Singularity models for the Ricci flow: an introductory survey

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At the time this article is being written (May 2003) much of the mathematical world is waiting with intense interest to see the results of Grisha Perelman's effort [**22, 23**] to resolve William Thurston's Geometrization Conjecture [**28**] for closed 3-manifolds by completing the program [**16**] begun by Richard Hamilton. It is still too early to give an accurate and fair assessment of the full impact of Perelman's work. But in order to aid the many mathematicians who may be inspired by that work to look more closely at the Ricci flow, this does seem like an appropriate time to write a brief and purely expository introduction to the topic, intended for the non-expert. Readers desiring more information are encouraged to read the more advanced survey articles [**6**] and [**7**], as well as to consult Hamilton's and Perelman's original papers.

1. Heuristics

There are two heuristic principles which are useful to keep in mind when one first studies the (unnormalized) Ricci flow, wherein one starts with a smooth Riemannian manifold $(\mathcal{M}^n, g_0)$ and evolves it by the equation

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Rc}, \tag{1.1}$$

where Rc denotes the Ricci tensor of the metric g. The first principle is that equation (1.1) is morally the heat equation for a Riemannian metric. The best way to see this is by writing the right-hand side of equation (1.1) in harmonic coordinates. Recall that a coordinate chart $\varphi : \mathcal{U} \to \mathbb{R}^n$ defined in a neighborhood $\mathcal{U}$ of a point x_0 on a smooth Riemannian manifold $(\mathcal{M}^n, g)$ is called *harmonic* if the coordinate functions $\{x^k : 1 \leq k \leq n\}$ it induces are harmonic throughout $\mathcal{U}$:

$$\Delta x^k = 0.$$

It follows from standard existence and regularity theory for elliptic PDE that a harmonic coordinate chart exists in a sufficiently small neighborhood $\mathcal{U}$ of any $x_0 \in \mathcal{M}^n$, and moreover that the metric enjoys optimal regularity in such coordinates. The feature of harmonic coordinates relevant to the Ricci flow is that the identity

$$0 = -\Delta x^k = -g^{ij}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma_{ij}^{\ell}\frac{\partial}{\partial x^\ell}\right) x^k = g^{ij}\Gamma_{ij}^k$$

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holds throughout $\mathcal{U}$. Using this, it is not hard to see that

$$-2R_{ij} = -2R^{m}_{mij} = g^{k\ell}\frac{\partial^2}{\partial x^k \partial x^\ell}(g_{ij}) + Q_{ij}\left(g^{-1}, \partial g\right), \tag{1.2}$$

where Q is quadratic in the inverse and first derivatives of g. In other words, the highest-order derivative term in $-2R_{ij}$ appears in a harmonic chart to be the Laplacian of the component g_{ij} of the metric, regarded as a scalar function in that chart.

There are at least two reasons why this principle is heuristic and not entirely rigorous. One is that equation (1.2) is not tensorial; indeed, all covariant derivatives of the metric vanish identically. Another reason is that when one evolves a metric by (1.1), coordinates which are harmonic at time t cannot be expected to be so at time $t+\varepsilon$, for any $\varepsilon > 0$.

Nonetheless, much of what this heuristic principle suggests is almost true. Equation (1.2) is a quasilinear parabolic equation. (It is quasilinear because the inverse of the unknown function $g(t)$ multiplies the highest-order derivatives in the equation.) In fact, the Ricci flow is quasilinear and almost parabolic. Its failure to be strictly parabolic stems from the fact that the Ricci flow is defined entirely in terms of natural geometric quantities, and hence is invariant under the full diffeomorphism group. This invariance is a great advantage from the standpoint of geometry. Fortunately, it is only a minor inconvenience from the standpoint of PDE, because Dennis DeTurck showed [**10, 11**] that the flow is equivalent to a strictly parabolic quasilinear equation, so that short-time existence and uniqueness follow from standard theory.

The heuristic of thinking of the Ricci flow as a heat equation is useful in another important sense. The heat equation seeks to regularize its initial data, so one expects equation (1.1) to improve the metric, at least for a short time. In fact, initial bounds on the curvature of a metric induce subsequent *a priori* bounds on all derivatives of the curvature. These derivative bounds were derived in [**3**], [**2**], and [**25, 26**]; most known proofs use the maximum principle for parabolic equations in a familiar technique pioneered by Bernstein.

