial to our work here, as follows:

Lemma 1. *A* $[\mathcal{S}^2\tilde{\times}\mathcal{S}^2]$*-warped Berger metric* (1) *is Kähler if and only if* $f = g g_s$.

If G is Kähler, then its sectional curvatures, which generally take the form (4), take the following special form:

$$\begin{aligned}
\kappa_{12} = \kappa_{31} = \kappa_{02} = \kappa_{03} &= -\frac{g_{ss}}{g},\\
\kappa_{23} &= 4\frac{1-g_s^2}{g^2},\\
\kappa_{01} &= -\frac{g_{sss}}{g_s} - 3\frac{g_{ss}}{g}.
\end{aligned}$$

As must be true for a Kähler metric on a complex surface, the Ricci endomorphism then has only two eigenvalues,

$$R_0^0 = R_1^1 = -\frac{g_{sss}}{g_s} - 5\frac{g_{ss}}{g} \qquad \text{and} \qquad R_2^2 = R_3^3 = -2\frac{g_{ss}}{g} + 4\frac{1-g_s^2}{g^2}.$$

Because Kähler–Ricci flow is strictly parabolic, no time-dependent choice of gauge $s(x,t)$ is needed to ensure parabolicity. Rather, one can write the Kähler–Ricci PDE with respect to a time-independent coordinate. The following observation is a particular instance of this general fact.

Lemma 2. *The evolution equation for the coordinate ϱ under Ricci flow takes the form*

$$\varrho_t = 2\int_0^{\varrho} \left\{ \frac{g_{ss}}{g} - \frac{f_s g_s}{fg} + \frac{f^2}{g^4} \right\} \mathrm{d}\bar\varrho. \tag{16}$$

For a Kähler geometry, the integrand in (16) vanishes pointwise; hence, for Kähler initial data, the coordinate ϱ is independent of t.

Remark 2. *For Kähler initial data, one may therefore assume without loss of generality that ϱ is identical to Calabi's coordinate $r = \log(|z_1|^2 + |z_2|^2)$.*

Proof of Lemma 2. It follows from equation (56) in [IKS16] and from (4) above that the gauge quantity $\frac{\partial s}{\partial x}$ evolves according to the equation

$$\left(\frac{\partial s}{\partial x}\right)_t = \left\{ \frac{f_{ss}}{f} + 2\frac{g_{ss}}{g} \right\} \frac{\partial s}{\partial x}.$$

Hence, using equations (13) and (5a), we determine that the time derivative of ϱ at fixed x is given by

$$\begin{aligned} \frac{1}{2}\varrho_t &= \frac{\partial}{\partial t}\left(\int_0^x f^{-1}\big(s(\bar x,t),t\big)\, \frac{\partial s}{\partial \bar x}\,\mathrm{d}\bar x \right) \\ &= \int_0^x \left\{ f^{-1}\Big(\frac{\partial s}{\partial \bar x}\Big)_t - f^{-2} f_t \Big(\frac{\partial s}{\partial \bar x}\Big) \right\} \mathrm{d}\bar x \\ &= 2\int_0^s \left\{ \frac{g_{\bar s\bar s}}{fg} - \frac{f_{\bar s} g_{\bar s}}{f^2 g} + \frac{f}{g^4} \right\} \mathrm{d}\bar s. \end{aligned}$$

This proves the first claim. The second follows by direct computation. □

3.3. **Ricci flow of Calabi metrics.** Lemma 1 states that the initial metric is Calabi if and only if $f = g g_s$. Because Ricci flow preserves the Kähler condition with respect to the original complex structure (here, the unique complex structure on $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$) and also preserves initial symmetries, a solution originating from Calabi initial data remains Calabi for as long as it exists. This can be seen directly, as we now observe.

In this section (which is not needed for the rest of the chapter) and occasionally below, we find it convenient to work with $u := f^2$ and $v := g^2$. The Ricci flow evolution equations for these quantities are given by

$$u_t = u_{ss} - \frac{u_s^2}{2u} + \frac{u_s v_s}{v} - 4\frac{u^2}{v^2}, \tag{17a}$$

$$v_t = v_{ss} + \frac{u_s v_s}{2u} + 4\frac{u - 2v}{v}. \tag{17b}$$

On a Calabi solution, one can use the relation $u = v_s^2/4$ (equivalent to $f = g g_s$) to simplify the evolution equation above for v, thereby obtaining

$$v_t = 2v_{ss} + \frac{v_s^2}{v} - 8. \tag{18}$$

One now has two ways of computing the evolution of u. Evaluating the equation above for u_t by using the Kähler condition $u = v_s^2/4$ to convert the RHS into terms involving only v and its derivatives, one obtains

$$u_t = \frac{1}{2} v_s v_{sss} + \frac{1}{2} \frac{v_{ss} v_s^2}{v} - \frac{1}{4} \frac{v_s^4}{v^2}. \tag{19}$$

On the other hand, one can differentiate the RHS of $u = v_s^2/4$ directly, use the commutator $[\partial_t, \partial_s]$ given in equation (6), and then apply (18), obtaining

$$\begin{aligned}\left(\frac{v_s^2}{4}\right)_t &= \frac{1}{2} v_s (v_s)_t \\ &= \frac{1}{2} v_s \left\{ (v_t)_s - \left(\frac{f_{ss}}{f} + 2\frac{g_{ss}}{g} \right) v_s \right\} \\ &= \frac{1}{2} v_s v_{sss} + \frac{1}{2} \frac{v_{ss} v_s^2}{v} - \frac{1}{4} \frac{v_s^4}{v^2},\end{aligned}$$

as above. This calculation verifies directly what one knows from general principles: that the Calabi condition is preserved by the flow. We note in particular that for a Calabi solution, the Ricci flow system reduces to a scalar PDE, in the sense that the evolution of u is completely determined by the evolution of v.

Remark 3. *For solutions with initial data satisfying our Closeness Assumptions, the fact that $g_s > 0$ everywhere except at $\mathbb{S}^2_\pm$ holds initially. For as long as this remains true (possibly only a short time for non-Kähler solutions), there is a well-defined function θ such that*

$$f = \theta \, g g_s.$$

We note that $\theta \equiv 1$ for a Kähler solution, and that the evolution equation for θ is

$$\theta_t = \theta_{ss} + 2\frac{f g_s - 2 f_s g}{g^2} (\theta^2 - 1),$$

which yields an easy direct proof that the Kähler condition is preserved for these geometries.