On a more intuitive level, the heat equation heuristic leads one to expect that the Ricci flow will mimic the diffusion properties of the heat equation, and thus will try to make a metric more homogeneous and isotropic. As we will see below, this expectation too is often justified.

The second heuristic principle is that we should expect the Ricci flow to develop singularities; in particular, it tells us that the first principle can be misleading if we take it too literally. One arrives at the second heuristic from the viewpoint of geometry when one observes that the normalized Ricci flow (defined in (2.1) below) can converge only to an Einstein metric. Since most Riemannian manifolds of dimension $n > 2$ are not Einstein, one should expect something to go wrong. The second principle is also supported by the viewpoint of analysis. Indeed, equation (1.1) implies that the scalar curvature evolves by

$$\frac{\partial}{\partial t}R = \Delta R + 2\left|\mathrm{Rc}\right|^2. \tag{1.3}$$

This is a reaction-diffusion equation: the reaction term $2\left|\mathrm{Rc}\right|^2$ may be regarded as fighting against the diffusion term ΔR. By a standard estimate, equation (1.3) implies that

$$\frac{\partial}{\partial t} R \geq \Delta R + \frac{2}{n} R^2.$$

By the parabolic maximum principle, one can then compare solutions of (1.3) with solutions of the ODE

$$\frac{d\rho}{dt} = \frac{2}{n}\rho^2$$

obtained by ignoring the diffusion term in (1.3). One concludes that that

$$R_{\min}(t) \geq \frac{1}{R_{\min}^{-1}(t_0) - \frac{2}{n}(t - t_0)}$$

for all $t \geq t_0$ that the solution $g(t)$ exists. Hence if the scalar curvature ever becomes everywhere positive, a finite-time singularity is inevitable.

The second heuristic principle suggests correctly that a positive resolution of the Geometrization Conjecture depends on developing an adequate understanding of singularities. One must learn what the behavior of a solution to the Ricci flow that becomes singular reveals about the topology of the underlying manifold. The reason why current research into geometrization concentrates on the analysis of singularities is because any solutions which remain nonsingular behave very nicely. Indeed, these are the solutions for which the diffusion term dominates, in accord with the first heuristic principle. Hence our informal survey begins with such solutions.

2. Nonsingular solutions

By rescaling space and time, one can modify the Ricci flow so that it preserves volume. This leads to the equation

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Rc} + \frac{2r}{n} g, \tag{2.1}$$

where

$$r(t) = \frac{\int_{\mathcal{M}^n} R\, d\mu}{\int_{\mathcal{M}^n} d\mu}$$

denotes the average scalar curvature. Equation (1.1) (the unnormalized Ricci flow) is in a sense the more natural PDE, but equation (2.1) (the normalized Ricci flow) can be more convenient for taking limits and for establishing certain convergence properties.

For example, in proving Hamilton's seminal result [**14**] that any manifold which admits a metric of positive Ricci curvature is in fact a space form, one first studies the solution of equation (1.1). One shows that the solution exists on a maximal time interval $0 \leq t < T < \infty$, that the curvature becomes unbounded as $t \nearrow T$, and that the metric becomes nearly Einstein at points where the curvature is large. Then one derives a crucial estimate on the gradient of the scalar curvature. Together with the observation that the diameter of the solution is uniformly bounded, this allows one to show that the ratio $R_{\max}/R_{\min}$ approaches 1 as the singularity time is approached. Then one rescales space and time, converting the solution $g(t)$ of equation (1.1) into a solution $\bar{g}(\bar{t})$ of equation (2.1). In the final step, one shows that $\bar{g}(\bar{t})$ exists for all time and converges exponentially in every C^k norm to a metric of constant sectional curvature.

One way of understanding the role of the normalized flow in this proof is as follows: because the minimum curvature of the original solution approaches the maximum curvature as the maximum becomes large, the average curvature itself becomes large. Intuitively this means that when one rescales, the dilation term $2r/n$ is large enough to keep the solution nonsingular. So in this case, the diffusion effect suggested by the first heuristic principle wins: it tames the singularity one expects from the second heuristic principle. On the other hand, if the curvature of a solution of (2.1) ever becomes large on a set of small volume, one would not expect the dilation term to be adequate to avoid a singularity.

One says a solution $(\mathcal{M}^3, g(t))$ of the normalized flow (2.1) on a compact 3-manifold is *nonsingular* if it exists for all positive time and satisfies a uniform curvature bound

$$\sup_{\mathcal{M}^3\times[0,\infty)} |\mathrm{Rm}| \le C < \infty.$$

In this case, $\mathcal{M}^3$ is geometrizable. This result was proved in [**18**]. The proof uses the Gromov-type convergence results in [**17**] as well as the advanced tensor maximum principle of [**15**], which lets one compare a tensor evolving by a PDE with a the solution to a system of ODE.