3.4. **The blowdown soliton.** Under Kähler–Ricci flow, the evolution of an arbitrary Calabi metric

$$h = \left\{ e^{-r} \phi \, \delta_{\alpha\beta} + e^{-2r} (\phi_r - \phi) \, \bar{z}^\alpha z^\beta \right\} \mathrm{d}z^\alpha \otimes \mathrm{d}\bar{z}^\beta, \tag{20}$$

written in terms of Calabi's fixed r-coordinate on $\mathcal{L}^2_{-1}$ or on $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, is determined by the PDE

$$\phi_t = \frac{\phi_{rr}}{\phi_r} + \frac{\phi_r}{\phi} - 2. \tag{21}$$

The blowdown soliton is specified by setting ϕ in (20) equal to a function φ which (following Lemma 6.1 and equation (27) of [FIK03] with $\lambda = -1$, $\mu = \sqrt{2}$, and $\nu = 0$) satisfies the separable first-order ODE

$$\varphi_r = \frac{1}{\sqrt{2}} \varphi - (\sqrt{2} - 1) - \left(1 - \frac{1}{\sqrt{2}} \right) \varphi^{-1}. \tag{22}$$

Rewriting this ODE in the form

$$\mathrm{d}r = \frac{\varphi \, \mathrm{d}\varphi}{\varphi - 1} - \frac{\varphi \, \mathrm{d}\varphi}{\varphi + \sqrt{2} - 1},$$

one can solve it implicitly, obtaining

$$e^{r+\chi} = \frac{\varphi - 1}{\left(\varphi + \sqrt{2} - 1\right)^{\sqrt{2}-1}}. \tag{23}$$

The arbitrary constant χ above reflects the fact that the soliton is unique only modulo translations in r. Examination of formula (23) also shows that the soliton is cone-like at spatial infinity and hence complete.

Equation (24) of [FIK03] implies that the blowdown soliton function φ also satisfies the second-order ODE

$$\frac{\varphi_{rr}}{\varphi_r} + \frac{\varphi_r}{\varphi} - \sqrt{2}\varphi_r + \varphi - 2 = 0. \tag{24}$$

It follows from (24) that φ evolves by

$$\varphi_t = \sqrt{2}\varphi_r - \varphi \tag{25}$$

In particular, the soliton evolves by translation and scaling.

4. Basic estimates

In this section, we prove several estimates that support our Main Conjecture.

4.1. **A weak one-sided Kähler stability result.**

We begin by introducing the useful quantity

$$\psi := \left(\frac{g g_s}{f}\right)^2 - 1. \tag{26}$$

This quantity ψ is well defined at $\mathcal{S}^2_\pm$, because it follows from l'Hôpital's rule that $\frac{g g_s}{f}\big|_{\mathcal{S}^2_\pm} = g g_{ss}$.

Lemma 1 tells us that $\psi \equiv 0$ if and only if the metric G from (1) is Kähler. Therefore, we use ψ to measure, in a precise sense, how far away a solution is from being Kähler. The following result is thus a statement of weak (one-sided) stability for the Kähler condition. Note that part (b) of our Closeness Assumptions ensures that $\psi \le 0$ at $t = 0$.

Lemma 3. *If $-1 \le \psi \le 0$ initially, then $-1 \le \psi \le 0$ as long as the flow exists.*

Proof. It is only necessary to prove the upper bound. The quantity ψ evolves under Ricci flow by

$$\psi_t = \psi_{ss} + \left\{3\frac{f_s}{f} - 2\frac{g_s}{g}\right\}\psi_s - \frac{\psi_s^2}{2(\psi+1)} + \left\{4\frac{g_s^2}{g^2} - 8\frac{f_s g_s}{fg}\right\}\psi. \tag{27}$$

From this equation, it is clear that the condition $\psi \le 0$ is preserved if all maxima of ψ occur away from $\mathcal{S}^2_\pm$.

If a maximum occurs instead at $\mathcal{S}^2_\pm$, then we apply l'Hôpital to determine that

$$\frac{f_s \psi_s}{f}\Big|_{\mathcal{S}^2_\pm} = \psi_{ss} \qquad \text{and} \qquad \frac{f_s g_s}{fg}\Big|_{\mathcal{S}^2_\pm} = \frac{g_{ss}}{g}.$$

Hence

$$\psi_t\big|_{\mathcal{S}^2_\pm} = 4\psi_{ss} - 8\frac{g_{ss}}{g}\psi.$$

However, smoothness of either function $\psi_\pm(s,\cdot) := \psi(s - s_\pm, \cdot)$ at a maximum on $\mathcal{S}^2_\pm$ shows that $\psi_{ss}\big|_{\mathcal{S}^2_\pm} = (\psi_\pm)_{ss}\big|_{\mathcal{S}^2_\pm} \le 0$. The result follows. □

4.2. **First-derivative estimates.** Based on the one-sided Kähler stability established in Lemma 3, we now derive estimates on the first derivatives of f and g, and consequently on the curvatures which depend on these first derivatives. We first state an immediate corollary of Lemma 3, which controls $|g_s|$.

Corollary 4. *Solutions originating from initial data satisfying our Closeness Assumptions have $|g_s| \leq 1$ for as long as they exist.*

Proof. Because, as noted above, the ordering $f \leq g$ is preserved by the flow, it follows from Lemma 3 that $f^2 g_s^2 \leq g^2 g_s^2 \leq f^2$. The result follows. $\square$

Next, we obtain a bound for $|f_s|$.

Lemma 5. *If $f \leq g$ initially, then for as long as the flow exists,*

$$|f_s| \leq \max\left\{\frac{2}{\sqrt{3}}, \max|f_s(\cdot,0)|\right\}.$$

Proof. Using equation (21) of [IKS16] together with the fact that

$$\Delta\zeta = \zeta_{ss} + (f_s/f + 2g_s/g)\zeta_s$$

holds for any smooth function $\zeta(s,t)$, we see that f_s evolves by

$$(f_s)_t = (f_s)_{ss} + \left\{2\frac{g_s}{g} - \frac{f_s}{f}\right\}(f_s)_s - \left\{6\frac{f^2}{g^4} + 2\frac{g_s^2}{g^2}\right\} f_s + 8\frac{f^3}{g^5}\, g_s. \tag{28}$$

Because $f_s\big|_{\mathcal{S}^2_\pm} = \pm 1$, we do not need to worry about a maximum of $|f_s|$ on $\mathcal{S}^2_\pm$.

We apply the weighted Cauchy–Schwarz inequality $|ab| \leq \epsilon a^2 + (1/4\epsilon)b^2$ to the term $8f^3 g_s/g^5$ above, with $a = g_s/g$ and $b = f^3/g^4$. Thus if $(f_s)_{\max} = C > 0$ at some time, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(f_s)_{\max} &\leq -(f_s)_{\max}\left(6\frac{f^2}{g^4} + 2\frac{g_s^2}{g^2}\right) + 8\frac{f^3}{g^5} g_s \\
&\leq \left(\frac{4}{\sqrt{3}} - 2C\right)\frac{g_s^2}{g^2} + \left(\frac{12}{\sqrt{3}}\frac{f^4}{g^4} - 6C\right)\frac{f^2}{g^4} \\
&\leq 0
\end{aligned}$$

if $C \geq 2/\sqrt{3}$, because $f/g \leq 1$.

A similar argument shows that $\frac{\mathrm{d}}{\mathrm{d}t}(f_s)_{\min} \geq 0$ if $(f_s)_{\min} = -C$ at some time. $\square$

These uniform bounds on the first derivatives of f and g lead to the following.

Lemma 6. *For any solution originating from initial data satisfying our Closeness Assumptions, there exists a uniform constant C such that*

$$|\kappa_{12}| + |\kappa_{31}| + |\kappa_{23}| \leq \frac{C}{g^2}$$

for as long as the flow exists.

Proof. It follows from Lemma 3 that the inequality $(gg_s/f)^2 \leq 1$ persists if it is true initially. This implies that $|g_s| \leq f/g$ for as long as the flow exists. Combining this estimate with the identities in (4), using Corollary 4, Lemma 5, and the fact that $f \leq g$, we obtain

$$|\kappa_{12}| + |\kappa_{31}| + |\kappa_{23}| \leq \frac{C}{g^2}.$$

$\square$

4.3. **Second-derivative estimates.** Here we derive estimates for the remaining curvatures — those that depend on second-order derivatives of (f, g).

Lemma 7. *For any solution originating from initial data satisfying our Closeness Assumptions, there exists a uniform constant C such that*

$$|\kappa_{02}| = \left|\frac{g_{ss}}{g}\right| \leq \frac{C}{g^2}$$

for as long as the flow exists.

Proof. We define $Q = gg_{ss} - Ag_s^2 - Bf_s^2$, where $A, B > 0$ are to be suitably chosen below. We first show that there exists a uniform constant C so that $Q \geq -C$ for as long as the flow exists. A straightforward computation shows that the evolution of Q is given by

$$\begin{aligned} \frac{\partial Q}{\partial t} &= \Delta Q + \frac{12Bf^2f_s^2}{g^4} + \frac{4f_s^2}{g^2} + \frac{24f^2g_s^2}{g^4} + \frac{12Af^2g_s^2}{g^4} + \frac{2Af_s^2g_s^2}{f^2} + \frac{4Bf_s^2g_s^2}{g^2} \\ &\quad + \frac{2g_s^4}{g^2} + \frac{2Ag_s^4}{g^2} + 2Bf_{ss}^2 + 2(A-1)g_{ss}^2 - gg_{ss}\left(\frac{4f^2}{g^4} + \frac{2f_s^2}{f^2} + \frac{4Ag_s^2}{g^2}\right) \\ &\quad - \frac{16Bf^3f_sg_s}{g^5} - \frac{24ff_sg_s}{g^3} - \frac{8Aff_sg_s}{g^3} + \frac{2gf_s^3g_s}{f^3} - \frac{8g_s^2}{g^2} - \frac{8Ag_s^2}{g^2} \\ &\quad + \frac{4ff_{ss}}{g^2} + \frac{4Bf_s^2f_{ss}}{f} - \frac{8Bf_sg_sf_{ss}}{g} - \frac{2gf_sg_sf_{ss}}{f^2}, \end{aligned} \tag{29}$$

where as noted above, $\Delta Q = Q_{ss} + (f_s/f + 2g_s/g)Q_s$. We observe that l'Hôpital's rule implies that the terms

$$\frac{Q_sf_s}{f}, \qquad \frac{2Af_s^2g_s^2}{f^2}, \qquad 2gf_s^2\,\frac{(f_sg_s - fg_{ss})}{f^3}, \qquad \frac{4Bf_s^2f_{ss}}{f},$$

appearing in equation (29) are well defined and smooth at $\mathcal{S}^2_\pm$. We now distinguish between two cases.

Case 1. A minimum of Q occurs away from $\mathcal{S}^2_\pm$.

We assume that at a minimum of Q at some time t, we have $gg_{ss} - Ag_s^2 - Bf_s^2 \leq -\bar{C}$ for a large constant $\bar{C} > 0$ to be chosen. Because we are bounding Q from below, we may assume that $g_{ss} \leq 0$. Then since Corollary 4 and Lemma 5 give uniform bounds for $|f_s|$ and $|g_s|$, we may choose $\bar{C}$ sufficiently large relative to A and B such that

$$-gg_{ss}\left(\frac{4f^2}{g^4} + \frac{2f_s^2}{f^2} + \frac{4Ag_s^2}{g^2}\right) \geq \frac{\bar{C}f^2}{2g^4} + \frac{\bar{C}f_s^2}{2f^2} + \frac{\bar{C}Ag_s^2}{g^2}. \tag{30}$$

It then follows from (29) that at a minimum of Q at time t, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}Q_{\min} &\geq 2Bf_{ss}^2 + \frac{\bar{C}f^2}{2g^4} + \frac{\bar{C}f_s^2}{2f^2} + \frac{\bar{C}Ag_s^2}{g^2} \\ &\quad - \frac{16Bf^3f_sg_s}{g^5} - \frac{24ff_sg_s}{g^3} - \frac{8Aff_sg_s}{g^3} + \frac{2gf_s^3g_s}{f^3} - \frac{8g_s^2}{g^2} - \frac{8Ag_s^2}{g^2} \\ &\quad + \frac{4ff_{ss}}{g^2} + \frac{4Bf_s^2f_{ss}}{f} - \frac{8Bf_sg_sf_{ss}}{g} - \frac{2gf_sg_sf_{ss}}{f^2}. \end{aligned} \tag{31}$$

To estimate the terms in (31) containing f_{ss}, we use Lemma 3, Corollary 4, Lemma 5, the facts that $f \le g$ and $|g_s| \le f/g$, and a weighted Cauchy–Schwarz inequality to determine that there exists a uniform constant C' such that

$$\begin{aligned}
&\left|\frac{4ff_{ss}}{g^2}\right| + \left|\frac{4Bf_s^2 f_{ss}}{f}\right| + \left|\frac{8Bf_s g_s f_{ss}}{g}\right| + \left|\frac{2gf_s g_s f_{ss}}{f^2}\right| \\
&\quad \le \left(\frac{1}{2}f_{ss}^2 + \frac{C'f^2}{g^4}\right) + \left(\frac{B}{2}\, f_{ss}^2 + C'B\frac{f_s^2}{f^2}\right) + \left(\frac{B}{2} f_{ss}^2 + C'B\frac{g_s^2}{g^2}\right) + \left(\frac{1}{2} f_{ss}^2 + C'\frac{f_s^2}{f^2}\right) \\
&\quad \le (B+1)f_{ss}^2 + C'(B+1)\left(\frac{f_s^2}{f^2} + \frac{g_s^2}{g^2} + \frac{f^2}{g^4}\right).
\end{aligned}$$

The remaining terms in (31) can be estimated in a similar manner. Thus we find that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} Q_{\min} &\ge 2Bf_{ss}^2 + \frac{\bar{C}f^2}{2g^4} + \frac{\bar{C}f_s^2}{2f^2} + \frac{\bar{C}Ag_s^2}{g^2} \\
&\qquad - C'(1+A+B)\left(\frac{f^2}{g^4} + \frac{f_s^2}{f^2} + \frac{g_s^2}{g^2}\right) - (B+1)f_{ss^2} \\
&\ge 0,
\end{aligned}$$

if we choose $A = 1$, $B = 2$ and $\bar{C}$ sufficiently large so that $\bar{C} > C'(1 + A + B)$. Therefore, in this case, either $Q \ge -\bar{C}$ or $\frac{\mathrm{d}}{\mathrm{d}t} Q_{\min} \ge 0$.