The results in [**18**] classify nonsingular solutions. The behaviors one observes for such solutions turn out to be instructive when one later studies singular solutions. Hence it will be useful to recall some aspects of that classification here.

If

$$\lim_{t\to\infty}\left(\sup_{x\in\mathcal{M}^3} \mathrm{inj}\,(x,t)\right) = 0, \tag{2.2}$$

one says the solution exhibits *collapse with bounded curvature.* In this case, results of Cheeger–Gromov imply that $\mathcal{M}^3$ admits an $\mathcal{F}$-structure, hence is a graph manifold, hence can be decomposed into a union of Seifert fiber-space pieces, all of which are known to be geometrizable. (An excellent survey is [**24**].) In case (2.2) does not hold, one can find $\varepsilon > 0$ and sequences of points $x_j \in \mathcal{M}^3$ and times $t_j \to \infty$ such that

$$\inf_{j\in\mathbb{N}} (\mathrm{inj}\,(x_j, t_j)) \ge \varepsilon. \tag{2.3}$$

Then there exist diffeomorphisms $\varphi_j : \mathcal{M}^3 \to \mathcal{M}^3$ such that the pointed sequence of solutions

$$\left(\mathcal{M}^3, g_j(t), x_j\right)$$

defined by

$$g_j(t) = \left(\varphi_j^* g\right)(t_j + t)$$

converges locally smoothly in the pointed category to a limit solution

$$\left(\mathcal{M}^3_\infty, g_\infty(t), x_\infty\right).$$

If $\mathcal{M}^3_\infty$ is compact, it is necessarily diffeomorphic to $\mathcal{M}^3$. In this case, results in [**18**] prove that g_∞ is a metric of constant sectional curvature K, hence that $\mathcal{M}^3$ is a space form. Moreover, one can in some cases prove a stronger convergence than was stated above. Indeed, if $K > 0$, then there exists t_j large enough so that the Ricci curvature of $g(t_j)$ is strictly positive; it then follows from Hamilton's original result [**14**] that the original solution converges exponentially fast to a space form. (In other words, one needed neither to modify by diffeomorphisms nor to restrict to a sequence of times.) If $K = 0$, the same statement holds by [**13**]. It is conjectured but not yet proven that a similar statement holds for the case $K < 0$.

The remaining possibility is that $\mathcal{M}^3_\infty$ is noncompact, which is in a sense the most interesting case. A particular consequence of the results proved in [**18**] is that there exists a finite collection of finite-volume noncompact hyperbolic manifolds $(\mathcal{H}_i, h_i)$ with the following properties. For each $k \in \mathbb{N}$, there exist truncations $\mathcal{H}_{i,k}$ of $\mathcal{H}_i$ along constant mean curvature tori of area less than $1/k$, a time t_k, and embeddings $\psi_{i,k}(t) : \mathcal{H}_{i,k} \hookrightarrow \mathcal{M}_\infty$ defined for all $t \geq t_k$ such that the images $\psi_{i,k}(t)(\mathcal{H}_{i,k})$ are mutually disjoint submanifolds for all $t \geq t_k$ and

$$\sup_{t \geq t_k} \left\| \psi^*_{i,k}(t)(g(t)) - h_i \right\|_{C^k(\mathcal{H}_{i,k})} \leq 1/k$$

in the norm of C^k convergence on compact subsets of $\mathcal{H}_{i,k}$. From the perspective of topology, the most important aspect of this case is that the fundamental group of each $\mathcal{H}_i$ injects into $\mathcal{M}^3_\infty$ under the maps $\psi_{i,k}$. This proves that $\mathcal{M}^3_\infty$ is Haken, hence is geometrizable by Thurston's result [**28**]. In fact, one can show directly that the injectivity radius at any point in the complement $\mathcal{M}_\infty \backslash \cup_i \psi_{i,k}(t)(\mathcal{H}_{i,k})$ of the hyperbolic pieces in the limit remains less than $1/k$ for all $t \geq t_k$. In particular, this implies that the complement is a manifold-with-boundary which admits an $\mathcal{F}$-structure, hence may be decomposed into a union of Seifert fiber-space pieces.