Case 2. A minimum of Q occurs on $\mathcal{S}^2_\pm$.

The only difference from Case 1 is that one must deal with the term $\frac{Q_s f_s}{f}$ at $\mathcal{S}^2_\pm$. We apply l'Hôpital's rule to see that

$$\frac{Q_s f_s}{f}\Big|_{\mathcal{S}^2_\pm} = \left(Q_{ss} + Q_s\frac{f_{ss}}{f_s}\right)\Big|_{\mathcal{S}^2_\pm} = Q_{ss}\big|_{\mathcal{S}^2_\pm}.$$

However, smoothness of either function $Q_\pm(s,\cdot) := Q(s - s_\pm, \cdot)$ at a minimum on $\mathcal{S}^2_\pm$ shows that $Q_{ss}|_{\mathcal{S}^2_\pm} = (Q_\pm)_{ss}|_{\mathcal{S}^2_\pm} \ge 0$. A similar computation as in Case 1 then yields

$$\frac{\mathrm{d}}{\mathrm{d}t} Q|_{\mathcal{S}^2_\pm} \ge 0,$$

unless $Q_{\min}(t) = Q(\cdot, t)|_{\mathcal{S}^2_\pm} \ge -\bar{C}$, for the constant $\bar{C}$ chosen in Case 1.

Combined, Case 1 and Case 2 show that

$$Q(\cdot, t) \ge \min\big\{-\bar{C},\, Q_{\min}(0)\big\}.$$

In particular, this implies that

$$\frac{g_{ss}}{g} \ge -\frac{C}{g^2},$$

for a uniform constant C as long as the flow exists.

Finally, considering the quantity $\tilde{Q} := gg_{ss} + Ag_s^2 + Bf_s^2$ and bounding $\tilde{Q}$ from above using similar arguments yields a uniform constant C such that

$$\frac{g_{ss}}{g} \le \frac{C}{g^2}$$

for as long as the flow exists. This concludes the proof of the Lemma. □

We now define

$$\mu(t) := \min_{\mathbb{S}^2 \tilde{\times} \mathbb{S}^2} g(\cdot, t), \tag{32}$$

observing that Lemmas 6 and 7 imply that there exists a uniform constant C such that as long as the flow exists, one has

$$|\kappa_{12}| + |\kappa_{13}| + |\kappa_{23}| + |\kappa_{02}| + |\kappa_{03}| \leq \frac{C}{\mu^2}. \tag{33}$$

Remark 4. *Controlling the curvature κ_{01} is considerably more subtle. This is because, even for Kähler solutions, the alternative in statement* (II) *of Lemma 8 below is truly necessary: estimate* (35) *need not hold unless such solutions originate from initial data satisfying part (d) of our Closeness Assumptions. Solutions for which part (d) is false can have $f \searrow 0$ uniformly as $t \nearrow T$, with $g(\cdot, T) > 0$ everywhere. Each such (unrescaled) solution converges in the Gromov–Hausdorff sense to a $\mathbb{CP}^1$ of multiplicity two; see Theorem 1.1 of* [SW11].

Lemma 8. *For any solution originating from initial data satisfying our Closeness Assumptions, the following are true:*
(I) *The sectional curvature $\kappa_{01} = -f_{ss}/f$ satisfies*

$$\kappa_{01} \geq -\frac{C}{g^2} \tag{34}$$

for a uniform constant C.
(II) *Either there is an analogous upper bound*

$$\kappa_{01} \leq \frac{C}{\mu^2}, \tag{35}$$

or any finite-time singularity is Type-I.

Proof. Because the scalar curvature R is a supersolution of the heat equation (in the sense that $(\partial_t - \Delta) R \geq 0$), there exists a constant r_0 depending only on the initial data such that for as long as the flow exists, one has

$$r_0 \leq R = \kappa_{01} + \kappa_{02} + \kappa_{03} + \kappa_{12} + \kappa_{23} + \kappa_{31},$$

where $\kappa_{02} = \kappa_{03}$. Using this together with Lemma 6, Lemma 7, and the fact that $\frac{\mathrm{d}}{\mathrm{d}t} g_{\max} \leq 0$, we get the lower bound (34).

To prove (II), we assume that (35) fails and use a blow-up argument at a finite-time singularity. In particular, we assume that $T < \infty$ is a singular time for the flow, and that

$$\limsup_{t \to T} \left(\sup_{\mathbb{S}^2 \tilde{\times} \mathbb{S}^2} \kappa_{01}(\cdot, t) \mu(t)^2 \right) = \infty. \tag{36}$$

We now let $t_i \to T$ as $i \to \infty$ such that

$$\sup_{t \in [0, t_i]} \left(\sup_{\mathbb{S}^2 \tilde{\times} \mathbb{S}^2} \kappa_{01}(\cdot, t) \mu(t)^2 \right) = \kappa_{01}(p_i, t_i) \mu(t_i)^2$$

for some $p_i \in M$, and we let $K_i := \kappa_{01}(p_i, t_i)$. It follows from our choice of t_i that

$$K_i \mu(t_i)^2 \to \infty \quad \text{as} \quad i \to \infty. \tag{37}$$

We define the blow-up sequence of solutions G_i of the metric of the form (1) by

$$G_i(\cdot, t) := K_i \, G(\cdot, t_i + t K_i^{-1}),$$

for t satisfying
$$-K_i t_i \le t < (T - t_i) K_i.$$
We claim that the curvatures of the rescaled metrics G_i are uniformly bounded. To prove the claim for κ_{12}, say, we begin by noting that estimate (33) implies that

$$\left|\kappa^i_{12}(\cdot,t)\right| = \frac{|\kappa_{12}(\cdot, t_i + tK_i^{-1})|}{K_i} \le \frac{C}{K_i \mu(t_i + tK_i^{-1})^2}. \tag{38}$$

It follows from Remark 1 of [IKS16] that the evolution equation for $g(\cdot,t)$ can be written as
$$\frac{\partial}{\partial t}(\log g) = -\kappa_{02} - \kappa_{23} - \kappa_{31},$$
which implies that
$$\left|\frac{\partial}{\partial t}\log g\right| \le \frac{C}{g^2},$$
and therefore that
$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mu^2\right| \le C.$$
Integrating this over $[t_i + tK_i^{-1}, t_i]$ yields
$$|\mu(t_i + tK_i^{-1})^2 - \mu(t_i)^2| \le \frac{C}{K_i},$$
for, say, $t \in [-1,0]$. This implies that
$$\mu(t_i + tK_i^{-1})^2 \ge \mu(t_i)^2 - \frac{C}{K_i},$$
whereupon (38) implies for $t \in [-1,0]$ that
$$|\kappa^i_{12}(\cdot,t)| \le \frac{C}{K_i\mu(t_i)^2 - C} \to 0$$
as $i \to \infty$, because (37) holds.

To bound the remaining curvatures of the rescaled metrics, we use similar arguments together with (33) to conclude that
$$|\kappa^i_{12}(\cdot,t)| + |\kappa^i_{13}(\cdot,t)| + |\kappa^i_{23}(\cdot,t)| + |\kappa^i_{02}(\cdot,t)| + |\kappa^i_{03}(\cdot,t)| \le \frac{C}{K_i\mu(t_i)^2 - C} \to 0$$
as $i \to \infty$, and we use (34) to show that
$$\kappa_{01} \ge -\frac{C}{K_i\mu(t_i)^2 - C} \to 0,$$
as $i \to \infty$.

After extracting a convergent subsequence, we determine that $(\mathcal{S}^2 \tilde{\times} \mathcal{S}^2, G_i(t), p_i)$ converges in the pointed Cheeger–Gromov–Hamilton sense to a complete ancient solution
$$(\mathcal{M}^4_\infty, \mathcal{G}_\infty(t), p_\infty)$$
that exists for $t \in (-\infty, t^*)$ where $t^* := \lim_{i\to\infty}(T - t_i)\,K_i \le \infty$. Moreover, one has

$$\kappa^\infty_{12} = \kappa^\infty_{13} = \kappa^\infty_{23} = \kappa^\infty_{02} = 0, \qquad \text{and} \qquad \kappa^\infty_{01} \ge 0, \tag{39}$$

with $\kappa_{01}(p_\infty, 0) = 1$. By applying Hamilton's splitting theorem [Ham93] twice, we find that the universal cover $(\tilde{\mathcal{M}}^4_\infty, \mathcal{G}_\infty, p_\infty)$ splits isometrically as the product of

$\mathbb{R}^2$ and a complete ancient solution $(\mathcal{N}^2, \mathcal{G}_\infty|_{\mathcal{N}^2})$ with bounded positive scalar curvature. It follows from the classification in [DHS12] and [DS06][6] that $(\mathcal{N}^2, \mathcal{G}_\infty|_{\mathcal{N}^2})$ is either the King–Rosenau solution, the cigar, or the round sphere $\mathcal{S}^2$. In the former case, it is a standard fact that by choosing a modified sequence $\tilde{p}_i$ of blow-up points, one can obtain the cigar as a limit. But this is impossible by Perelman's κ-non-collapsing result [Per02]. So the limit must be isometric to one of the products $\mathcal{S}^2 \times \mathbb{R}^2$ or $\mathbb{R}^2 \times \mathcal{S}^2$. In either case,[7] the singularity is Type-I, and we have $\kappa_{01} \le C/(T-t)$. □

5. Singularity formation

In this section, we investigate finite-time singularity formation for Ricci flow solutions originating from initial data satisfying our Closeness Assumptions, with the objective — not fully achieved — of proving that all such singularities are Type-I, with $|\mathcal{S}^2_-| = 0$ at the singular time $T < \infty$. This requires some work, for the following reason. Away from the special fibers $\mathcal{S}^2_\pm$, the geometry of $(\mathcal{S}^2 \tilde{\times} \mathcal{S}^2, G)$ is that of $(a,b) \times \mathcal{S}^3$. So without appropriate assumptions on the initial data, it is highly plausible that neckpinch singularities like those analyzed in [IKS16] could develop at a fiber $\{s_0\} \times \mathcal{S}^3$ far from $\mathcal{S}^2_-$. As explained below, we do not expect this possibility to occur for solutions originating from initial data satisfying our Closeness Assumptions.

As proved in [SW11] and as noted above, the behavior of Kähler solutions depends strongly on whether $|\mathcal{S}^2_+| < 3|\mathcal{S}^2_-|$, $|\mathcal{S}^2_+| = 3|\mathcal{S}^2_-|$, or $|\mathcal{S}^2_+| > 3|\mathcal{S}^2_-|$. It follows from part (d) of our Closeness Assumptions that the solutions we study have $|\mathcal{S}^2_+| > 3|\mathcal{S}^2_-|$ initially. Our first result in this section proves that this threshold condition is preserved by the flow, even for non-Kähler solutions, provided they originate from initial data satisfying the Closeness Assumptions.

Lemma 9. *Solutions originating from initial data satisfying our Closeness Assumptions satisfy*

$$g^2(s_+, t) - 3g^2(s_-, t) \ge \delta^2$$

for as long as they exist.

Proof. We recall that

$$g_t = g_{ss} + \left(\frac{f_s}{f} + \frac{g_s}{g}\right) g_s + 2\,\frac{(f^2 - 2g^2)}{g^3}.$$

Using l'Hôpital's rule, we compute at s_+ that

$$\lim_{s \to s_+} \frac{f_s g_s}{f} = g_{ss}(s_+, t).$$

Because $g_s(s_+, t) = 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} g(s_+, t) = 2g_{ss}(s_+, t) - \frac{4}{g(s_+, t)}.$$

[6] It is proved in [DH04] that solutions that vanish at $T < \infty$ satisfy the width condition needed to apply [DS06].

[7] For the metrics we study here, the case $\mathcal{S}^2 \times \mathbb{R}^2$ corresponds to the $g^2(\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3)$ factor becoming flat after rescaling, while the case $\mathbb{R}^2 \times \mathcal{S}^2$ corresponds to the $(\mathrm{d}s \otimes \mathrm{d}s + f^2\,\omega^1 \otimes \omega^1)$ factor becoming flat.

Lemma 3 tells us that $g|g_s| \leq f$, which as a consequence of l'Hôpital's rule, implies at s_+ that

$$gg_{ss} \geq -1.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} g^2(s_+, t) \geq -12. \tag{40}$$

Similarly, using the fact that $g|g_s| \leq f$ and using l'Hôpital's rule, we have that $gg_{ss} \leq 1$ at s_-, from which we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} g^2(s_-, t) \leq -4. \tag{41}$$

Estimates (40) and (41) together imply that

$$\frac{\mathrm{d}}{\mathrm{d}t} \big(g^2(s_+, t) - 3g^2(s_-, t) \big) \geq 0,$$

which yields

$$g^2(s_+, t) - 3g^2(s_-, t) \geq g^2(s_+, 0) - 3g^2(s_-, 0).$$

□

Our second result in this section proves that solutions originating from initial data satisfying our Closeness Assumptions become singular at $T < \infty$ only if g vanishes somewhere.

Lemma 10. *If a solution originating from initial data satisfying our Closeness Assumptions becomes singular at time T, then $\mu(T) = 0$.*

Proof. Lemma 8 proves that either there is a two-sided curvature bound for κ_{01} or the singularity is Type-I.

If there is a two-sided bound $|\kappa_{01}| \leq C/\mu^2$, then combining this with estimate (33) we obtain a uniform constant C such that

$$|\operatorname{Rc}(G(t))| \leq \frac{C}{\mu^2},$$

for as long as the flow exists. Because [Ses05] proves that $\limsup_{t \nearrow T} |\operatorname{Rc}| = \infty$ if $T < \infty$ is the singularity time, it follows that $\mu(T) = 0$.

To complete the proof, we may assume, to obtain a contradiction, that a solution encounters a finite-time Type-I singularity for which $\lim_{t \to T} \mu(t) = 0$ is false.