3. Singular solutions

Having seen that any manifold which admits a nonsingular solution of the Ricci flow is known to be geometrizable, one is led to ask the following question: What does a singularity of the flow reveal about the topology of the underlying manifold? A powerful technique in geometric analysis for answering this question is to 'blow-up' the singularity, obtaining a limit flow whose properties should yield information about the geometry of the original manifold near the singularity just prior to its formation.

To see how this is done, consider a solution $(\mathcal{M}^n, g(t))$ of the unnormalized Ricci flow on a maximal time interval $0 \leq t < T$. For simplicity, we shall discuss the case that $T < \infty$, although an analogous theory is available for singularities which form in infinite time. In the case of a finite-time singularity, it follows from short-time existence results for the flow that

$$\lim_{t \nearrow T} \left(\sup_{x \in \mathcal{M}^n} |\mathrm{Rm}(x,t)| \right) = \infty. \tag{3.1}$$

To develop intuition, consider what is perhaps the simplest example of a finite-time singularity. Let g_{can} denote the canonical metric of constant sectional curvature $K = 1$ on the sphere $\mathcal{S}^n$. Consider a 1-parameter family $g(t)$ of conformal metrics defined by

$$g(t) = r(t)^2 g_{\mathrm{can}}, \tag{3.2}$$

noting that $g(t)$ has constant sectional curvature $r(t)^{-2}$. Observe that $g(t)$ is a solution of the Ricci flow if and only if

$$2r\frac{dr}{dt} \cdot g_{\mathrm{can}} = \frac{\partial}{\partial t} g = -2\,\mathrm{Rc}[g] = -2\,\mathrm{Rc}[g_{\mathrm{can}}] = -2(n-1) \cdot g_{\mathrm{can}},$$

hence if and only if

$$r(t) = \sqrt{r_0^2 - 2(n-1)t} = \sqrt{2(n-1)}\sqrt{T-t}. \tag{3.3}$$

So if $g(t)$ solves the Ricci flow, it must become singular at $T = r_0^2/(2(n-1))$.

In general, one says that a finite-time singularity of $(\mathcal{M}^n, g(t))$ is of *Type I* or *fast-forming* if it occurs at the natural rate suggested by this example, hence if there exists $C < \infty$ such that

$$\sup_{\mathcal{M}^n \times [0,T)} |\mathrm{Rm}| \cdot (T-t) \leq C < \infty.$$

On the other hand, one says the singularity is of *Type IIa* if

$$\sup_{\mathcal{M}^n \times [0,T)} |\mathrm{Rm}| \cdot (T-t) = \infty.$$

Solutions with this property are also called *slowly-forming singularities.* (This terminology may be somewhat nonintuitive initially, but is actually perfectly natural for reasons involving the sharpness of certain *a priori* estimates which guarantee short-time existence of the flow.)

It is common to study a singularity by the technique of *parabolic dilation.* One chooses sequences of points $x_j \in \mathcal{M}^n$, times $t_j \in [0,T)$ increasing monotonically to T, and dilating factors $\lambda_j > 0$. Using these, one defines a sequence $(\mathcal{M}^n, g_j(t), x_j)$ of pointed solutions to the Ricci flow, where

$$g_j(t) = \lambda_j g\left(t_j + \frac{t}{\lambda_j}\right).$$

Notice that each solution $g_j(t)$ exists on the time interval $-\lambda_j t_j \leq t < \lambda_j (T - t_j)$.

One chooses the sequences $\{x_j\}$, $\{t_j\}$, and $\{\lambda_j\}$ in order to get a good limit. A flat limit would not be desirable because it would not reveal enough about the geometry of the original solution. (Intuitively, getting a flat limit means one has dilated too much.) On the other hand, one wants uniform bounds on the curvatures of the metrics $g_j(0)$ in order to take advantage of the derivative estimates described in Section 1. (Bounds on the curvature of a solution to the Ricci flow imply bounds on all derivatives of the curvature.) Accordingly, one usually chooses x_j and λ_j so that the curvature $|\mathrm{Rm}(x_j, t_j)|$ is comparable to $\sup_{x \in \mathcal{M}^n} |\mathrm{Rm}(x, t_j)|$ and so that

$$0 < c \leq \lambda_j^{-1} |\mathrm{Rm}(x_j, t_j)| \leq C < \infty. \tag{3.4}$$

Recalling (3.1), one chooses the sequence of times so that $\sup_{x \in \mathcal{M}^n} |\mathrm{Rm}(x, t_j)|$ is comparable to the maximum curvature over a sufficiently large interval of earlier times.