We first claim that this assumption implies that there exists $\eta > 0$ such that $\mu(t) \geq \eta > 0$ for $t \in [0, T)$. We prove this claim by contradiction. Observe that the maximum principle implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mu(t) \geq -\frac{4}{\mu(t)}.$$

So for $t \geq \tau$ in $[0, T)$, one has

$$\mu(t)^2 \geq \mu(\tau)^2 - 8(t - \tau). \tag{42}$$

If it is not true that $\lim_{t \to T} \mu(t) = 0$, then there exists a sequence $\tau_i \to T$ along which $\mu(\tau_i) \geq \eta > 0$ for all i. On the other hand, if there exists another sequence $t_i \to T$ along which $\lim_{i \to \infty} \mu(t_i) = 0$, then by passing to subsequences, we may assume that $t_i \geq \tau_i$, and hence that

$$\mu(t_i)^2 \geq \mu(\tau_i)^2 - 8(t_i - \tau_i) \geq \eta^2 - 8(t_i - \tau_i).$$

But this is impossible, because $\lim_{i\to\infty}\mu(t_i)=0$ and $\lim_{i\to\infty}(t_i-\tau_i)=0$. This contradiction proves the claim.

The proof of Lemma 8 tells us that the inequality $\mu(t)\geq\eta>0$ implies that the universal cover of any Type-I singularity model must be $\mathcal{S}^2\times\mathbb{R}^2$. Compactness of the $\mathcal{S}^2$ factor implies there is a sequence $t_i\to T$ along which $\sup_{s_-\leq s\leq s_+} f(s,t_i)\leq C\sqrt{T-t_i}$. On the other hand it follows from Lemma 3 that

$$g|g_s|\leq f\leq C\sqrt{T-t_i}$$

at those times, which implies that

$$g^2(s_+,t_i)-g^2(s_-,t_i)\leq C\sqrt{T-t_i}(s_+-s_-). \tag{43}$$

We recall that

$$\frac{\mathrm{d}}{\mathrm{d}t}(s-s_-)=\int_{s_-}^{s}\left(\frac{f_{ss}}{f}+2\frac{g_{ss}}{g}\right)\mathrm{d}s=-\int_{s_-}^{s}(\kappa_{01}+2\kappa_{02})\,\mathrm{d}s.$$

Combining Lemma 7 and part (I) of Lemma 8, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(s-s_-)\leq\frac{C(s-s_-)}{\mu(t)^2}\leq C'(s-s_-),$$

because $\mu(t)\geq\eta>0$. Integrating this over $[0,T)$ yields a constant C'' such that

$$|s-s_-|\leq C''\qquad\text{for all}\quad t\in[0,T)\quad\text{and}\quad s\in[s_-,s_+]. \tag{44}$$

Combining (43) and (44) then gives us

$$g^2(s_+,t_i)-g^2(s_-,t_i)\leq C\sqrt{T-t_i}.$$

But this is incompatible with the conclusion of Lemma 9 that

$$g^2(s_+,t_i)-3g^2(s_-,t_i)\geq\delta^2>0.$$

This contradiction proves the result. □

For Kähler solutions, monotonicity of g is preserved automatically for as long as the metric remains smooth. We do not know if this is true for the non-Kähler solutions studied here. However, it follows from the evolution equation for g_s,

$$(g_s)_t=\Delta(g_s)-2\frac{g_s}{g}(g_s)_s+\left\{\frac{4}{g^2}-\frac{g_s^2}{g^2}-\frac{f_s^2}{f^2}-6\frac{f^2}{g^4}\right\}g_s+4\frac{f}{g^3}f_s,$$

that monotonicity can fail only where $f_s<0$, *i.e.,* only in a proper neighborhood of $\mathcal{S}^2_+$. In our construction of initial data in Section 2.4, we are free to choose the parameter $\alpha^2=|\mathcal{S}^2_-|$ as small as possible, and the parameter A, which controls the size of $\mathcal{S}^2_+$ up to an ε error, as large as possible. Moreover, estimate (41) shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}g^2(s_-,t)\leq-4,$$

while at an interior minimum s_{neck} of g, it is easy to see that

$$(g^2)_t\big|_{s=s_{\text{neck}}}\geq-8.$$

This line of reasoning strongly suggests that it should be possible to construct an open set of initial data for which g^2 vanishes at s_- before it can vanish at an interior point. What keeps this formal argument from being a rigorous proof is that in order to obtain a uniform lower bound for g in the neighborhood where a local minimum can form, one needs a uniform bound from below on the distance between $\mathcal{S}^2_-$ and the first critical point of f. However, it is notoriously difficult to control the location of a critical point of a solution of a parabolic PDE. Nevertheless, we believe the following to be true:

Conjecture A. *For appropriate choices of $\alpha \ll A$, a solution originating from initial data satisfying our Closeness Assumptions satisfies*

$$\mu(t) = g(s_-, t)$$

for as long as it exists.

We now proceed under the **assumption** that Conjecture A is true. If so, then recalling Lemma 10, ones sees that solutions originating from initial data satisfying our Closeness Assumptions become singular only by crushing the fiber $\mathcal{S}^2_-$. We state this as follows:

Corollary 11. *If a solution originating from initial data satisfying our Closeness Assumptions becomes singular at time T, then $g(s_-, T) = 0$.*

Corollary 12. *All solutions originating from initial data satisfying our Closeness Assumptions develop finite-time Type-I singularities.*

Proof. Assuming Conjecture A, it follows from estimate (41) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mu^2(t)\big) \leq -4. \tag{45}$$

So a finite-time singularity is inevitable. As a consequence of Lemma 8, to prove that the singularity is Type-I, we may assume there is a two-sided curvature bound for κ_{01}. Such a bound, together with estimate (33), gives a uniform constant C such that $|\operatorname{Rc}(G(t))| \leq C\mu^{-2}(t)$ for as long as the flow exists. But then the result follows easily from estimate (45). □

6. Convergence to the blowdown soliton

Corollaries 11 and 12 tell us that any point $p \in \mathcal{S}^2_-$ is a *special Type-I singular point* in the sense of Enders–Müller–Topping [EMT11]. It follows from that work that every blow-up sequence $\big(\mathcal{S}^2 \tilde{\times} \mathcal{S}^2, G_k(t), p\big)$ subconverges to a smooth nontrivial gradient shrinking soliton $\big(\mathcal{M}, G_\infty(\tau)\big)$ defined for $-\infty < \tau < 0$. Using Lemma 9, we determine that the limit is noncompact. So $\mathcal{M}$ is diffeomorphic to $\mathbb{C}^2$ blown up at the origin; that is, $\mathcal{O}(-1)$. Moreover, the symmetries of $G(t)$ are preserved in the limit, so the metric retains the form exhibited in (1):

$$G_\infty = \mathrm{d}s \otimes \mathrm{d}s + \Big\{ f^2\, \omega^1 \otimes \omega^1 + g^2 \big(\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3\big) \Big\}. \tag{46}$$

Here and in the remainder of this section, we abuse notation by using s to represent arclength from $\mathcal{S}^2_-$ in the limit soliton, and using f and g for the other components of the limit soliton metric.