Having made suitable choices, one wants to prove that $(\mathcal{M}^n, g_j(t), x_j)$ converges locally smoothly in the pointed category to a limit solution $(\mathcal{M}^n_\infty, g_\infty(t), x_\infty)$ of the Ricci flow, called a *singularity model* (also called a 'final time limit flow' in the literature). As we shall see below, such solutions are typically very special. (For example, a singularity model formed from a Type IIa singularity will exist for all times $-\infty < t < \infty$, which helps explain why such singularities are called 'slowly forming'.) By developing an adequate understanding of these model solutions, one hopes to obtain local information about the geometry of the original sequence just prior to the formation of the singularity. In particular, the strategy for extracting topological information about a 3-manifold $\mathcal{M}^3$ from a solution $(\mathcal{M}^3, g(t))$ of the Ricci flow that becomes singular is to perform a geometric-topological surgery on

$\mathcal{M}^3$ just prior to the singularity in such a way that the maximum curvature of the solution is reduced by an amount large enough to permit the flow to be continued on the piece or pieces that remain after the surgery. In the final step of this program, one needs to argue that only geometrically recognizable pieces will remain after finitely many surgeries.

The main difficulty in obtaining convergence to a singularity model is obtaining an adequate injectivity radius estimate for the sequence $(\mathcal{M}^n, g_j(t), x_j)$, namely an estimate of the form

$$\inf_{x\in\mathcal{M}^3} (\mathrm{inj}\,(x,t)) \geq \frac{c}{\sqrt{\sup_{x\in\mathcal{M}^3} |\mathrm{Rm}\,(x,t)|}}.$$

Until recently, such estimates had to be derived by *ad hoc* means. For example, in Section 23 of [**16**], Hamilton proved an isoperimetric inequality that implies an injectivity radius estimate for appropriately chosen sequences of dilations approaching a Type I singularity in dimension three. In Section 22 of the same paper, he proved an injectivity radius estimate for odd-dimensional solitons that is useful for dimension reduction. (See Section 5, below). The paper [**8**] derives an injectivity estimate useful for forming noncompact limits of Type IIa singularities. One could cite other examples, but the recent work [**22**] of Perelman yields the most powerful and general estimate known to date, which enables one to construct singularity models under very general hypotheses.

4. Singularity models

By analogy with other geometric evolution equations (especially the mean curvature flow) one expects singularity models to have special properties. Indeed, such solutions are generally *ancient* (existing for $-\infty < t < \omega$) or *immortal* (existing for $\alpha \leq t < +\infty$) or even *eternal* (existing for $-\infty < t < +\infty$). Many singularity models also exhibit special symmetries or asymptotic symmetries. (See [**9**].)

An important class of singularity models is the set of *generalized fixed points.* Such solutions can be described in either of two equivalent ways. One says $g(t)$ is a *self-similar solution* of the Ricci flow if there exist scalars $\sigma(t)$ and diffeomorphisms ψ_t of $\mathcal{M}^n$ such that

$$g(t) = \sigma(t)\ \psi_t^*(g_0)\,. \tag{4.1}$$

Notice that a self-similar solution changes only by diffeomorphism and rescaling, hence may be regarded as a fixed point of the Ricci flow modulo diffeomorphisms. On the other hand, one says that a fixed Riemannian manifold $(\mathcal{M}^n, g_0)$ is a *Ricci soliton* if the identity

$$-2\,\mathrm{Rc}\,(g_0) = \mathcal{L}_X g_0 + 2\lambda g_0 \tag{4.2}$$

holds for some constant λ and some complete vector field X on $\mathcal{M}^n$. Notice that (4.2) is a coupled elliptic system for g_0 and X. Because a Ricci soliton is Einstein if the vector field X vanishes identically, any solution of (4.2) may be regarded as a generalization of an Einstein metric. One says a Ricci soliton is *shrinking, translating,* or *expanding* in the cases $\lambda < 0$, $\lambda = 0$, $\lambda > 0$ respectively.

It is not hard to show that there is a one-to-one relationship between self-similar solutions of the flow and Ricci solitons. Hence the two concepts are commonly regarded as equivalent.

The best-known Ricci soliton is the *cigar* $(\mathbb{R}^2, g_\Sigma)$ found by Hamilton. This is the complete metric

$$g_\Sigma = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

on $\mathbb{R}^2$ of positive scalar curvature

$$R_\Sigma = \frac{4}{1 + x^2 + y^2}.$$

It is not hard to show that the curvature of the cigar decays like e^{-2s}, where $s = \operatorname{arcsinh} \sqrt{x^2 + y^2}$ is the metric distance from the origin. (The cigar is actually one representative of a family of Kähler-Ricci metrics that exist on $\mathbb{C}^{2m}$ and certain other complex manifolds. These are studied in [**4, 5**] and [**12**].) The reason why the cigar is of particular relevance to Geometrization is that in dimension $n = 3$, singularity models corresponding to the soliton metric

$$g_0 = g_\Sigma + dz^2 \tag{4.3}$$

on quotients of $\mathbb{R}^3$ would represent a serious obstacle to proving Geometrization via the Ricci flow, because it is not known how to perform surgery at singularities tending to such limits. (We will see the cigar again in Section 5 below.)

Another family of Ricci solitons of special interest comprises the complete metrics $(\mathbb{R}^n, g_0)$ found for $n \geq 3$ by Robert Bryant and Tom Ivey [**21**]. These are called *gradient solitons* because the vector field X in (4.2) is the gradient field of a potential function. Although the metrics g_0 cannot be written down explicitly, one can compute that the curvature decays like $1/s$ as one moves away from the origin.

In spite of the special properties possessed by singularity models in all dimensions, singularities of the Ricci flow $(\mathcal{M}^n, g(t))$ in high dimensions are expected to be very complex. In dimension $n = 3$ however, there are three observations that lead one to expect singularities to be relatively tractable.

The first observation is a pinching estimate proved independently by Hamilton [**16**] and by Ivey [**20**] and later improved by Hamilton [**18**]. Recall that one may regard the Riemann curvature tensor as a self-adjoint operator

$$\mathrm{Rm} : \wedge^2 T\mathcal{M}^n \to \wedge^2 T\mathcal{M}^n.$$

On a 3-manifold, $\wedge^2 T_x\mathcal{M}^3$ is a 3-dimensional vector space for each $x \in \mathcal{M}^3$, so one may denote the eigenvalues of $\mathrm{Rm}(x)$ by $\lambda_1 \leq \lambda_2 \leq \lambda_3$. The *curvature pinching estimate* says that if $(\mathcal{M}^3, g(t))$ is a solution of the Ricci flow for $0 \leq t < T$ such that $\lambda_1 \geq -1$ everywhere on $\mathcal{M}^3$ at $t = 0$ (which can always be achieved by scaling), then at any point $x \in \mathcal{M}^3$ and time $t \in [0, T)$ such that $\lambda_1 < 0$, the scalar curvature satisfies

$$R \geq |\lambda_1| (\log |\lambda_1| + \log(1 + t) - 3).$$

This estimate says that at any point and time where a sectional curvature is negative and large in absolute value, one finds a much larger positive sectional curvature. It implies in particular that any singularity model $(\mathcal{M}^3_\infty, g_\infty(t), x_\infty)$ must have nonnegative sectional curvature at $t = 0$.

The second observation which restricts the possible singularity models one may see in dimension $n = 3$ is the fact that $\mathfrak{so}(2)$ is the only proper nontrivial Lie subalgebra of $\mathfrak{so}(3)$. This fact says that at the origin of any singularity model, the eigenvalues of the curvature operator (after scaling) must conform to either the signature $(+,+,+)$ or else $(0,0,+)$. The other possible pattern, $(0,0,0,)$, is ruled out by (3.4).

The third observation is the fact that a strong maximum principle holds for tensors. Because the curvature operator of any singularity model $\left(\mathcal{M}_\infty^3, g_\infty(t), x_\infty\right)$ is nonnegative at $t=0$ and has either the signature $(+,+,+)$ or else $(0,0,+)$ at the single point $(x_\infty, 0)$, the strong maximum principle says that the curvature operator must possess either the signature $(+,+,+)$ or else $(0,0,+)$ respectively at *all* points $x \in \mathcal{M}_\infty^3$ and times $t>0$ such that the limit solution $g_\infty(t)$ exists.

When viewed at an appropriate length scale, the geometry of a solution to the Ricci flow $\left(\mathcal{M}^3, g(t)\right)$ that becomes singular at time $T<\infty$ closely resembles the singularity model $\left(\mathcal{M}_\infty^3, g_\infty(t), x_\infty\right)$ one obtains by blowing-up the singularity, at least for points near the singularity and times just before T.