The quantity ψ that we estimate in Lemma 3 is scale-invariant, so the limit soliton satisfies

$$-1 \leq \frac{g g_s}{f} \leq 1, \tag{47}$$

which implies that the limit is "not too far" from Kähler in a precise sense. It is a general principle that shrinking solitons appear in discrete rather than continuous families, modulo scaling and isometry. So it is reasonable to expect that there are no other cohomogeneity-one shrinking solitons in the neighborhood of such metrics satisfying estimate (47). (For a related rigidity result, see work [Kot18] of Kotschwar.) To obtain this result, however, we require another assumption.

We now introduce that second conjecture and then present the formal argument that motivates us to believe it is true:

Conjecture B. *If $\big(\mathcal{M}, G_\infty(\tau)\big)$ is a smooth gradient shrinking soliton having the form (46) obtained as a limit of parabolic rescalings of a solution originating from initial data that satisfy our Closeness Assumptions for suitable $\alpha \ll A$, then*

$$g g_{ss}\big|_{\mathcal{S}^2_-} = 1.$$

Since by l'Hôpital's rule, $g g_{ss}\big|_{\mathcal{S}^2_-} = \lim_{s\searrow s_-}(g g_s/f)$, we call this an "infinitesimal Kähler condition". Our formal argument that it should hold on the limit soliton is based on a parabolic rescaling of the original solution in a neighborhood of the developing singularity on $\mathcal{S}^2_-$. For clarity in the argument, we write $\zeta_t\big|_\xi$ to indicate that we are taking the time derivative of a smooth space-time function ζ with a spatial variable ξ held fixed. All time derivatives computed thus far have been derived from (5), in which x is held fixed.

Using (14) along with equations (17a) and (17b), one computes that the evolution equations for u and v with x held fixed may be written with respect to the ϱ variable as

$$\frac{1}{4}\, u_t\big|_x = \frac{u_{\varrho\varrho}}{u} - \frac{u_\varrho^2}{u^2} + \frac{u_\varrho v_\varrho}{uv} - \frac{u^2}{v^2}, \tag{48a}$$

$$\frac{1}{4}\, v_t\big|_x = \frac{v_{\varrho\varrho}}{u} + \frac{u}{v} - 2, \tag{48b}$$

respectively. Motivated by Corollary 12 and the geometry of the blowdown soliton, we introduce new time and space variables,

$$\tau := -\log\big\{4(T-t)\big\} \qquad \text{and} \qquad \sigma := \sqrt{2}\,\tau + \varrho,$$

where $T<\infty$ is the singularity time. We then define rescaled metric components,

$$U(\sigma,\tau) := e^\tau u(s,t) \qquad \text{and} \qquad V(\sigma,\tau) := e^\tau v(s,t),$$

noting that a solution is Kähler if and only if $U = V_\sigma$. We observe that equation (16) implies that

$$\sigma_\tau = \sqrt{2} + \mathcal{J},$$

where $\mathcal{J} := \varrho_\tau = e^{-\tau}\varrho_t/4$. To compute the nonlocal, nonlinear term $\mathcal{J}$, we note that

$$f_s = \frac{U_\sigma}{U}, \qquad g_s = \frac{V_\sigma}{\sqrt{UV}},$$

and

$$g_{ss} = e^{\tau/2}\left\{\frac{2V_{\sigma\sigma}}{UV^{1/2}} - \frac{V_\sigma\,(UV)_\sigma}{U^2V^{3/2}}\right\}.$$

Applying these transformations to formula (16) shows that in these coordinates, the nonlocal term is given by

$$\mathcal{J} = \int_{\sqrt{2}\tau}^{\sigma}\left\{\frac{V_{\bar\sigma\bar\sigma}}{UV} - \frac{U_{\bar\sigma}\,V_{\bar\sigma}}{U^2V} - \frac{1}{2}\frac{V_{\bar\sigma}^2}{UV^2} + \frac{1}{2}\frac{U}{V^2}\right\}\mathrm{d}\bar\sigma. \tag{49}$$

The conversion from time derivatives with x held fixed to time derivatives with σ held fixed is given by

$$U_\tau\big|_\sigma + \big(\sqrt{2}+\mathcal{J}\big)U_\sigma - U = \frac{1}{4}u_t\big|_x,$$
$$V_\tau\big|_\sigma + \big(\sqrt{2}+\mathcal{J}\big)V_\sigma - V = \frac{1}{4}v_t\big|_x.$$

Thus by using (48), we obtain the evolution equations

$$U_\tau\big|_\sigma = \frac{U_{\sigma\sigma}}{U} - \big(\sqrt{2}+\mathcal{J}\big)U_\sigma - \frac{U_\sigma^2}{U^2} + \frac{U_\sigma V_\sigma}{UV} - \frac{U^2}{V^2} + U, \tag{50a}$$

$$V_\tau\big|_\sigma = \frac{V_{\sigma\sigma}}{U} - \big(\sqrt{2}+\mathcal{J}\big)V_\sigma + \frac{U}{V} + V - 2. \tag{50b}$$

Remark 5. *By comparing equations (25) and (50b), one finds that if Φ is the rescaling of the blowdown soliton φ, then $V = \Phi$ evolves by $V_\tau\big|_\sigma = -\mathcal{J}\,V_\sigma$. But by Lemma 2, $\mathcal{J}$ vanishes on any Kähler solution. Hence $V = \Phi$ becomes a stationary solution in these coordinates.*

Motivated by the quantity ψ introduced in (26), we now define

$$\Omega := \frac{V_\sigma}{U}.$$

Then differentiating equations (49) and (50b), recalling (50a), and arranging terms, one computes that Ω evolves by

$$\Omega_\tau\big|_\sigma = \frac{\Omega_{\sigma\sigma}}{U} + \left\{\frac{U_\sigma}{U^2} - \frac{\Omega}{V} - \sqrt{2} - \mathcal{J}\right\}\Omega_\sigma + (1-\Omega^2)\left(\frac{U_\sigma}{UV} - \frac{1}{2}\frac{U\Omega}{V^2}\right). \tag{51}$$

We note that $\Omega \equiv 1$ is a stationary solution, which reflects the fact that the Kähler condition is preserved under Riemannian Ricci flow.

To linearize, we define $\omega := \Omega - 1$ and compute that

$$\omega_\tau\big|_\sigma = \frac{\omega_{\sigma\sigma}}{U} + \left\{\frac{U_\sigma}{U^2} - \frac{1}{V} - \sqrt{2}\right\}\omega_\sigma + \left\{\frac{U}{V^2} - 2\frac{U_\sigma}{UV}\right\}\omega + Q[\omega], \tag{52}$$

where the nonlinear terms on the RHS are given by

$$Q[\omega] = -\left\{\frac{\omega}{V} + \mathcal{J}\right\}\omega_\sigma + \left\{\frac{1}{2}\frac{U\,(3+\omega)}{V^2} - \frac{U_\sigma}{UV}\right\}\omega^2.$$

Here we use the fact that

$$\mathcal{J} = \int_{\sqrt{2}\tau}^{\sigma}\left\{\frac{\omega_{\bar\sigma}}{UV} - \frac{U(\omega+\omega^2/2)}{V^2}\right\}\mathrm{d}\bar\sigma.$$

If ω is small, we are close to a Kähler solution. It then follows from (11) that $V = 1 + a_1 e^\sigma + a_2 e^{2\sigma} + \cdots$ and $U = a_1 e^\sigma + 2a_2 e^{2\sigma} + \cdots$. Thus as $\sigma \searrow -\infty$, *i.e.,* in a neighborhood of $\mathcal{S}^2_-$, the factor multiplying ω in the linear reaction term of equation (52) satisfies

$$\frac{U}{V^2} - 2\frac{U_\sigma}{UV} \approx -2,$$

which leads us to expect "asymptotic approach to Kähler" in that neighborhood, and thus motivates us to make Conjecture B.