The standard example of a singularity of signature $(+,+,+)$ is the shrinking round 3-sphere with the metric $r(t)^2 g_{\text{can}}$, where $r(t)$ is given by (3.3) with $n=3$. As was remarked above, this singularity model itself exhibits a Type I singularity at some finite time.

Singularities of signature $(0,0,+)$ are called *neckpinches.* Near the singularity, a solution which encounters a finite-time neckpinch is expected to resemble the soliton solution

$$g(t) = ds^2 + r(t)^2 g_{\text{can}} \tag{4.4}$$

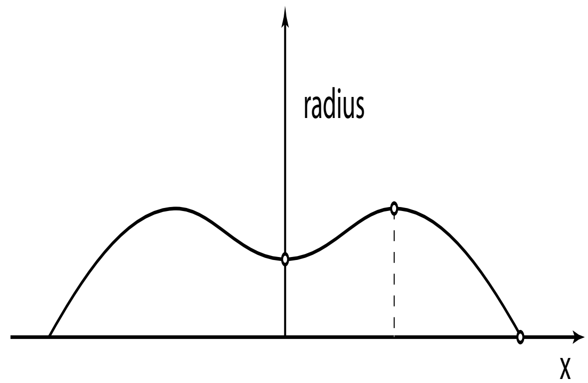
on $\mathbb{R}\times\mathcal{S}^2$. Here $r(t)$ is given by (3.3) with $n=2$. Notice that this model also forms a Type I singularity at some time $T<\infty$. Understanding neckpinches is a vital part of current efforts to obtain topological information from the Ricci flow: developing necks should be geometrically and topologically simple enough that one could remove a small piece of a neck just before a singularity forms in such a way that the curvature on the complement of this piece obeys a bound which allows the flow to be continued there, at least for a short time.

Remarkably, there were until recently no rigorous examples of neckpinch singularities on compact manifolds. The first examples of any sort were warped-product metrics constructed by Miles Simon [**27**] on $\mathbb{R}\times\mathcal{S}^n$. His construction used a supersolution as a barrier to force a singularity to occur in finite time on a compact subset of the manifold. Sigurd Angenent and I have recently studied neckpinch singularities on $\mathcal{S}^{n+1}$ for any $n\geq 2$. We consider metrics of the form

$$g(x,t) = \varphi(x,t)^2\, dx^2 + \psi(x,t)^2\, g_{\text{can}}$$

on $(-1,1)\times\mathcal{S}^n$, which we identify with the sphere $\mathcal{S}^{n+1}$ with its north and south poles removed.

A rough outline of our method is as follows. We call a local minimum of the radius a 'neck' and a local maximum a 'bump'. We consider $\mathrm{SO}(n+1)$-invariant metrics on $\mathcal{S}^{n+1}$ which possess bumps and necks, have positive scalar curvature everywhere, and also obey a gradient bound. One can construct simple examples of such data by removing a neighborhood of the equator of the standard unit sphere and replacing it with a narrow neck. (The figure below shows a reflection-symmetric metric having a single neck and two bumps, but our results are more general.) The hypothesis of $\mathrm{SO}(n+1)$ symmetry implies that the Riemann curvature tensor is completely determined by the sectional curvature K_0 of the 2-planes perpendicular to the spheres $\{x\} \times \mathcal{S}^n$ and the sectional curvature K_1 of the 2-planes tangential to these spheres. We derive pinching estimates for these curvatures which imply that a Type I (rapidly forming) singularity must develop at a neck at some time $T < \infty$. These estimates show that any sequence of parabolic dilations formed at the developing singularity converges to a shrinking cylinder soliton (4.4) on $\mathbb{R} \times \mathcal{S}^n$, with $r(t)$ given by (3.3). We prove that this convergence takes place uniformly in any ball of radius $o\left(\sqrt{-(T-t)\log(T-t)}\right)$ centered at the neck and obtain further estimates for the asymptotics of the developing singularity. (In forthcoming work, we intend to show that these estimates are in fact sharp.)



A sphere with one neck and two bumps.

5. The topology of singularities

As was mentioned above, there are reasons to believe that singularity models in dimension $n = 3$ are amenable to classification. In fact, using delicate geometric and analytic arguments, a partial classification was obtained in [**16**]. Let us now recall the part of that classification that deals with finite-time singularities.

(1) If a solution $(\mathcal{M}^3, g(t))$ of the unnormalized Ricci flow encounters a Type I singularity, then after performing dilations correctly and obtaining an injectivity radius estimate, one obtains a limit which is a quotient of either
   (a) a compact shrinking round sphere $(\mathcal{S}^3, g(t))$, where $g(t)$ is given by (3.2), or
   (b) a noncompact shrinking cylinder $(\mathbb{R} \times \mathcal{S}^2, g(t))$, where $g(t)$ is given by (4.4).

(2) If a solution $(\mathcal{M}^3, g(t))$ of the unnormalized Ricci flow encounters a Type IIa singularity, then after performing dilations correctly and obtaining an injectivity radius estimate, one gets a quotient of one of the following noncompact limits:
   (a) a translating self-similar solution $(\mathbb{R}^3, g(t))$ where $g(t)$ has the form given in (4.1),
   (b) a shrinking cylinder $(\mathbb{R} \times \mathcal{S}^2, g(t))$ as in Case 1b above, or
   (c) a cigar product $(\mathbb{R}^3, g(t))$, where $g(t)$ is the self-similar solution corresponding to the soliton metric g_0 given in (4.3).

An example of what one might see in Case 2a is the Bryant–Ivey soliton mentioned above. For any limit in Case 2a, one performs what is called *dimension reduction* to obtain an ancient solution. (Dimension reduction is a technique that involves taking a limit around a suitable sequence of points tending to spatial infinity; it will not be discussed further in this introductory survey.) If the ancient solution one obtains is not in fact an eternal solution which attains its maximum curvature, one then takes a third limit about a suitable sequence of points and times tending to $-\infty$ where the curvature is sufficiently near its maximum. Having done so, one sees either Case 2b or 2c above.

Since the limit in Case 1a is compact, the underlying manifold of the original solution must have been $\mathcal{S}^3$ or one of its quotients. In the other cases, the singularity model gives local information about the original solution near the singularity just prior to its formation. The recent work [**22**] of Perelman rules out Case 2c. This is highly significant, because (as was remarked above) it was not known how to perform surgery on the original solution if this case were to occur.

There are interesting connections between the topology of a manifold and the singularities it admits. Some of these are revealed by the method of performing geometric-topological surgeries just prior to singularity formation. Others may be found by more direct means. To conclude this survey, we offer one example of such a connection. In recent work [**19**], Tom Ilmanen and I were able to rule out product metrics on $\mathcal{S}^1 \times \mathcal{S}^2$ as possible singularity models, thereby answering affirmatively a conjecture made in [**16**]. Our result rests on a more general principle which yields a lower bound for the diameter of any solution $(\mathcal{M}^n, g(t))$ to the Ricci flow on certain manifolds of any dimension. Having such a bound implies in particular that any singularity model constructed from a finite-time singularity on one of those manifolds must be noncompact.

Here is an outline of the proof. Given a free homotopy class Γ on a compact Riemannian manifold $(\mathcal{M}^n, g)$, we define

$$\ell_g(\Gamma) = \inf_{\gamma \in \Gamma} \operatorname{length}_g(\gamma)$$

and

$$m_g(\Gamma) = \liminf_{k \to \infty} \frac{\ell_g(k\Gamma)}{k}.$$

Then we prove the monotonicity result that if $(\mathcal{M}^n, g(t))$ is a solution of the Ricci flow, then

$$m_{g(t_1)}(\Gamma) \geq m_{g(t_0)}(\Gamma)$$

for any $t_1 \geq t_0$. In certain cases (for example, if the image of Γ in $H_1(M^n; \mathbb{R})$ is nonzero, or if $\mathcal{M}^3$ is Haken, or if $\pi_1(\mathcal{M}^3)$ is word-hyperbolic) one knows that $m_{g(0)}(\Gamma)$ is positive. (Question: when is $m_{g(0)}(\Gamma)$ positive for a free homotopy class Γ in a homology 3-sphere?) Whenever one knows that $m_{g(0)}(\Gamma)$ is positive, our method yields a lower bound for the diameter of the solution $g(t)$ for however long it exists. In particular, if $\alpha \in H_1(M^n; \mathbb{Z})$ is any element of infinite order, one concludes that the infimum of the lengths measured with respect to $g(t)$ of all curves representing α is bounded below by some $c > 0$ depending only on α and g_0.

This survey has presented only a tiny fraction of what is known by experts in singularity formation for the Ricci flow. As was remarked in the introduction, anyone wishing more detailed information is urged to consult the papers of the original authors.

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