Assuming Conjecture A and Conjecture B are true, we now prove the following:

Lemma 13. *Any smooth gradient shrinking soliton* $\big(\mathcal{M}, G_\infty(\tau)\big)$ *having the form* (46) *obtained as a limit of parabolic rescalings at* $\mathcal{S}^2_-$ *is Kähler.*

Proof. We work at a fixed time $\tau < 0$ and so suppress time below. However, we continue to use subscripts to indicate spatial derivatives. We note here that smoothness requires that the closing conditions (2) hold at $s = 0$, a fact we use freely below.

We define

$$F(s) = f - gg_s.$$

We have $F(0) = 0$ by smoothness of the metric, and $F_s(0) = 0$ by Conjecture B, because $F_s(0) = 1 - gg_{ss}$. We proceed to show that $F = 0$ for all s.

We denote the soliton potential function by Γ and we set $\gamma = \Gamma_s$. Using equation (51) from [IKS16] to compute the Lie derivative, we find that the soliton equation

$$-\operatorname{Rc}[G_\infty] = \lambda G_\infty + \frac{1}{2}\mathcal{L}_{\nabla\Gamma} G_\infty \tag{53}$$

becomes the system

$$\gamma_s = \frac{f_{ss}}{f} + 2\frac{g_{ss}}{g} - \lambda, \tag{54a}$$

$$\frac{f_{ss}}{f} = \frac{f_s\gamma}{f} - 2\frac{f_s g_s}{fg} + 2\frac{f^2}{g^4} + \lambda, \tag{54b}$$

$$\frac{g_{ss}}{g} = \frac{g_s\gamma}{g} - \frac{f_s g_s}{fg} - \frac{g_s^2}{g^2} - 2\frac{f^2}{g^4} + \frac{4}{g^2} + \lambda, \tag{54c}$$

where $\lambda < 0$ depends on our choice of τ above.

Computing F_s using equation (54c), one finds that

$$\begin{aligned} F_s &= f_s - g_s^2 - gg_{ss} \\ &= f_s - gg_s\gamma + \frac{gf_sg_s}{f} + 2\frac{f^2}{g^2} - \lambda g^2 - 4 \\ &= \left(\gamma - \frac{f_s}{f}\right) F + 2f_s - f\gamma + 2\frac{f^2}{g^2} - \lambda g^2 - 4. \end{aligned} \tag{55}$$

Hence

$$F_{ss} = \left(\gamma - \frac{f_s}{f}\right) F_s + \left(\gamma - \frac{f_s}{f}\right)_s F + X,$$

where we use (54b) to rewrite the final term above as

$$\begin{aligned} X &= 2f_{ss} - f_s\gamma - f\gamma_s + 4\left(\frac{ff_s}{g^2} - \frac{f^2g_s}{g^3}\right) - 2\lambda gg_s \\ &= \left(4\frac{f^2}{g^4} + 4\frac{f_s}{g^2} + 2\lambda\right) F + \gamma^2\left(\frac{f}{\gamma}\right)_s. \end{aligned}$$

Therefore, F satisfies the linear second-order (seemingly inhomogeneous) ODE

$$F_{ss} - \left(\gamma - \frac{f_s}{f}\right) F_s - \left\{\left(\gamma - \frac{f_s}{f}\right)_s + 4\frac{f_s}{g^2} + 4\frac{f^2}{g^4} + 2\lambda\right\} F = \gamma^2\left(\frac{f}{\gamma}\right)_s. \tag{56}$$

We now show that the term on the RHS can be rewritten in terms of F and F_s. Using equations (54a), (54b), and (54c) in order, and then applying the identity $gg_s = f - F$, we obtain

$$\begin{aligned}
\frac{1}{2}\gamma^2\left(\frac{f}{\gamma}\right)_s &= \frac{1}{2}(f_s\gamma - f\gamma_s)\\
&= \frac{1}{2}\left(f_s\gamma - f_{ss} - 2\frac{fg_{ss}}{g} + \lambda f\right)\\
&= \frac{f_s g_s}{g} - \frac{f^3}{g^4} - \frac{fg_{ss}}{g}\\
&= 2\frac{f_s g_s}{g} - \frac{fg_s\gamma}{g} + \frac{fg_s^2}{g^2} + \frac{f^3}{g^4} - 4\frac{f}{g^2} - \lambda f\\
&= -\left(2\frac{f_s}{g^2} + \frac{f^2}{g^4} + \frac{fg_s}{g^3} - \frac{f\gamma}{g^2}\right)F + Y, \qquad (57)
\end{aligned}$$

where

$$Y = 2\frac{ff_s}{g^2} + 2\frac{f^3}{g^4} - 4\frac{f}{g^2} - \frac{f^2\gamma}{g^2} - \lambda f.$$

Using equation (55) to rewrite the first term on the RHS, it is easy to see that

$$Y = \frac{f}{g^2}F_s + \frac{f_s - f\gamma}{g^2}F. \qquad (58)$$

So by using equations (57) and (58), we find that equation (56) can be rewritten as the linear second-order *homogeneous* ODE

$$F_{ss} + \left(\frac{f_s}{f} - \gamma - 2\frac{f}{g^2}\right)F_s + \left\{\left(\frac{f_s}{f} - \gamma\right)_s + 2\frac{fg_s}{g^3} - 2\frac{f_s}{g^2} - 2\frac{f^2}{g^4} - 2\lambda\right\}F = 0.$$

Because $f_s/f \sim 1/s$ and $(f_s/f)_s \sim -1/s^2$ as $s \searrow 0$, this ODE has a regular singular point at $s = 0$. It is approximated in a neighborhood of $s = 0$ by the equidimensional Euler equation $s^2y''(s) - sy'(s) + y(s) = 0$, for which a fundamental set of solutions is $\{s,\, s\log(s))\}$. It then follows from a theorem of Frobenius that a fundamental set of solutions of the exact equation has the form

$$\sum_{n=0}^{\infty} a_n s^{n+1} \qquad \text{and} \qquad \sum_{n=0}^{\infty} b_n s^n \log(s),$$

where all coefficients except a_0 and b_0 are determined by recurrence relations. We conclude that F is identically zero for all $s \geq 0$, hence that the soliton is Kähler. □

Theorem 1.5 of [FIK03] tells us that the blowdown soliton is unique up to scaling and isometry among U(2)-invariant Kähler–Ricci solitons. Hence this completes our presentation of evidence in favor of our Main Conjecture.

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